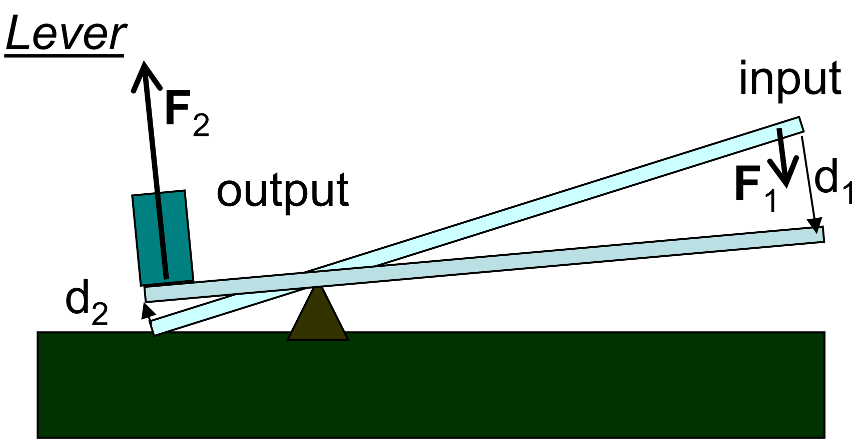
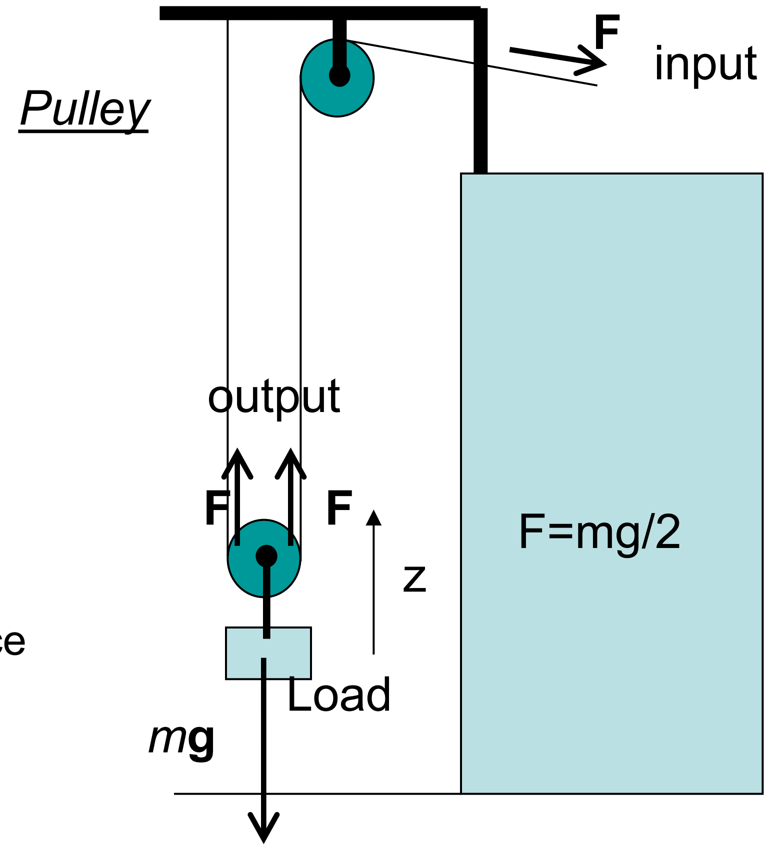
6 – Work and Energy

Work - The concept following from the analysis of simple mechanical devices





• Simple machines allow to gain in force:
achieve a greater *output* force with a smaller *input* force

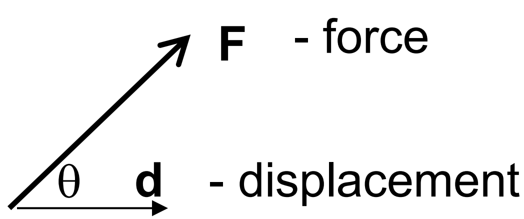
• Gaining in force one loses in displacement

$$F_1 d_1 = F_2 d_2$$

• Concept of *work*:

Work = *force* $\times$ *displacement* is conserved: (work input) = (work output)

Definition of Work for a constant force



Definition: $W = F_d d = F d_F = F d \cos\theta = \mathbf{F} \cdot \mathbf{d}$

(dot-product of two vectors)

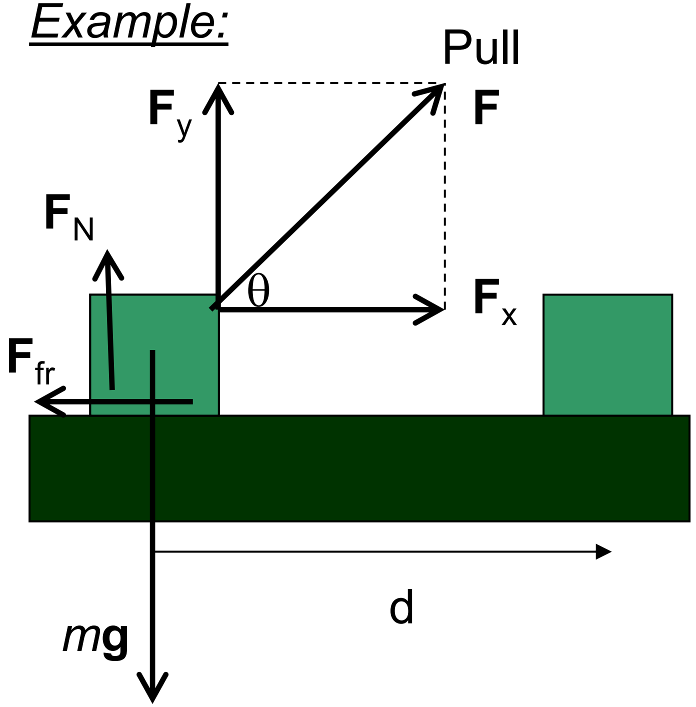
In components: $W = \mathbf{F} \cdot \mathbf{d} = F_x d_x + F_y d_y$

Force perpendicular to the displacement does not produce work as $\cos\theta = 0!$

Work is negative, if $\cos\theta < 0$

Unit of work: J(oule) = N(ewton) m = kg m^2/s^2

Example:



$m = 50$ kg, $F = 100$ N, $F_{fr} = 50$ N

$\theta = 37°$, $d = 40$ m

Work done by each force - ?

Solution

$$W_G = mgd \cos 90° = 0;$$
$$W_N = F_N d \cos 90° = 0;$$
$$W_{pull} = Fd \cos\theta = 100\ N \times 40\ m \times \cos 37° = 3200\ J$$
$$W_{fr} = F_{fr} d \cos 180° = 50\ N \times 40\ m \times (-1) = -2000\ J$$

Infinitesimal work and total work

If the force is not constant but changes from point to point, one has to consider the infinitesimal work corresponding to an infinitesimal displacement $\Delta\mathbf{r} \to \mathbf{0}$:

$$\Delta W = \mathbf{F} \cdot \Delta\mathbf{r}$$

The total work is the sum of all infinitesimal works along the trajectory:

$$W = \sum_i \Delta W_i = \sum_i \mathbf{F}_i \cdot \Delta\mathbf{r}_{\,i}$$

Power

- Rate of doing work

Instantaneous power: $P = \frac{\Delta W}{\Delta t}$

It can be expressed as $P = \frac{\mathbf{F} \cdot \Delta\mathbf{r}}{\Delta t} = \mathbf{F} \cdot \mathbf{v}$

Unit of power: W(att)

W = J(oule) /s = kg m^2/s^3

Car:

For $P = const$ one has $F = \frac{P}{v}$, so that the maximal acceleration $a = \frac{F}{m} = \frac{P}{mv}$ decreases with the speed

(Mechanical) Energy - Work stored in a body or ability of a body to do work

Mechanical energy = Kinetic energy + Potential energy
$E = E_{kin} + E_{pot}$

$$E_{kin} = \frac{mv^2}{2}, \qquad E_{pot} \text{ - different forms}$$

Work done → Increase of energy

Illustration for the linear motion with constant acceleration $(x_0 = v_0 = 0)$

Work (function of the process): $W = Fx = ma\frac{1}{2}at^2 = \frac{m(at)^2}{2} = \frac{mv^2}{2}$ Kinetic energy (function of the state)

In general: $W_{12} = \Delta E = E_2 - E_1$

Work of external forces done on the way from position 1 to position 2 equals the change of energy of the system

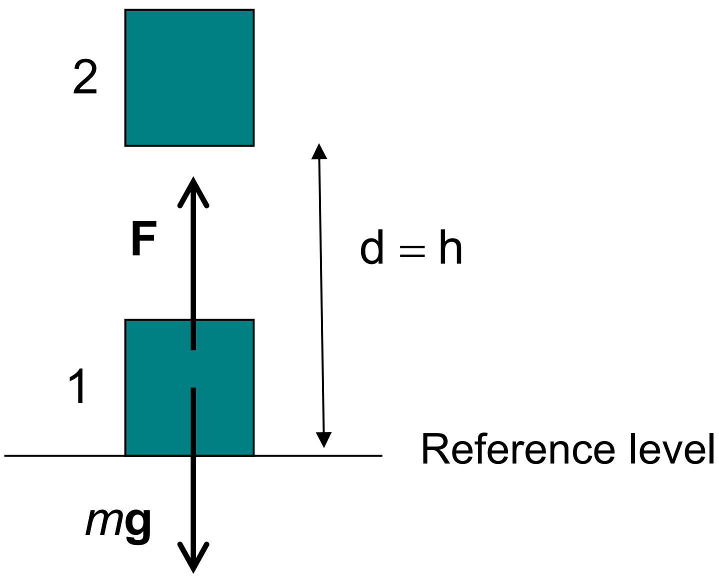
Or $W = E_f - E_i$, where E_f and E_i are the final and initial total energies and W is the work of the external forces on the system.

Potential Energy

Work of the external force needed to bring a system from the reference state into another state quasistatically ($v \to 0$)

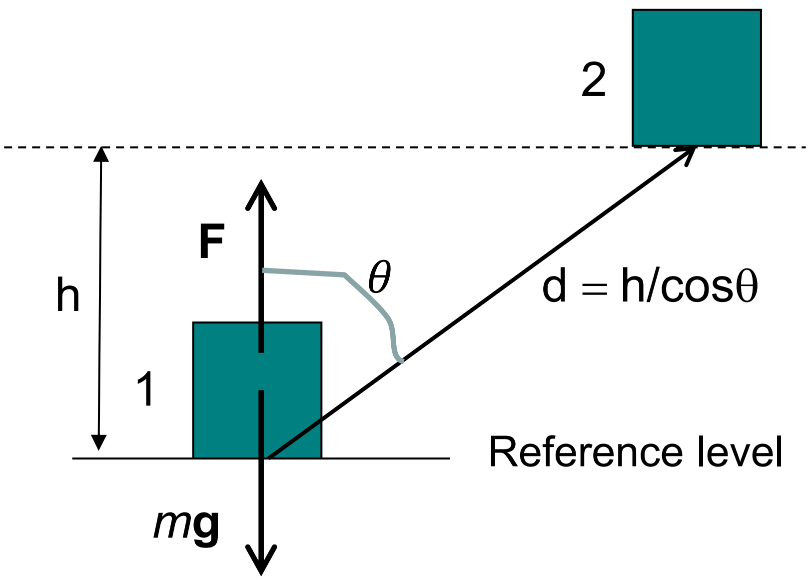
Potential energy is defined up to an arbitrary constant that can be understood as the potential energy of the reference state

Gravitational energy



$a = 0 \Rightarrow F = mg$

$$E_{pot} = W = Fh = mgh$$

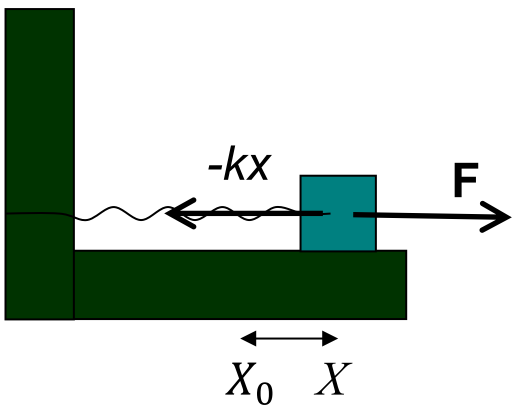


$$E_{pot} = W = Fd\cos\theta = mg\frac{h}{\cos\theta}\cos\theta = mgh$$

Elastic energy (the energy of a deformed spring)

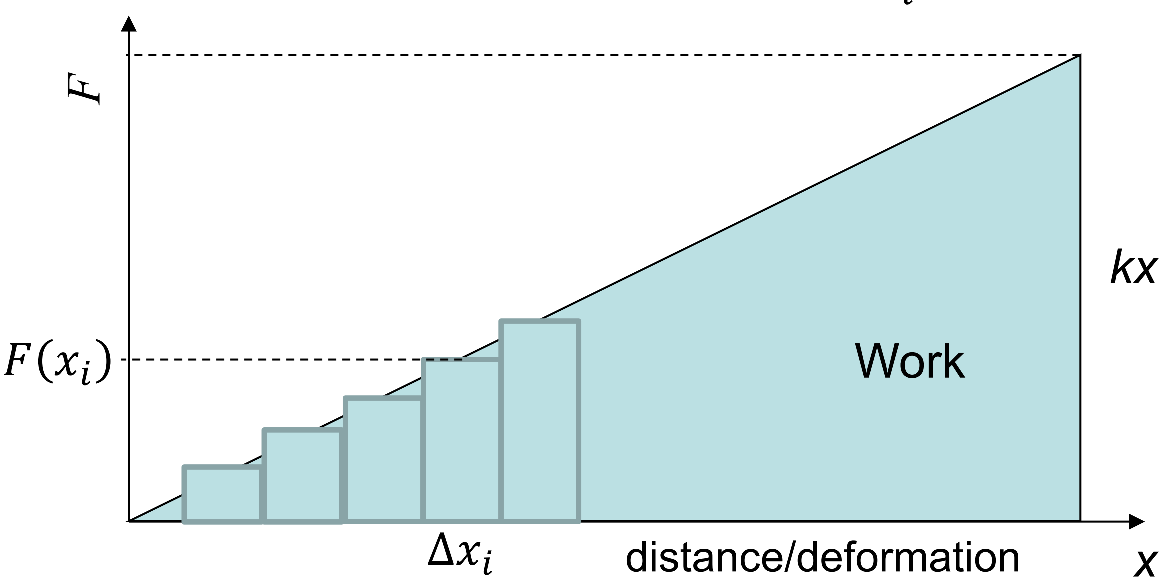
Hooke's law for the spring: $F_{Hooke} = -kx$

k – stiffness of the spring
x – elongation/kompression, $x \equiv X - X_0$
X_0 - the length of the free spring
X – the length of the deformed spring



$$F + F_{Hooke} = ma = 0 \;\; \rightarrow \;\; F = kx \text{ (external force)}$$

$$E_{pot} = W = \sum_i F(x_i)\Delta x_i$$ - area under the curve $F(x)$



$$E_{pot} = \frac{1}{2} x\, kx = \frac{kx^2}{2}$$

Conservation of Energy

In the absence of dissipation (friction) the *total energy* of an isolated system is conserved:

$$E \equiv E_{\text{tot}} = E_{\text{pot}} + E_{\text{kin}} = \text{const}$$

That is, initial energy is equal to the final energy

$$E_i = E_f$$

Energies of the two different kinds can be transformed into each other:

- potential energy can be released into kinetic energy
- kinetic energy can be absorbed into potential energy

Example: free fall from the height h

Show that the total energy is conserved, $E_f = E_i$.

Initial state: $E_{pot} = mgh$ and $E_{kin} = 0$, $E_i = mgh$

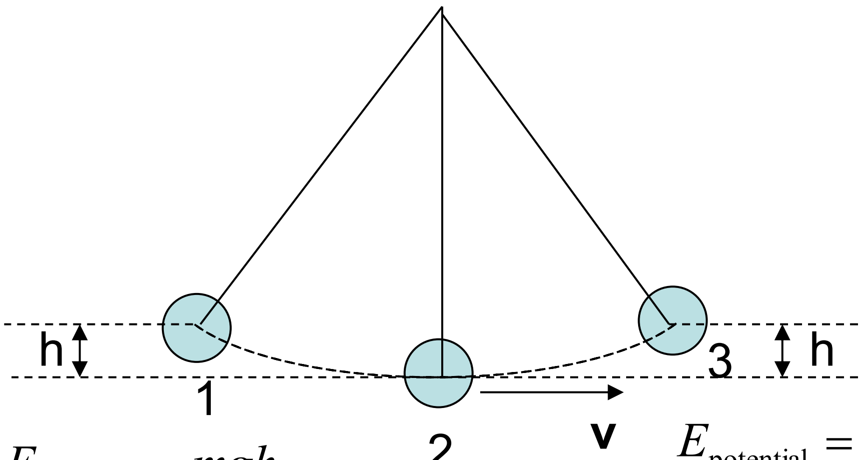
Final state: $z = 0$, $E_{pot} = mgz = 0$, $E_{kin} = \frac{mv^2}{2}$

$$v = -gt, \quad z = h - \frac{1}{2}gt^2, z = 0 \; \rightarrow t^2 = \frac{2h}{g}$$

$$E_f = E_{kin} = \frac{mv^2}{2} = \frac{mg^2t^2}{2} = \frac{mg^2\,2h/g}{2} = mgh = E_i.$$

Pendulum

(an example of energy conservation)



$E_{\text{potential}} = mgh$

$E_{\text{potential}} = mgh$

$E_{\text{kinetic}} = \dfrac{mv^2}{2}$

Oscillations!

Problem

At 1: m=0.5 kg, h=12 cm, v=0
Speed at 2 ?

Solution

Energy conservation in general:

$$E_1 = E_{\text{kin},1} + E_{\text{pot},1}$$
$$= E_2 = E_{\text{kin},2} + E_{\text{pot},2}$$

But here $E_{\text{kin},1} = E_{\text{pot},2} = 0$,

thus, the energy conservation has the form

$$E_{\text{pot},1} = mgh = E_{\text{kin},2} = \frac{mv_2^2}{2}$$

$$v_2 = \sqrt{2gh} = \sqrt{2 \times 9.8\ \text{m/s}^2 \times 0.12\ \text{m}} = 1.53\ \text{m/s}$$

Mathematical treatment of the pendulum problem and describing its harmonic (sinusoidal) motion requires calculus. This is because the Newton's second law in this case is not an algebraic equation but a differential equation.

Problem

A dart of a mass 0.100 kg is pressed against the spring of a toy dart gun. The spring with spring stiffness *k* = 250 N/m is compressed 6.0 cm and released. If the dart detaches from the spring when the spring is reaching ist natural length (*x*=0) what speed does the dart acquire?

Known: m = 0.1 kg, k = 250 N/m, x_1 = -6 cm = -0.06 m

To find: v_2 - ?

Solution:

The total energy of the system spring + dart is conserved

State 1: Deformed spring, potential energy

State 2: Flying dart, kinetic energy

$$E_1 = E_2 \quad \Rightarrow \quad \frac{1}{2}kx_1^2 = \frac{mv_2^2}{2} \quad \Rightarrow \quad v_2 = \sqrt{\frac{kx_1^2}{m}} = x_1\sqrt{\frac{k}{m}}$$

General analytical result

$$\sqrt{x^2} = \left(x^2\right)^{1/2} = x^{2\times 1/2} = x^1 = x$$

More accurately:

$$\sqrt{x^2} = |x|$$

Plugging numbers:

$$v_2 = 0.06\,\text{m}\sqrt{\frac{250\ \text{N/m}}{0.1\,\text{kg}}} = 0.06\sqrt{2500} = 0.06\times 50 = 3\ \text{m/s}$$

Check units separately:

$$\text{m}\sqrt{\frac{\text{N/m}}{\text{kg}}} = \text{m}\sqrt{\frac{\text{kg m/s}^2/\text{m}}{\text{kg}}} = \text{m}\sqrt{1/\text{s}^2} = \text{m/s, OK}$$