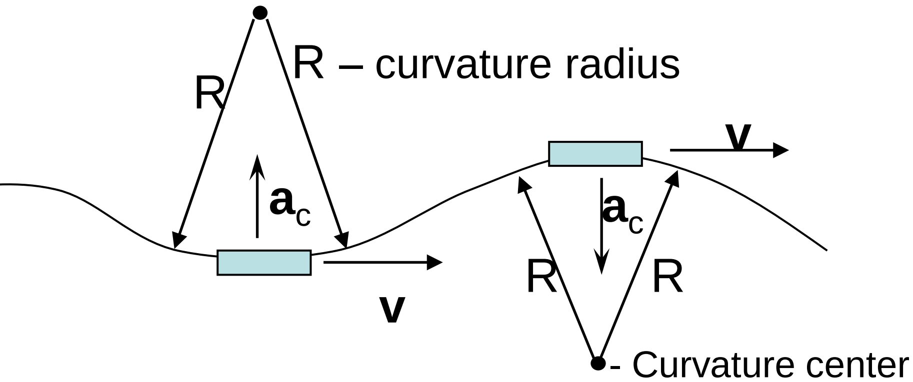
5 – Circular motion, Planets, Gravity

Centripetal acceleration

- Acceleration perpendicular to the velocity



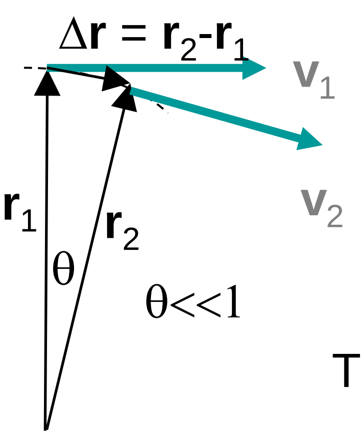
For any point of a smooth curve one can can define curvature center and curvature radius

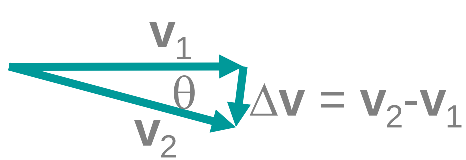
In general there are both centripetal acceleration and acceleration along the velocity

Let v = |**v**| = const. Then **v** can only change direction.

> **a**⊥**v** – centripetal acceleration, directed toward the curvature center

Magnitude of the centripetal acceleration





From similarity of triangles:

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{R}$$

Thus

$$a_c = \frac{|\Delta\mathbf{v}|}{\Delta t} = \frac{v}{R}\frac{|\Delta\mathbf{r}|}{\Delta t} = \frac{v}{R}v = \frac{v^2}{R}$$

We use

$$v_1 = v_2 = v, \qquad r_1 = r_2 = R$$

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t}, \qquad v \equiv |\mathbf{v}| = \frac{|\Delta\mathbf{r}|}{\Delta t}$$

$$\mathbf{a} = \frac{\Delta\mathbf{v}}{\Delta t}, \qquad a \equiv |\mathbf{a}| = \frac{|\Delta\mathbf{v}|}{\Delta t}$$

Radians

Radian is such an angle, for which the length of the arc is equal to the radius. In other words, the angle in radians is given by L/R and it is dimensionless. Revolution corresponds to $L=2\pi R$, thus $360°=2\pi$ radians. That is,

In radians, $\theta = L/R$ (no unit!)
Full circle: $\theta = 2\pi R/R = 2\pi = 360°$

Thus

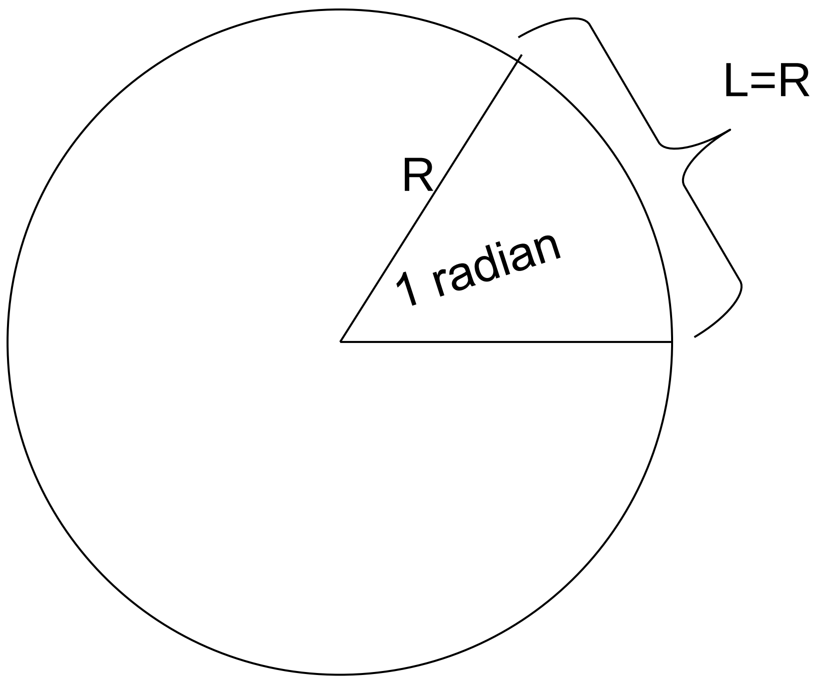
1 radian = $360°/(2\pi)=57.3°$

Useful formulas (for θ in radians):

$$\sin\theta \cong \tan\theta \cong \theta$$
$$\cos\theta \cong 1-\theta^2/2$$
for $\theta \ll 1$

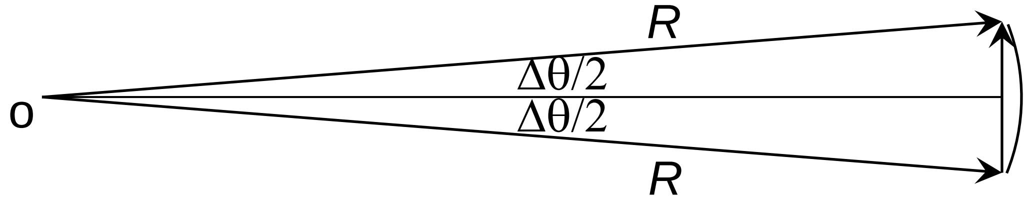
Angles θ and $\Delta\theta$ can be measured in

- degrees
- revolutions (360°) (used in engineering)
- radians (used in physics)



Displacement due to rotation

If a vector $\mathbf{R}$ is rotated by a small angle $\Delta\theta$, the change of the vector (the displacement of its end point) $\Delta\mathbf{r}$ is proportional to $\Delta\theta$, so that $\Delta r = R\Delta\theta$. This can be derived by approximating the small arc by a straight line and considering the two triangles with one angle equal to 90°, as shown

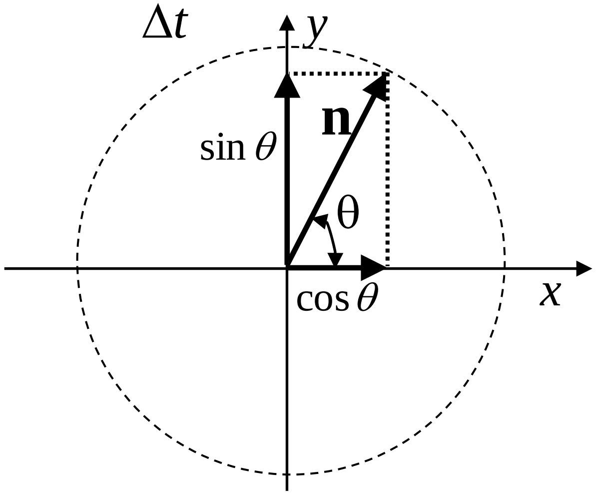


$$\Delta r = 2R\sin\frac{\Delta\theta}{2} \cong R\Delta\theta$$

for $\theta \ll 1$

Angular velocity

Angular velocity is the rate of change of the angle with time: $\omega = \dfrac{\Delta\theta}{\Delta t}$



Relation between the angular and linear velocities

can be derived using the displacement-rotation relation above:

$$v = \frac{\Delta r}{\Delta t} = \frac{R\Delta\theta}{\Delta t} = R\frac{\Delta\theta}{\Delta t} = R\omega, \qquad \text{or} \qquad \omega = \frac{v}{R}$$

Thus $a_c = \dfrac{v^2}{R} = \dfrac{(R\omega)^2}{R} = \omega^2 R$ - another formula for the centripetal acceleration

Angular velocity, frequency, period

The angular velocity ω is defined as

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where θ is the rotation angle in radians. The frequency of rotations f is defined as the number of rotations per second,

$$f = \frac{number\ of\ rotations}{\Delta t} \quad (special\ unit{:}\ Hertz\ (Hz)).$$

As one rotations corresponds to 2π radians, the number of rotations in the angle $\Delta\theta$ is given by $\Delta\theta/(2\pi)$. Thus

$$f = \frac{\Delta\theta/(2\pi)}{\Delta t} = \frac{1}{2\pi}\frac{\Delta\theta}{\Delta t} = \frac{\omega}{2\pi}$$

and $\omega = 2\pi f$. The period T of rotations is defined as the time needed for one rotation, that is,

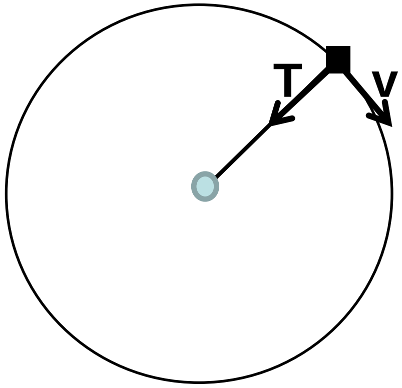
$$T = \frac{\Delta t}{number\ of\ rotations} = \frac{1}{f} = \frac{2\pi}{\omega}.$$

Centripetal forces

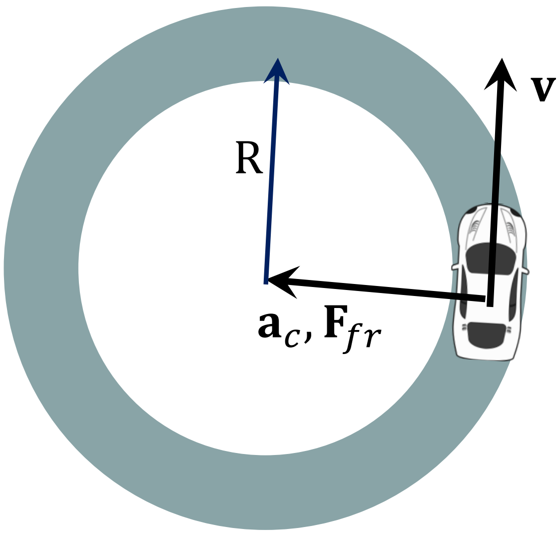
- Forces that create centripetal acceleration such as tension of a string, friction of tires against the road, etc., play the role of centripetal forces.

$$\mathbf{F}_c = m\mathbf{a}_c \Rightarrow F_c = m\frac{v^2}{R}$$

A stone on a string: tension force $\mathbf{T}$ plays the role of the centripetal force



A car on a curved road: friction force $\mathbf{F}_{\text{fr}}$ plays the role of the centripetal force

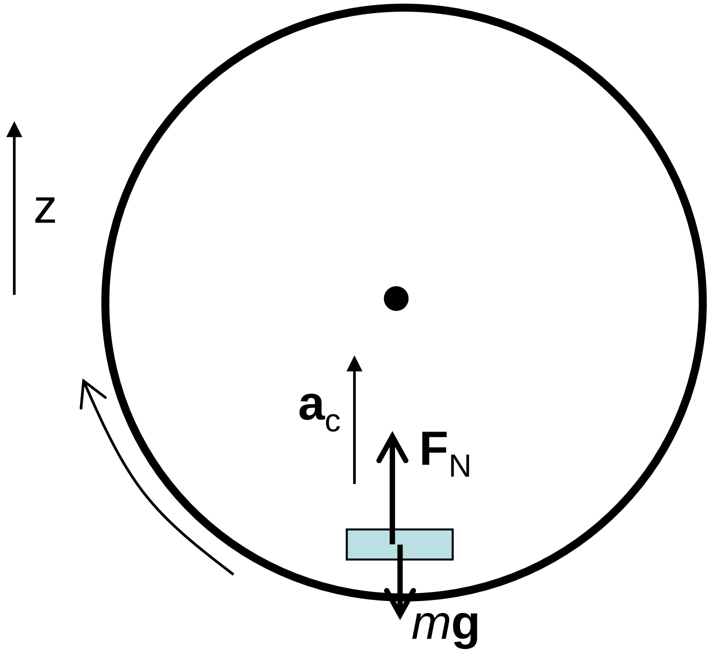


$$F_{fr} = m\frac{v^2}{R} \le \mu_s F_N = \mu_s mg$$

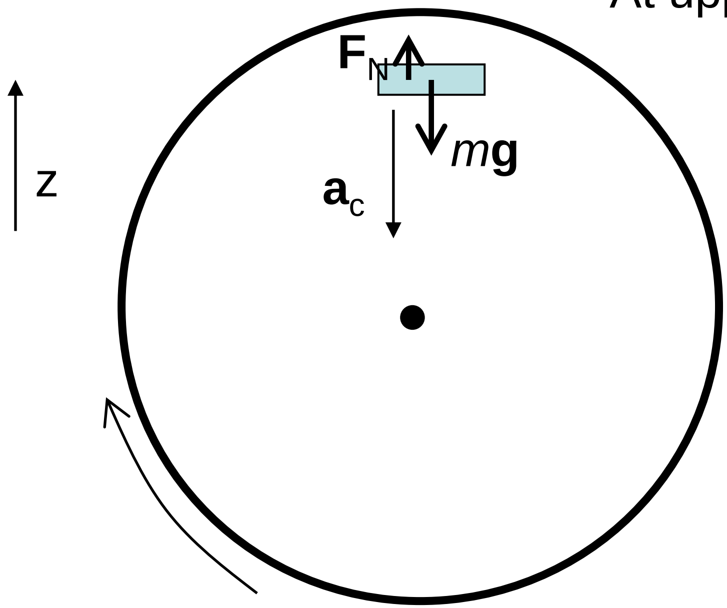
Driving is possible if the traction condition $v \le \sqrt{\mu_s R g}$ is satisfied, otherwise skidding

Ferris wheel

At lower point:



At upper point



$$F_N - mg = ma_c$$

$$F_N - mg = -ma_c$$

$$\Rightarrow F_N = m(g + a_c)$$

$$\Rightarrow F_N = m(g - a_c)$$

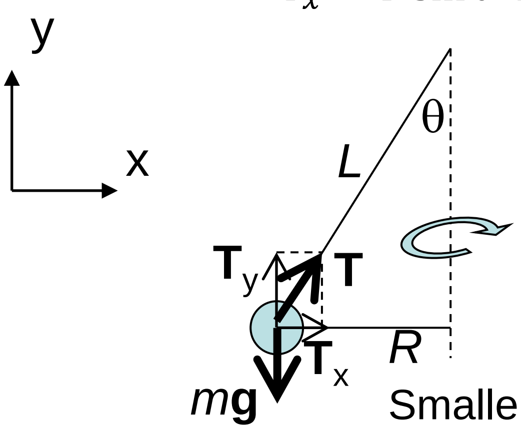
Larger pressure on seat
(apparent weight)

Smaller pressure on seat
(apparent weight)

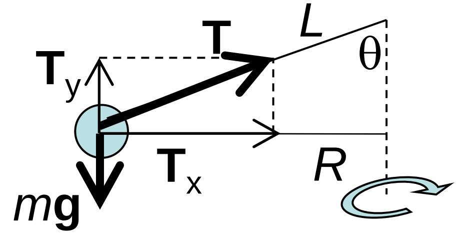
Conic pendulum (rotating with the angular velocity ω around the vertical axis)

L – length of the pendulum, *T* – tension of the string

$T_x = T\sin\theta$ is the centripetal force



Smaller angular velocity



Larger angular velocity, larger θ:

Newton's second law:

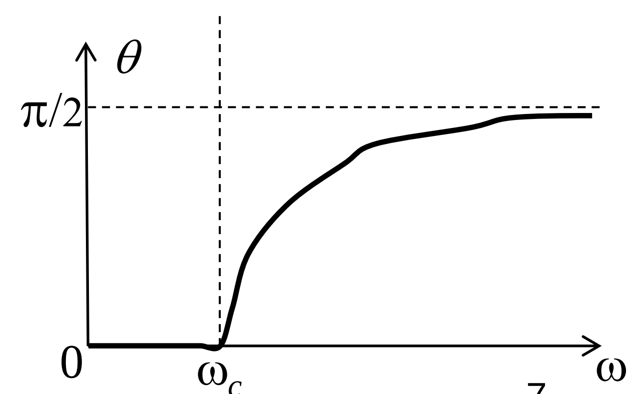
"y": $T\cos\theta - mg = 0 \;\rightarrow\; \theta = \arccos\dfrac{mg}{T}$

"x": $T\sin\theta = F_c = m\omega^2 R = m\omega^2 L\sin\theta \;\rightarrow\; T = m\omega^2 L \;\; (\text{if } \sin\theta \neq 0)$

Thus $\theta = \arccos\dfrac{mg}{m\omega^2 L} = \arccos\dfrac{g}{\omega^2 L} = \arccos\dfrac{\omega_c{}^2}{\omega^2}$,

under the condition $\omega \geq \omega_c = \sqrt{g/L}$.

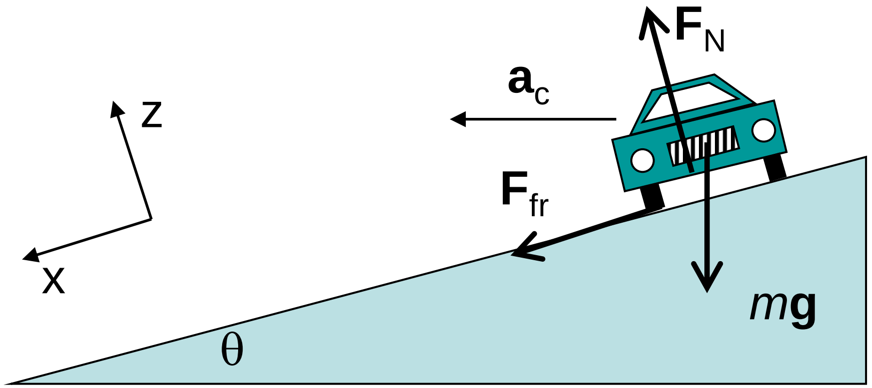
For $\omega \leq \omega_c$ the solution is $\theta = 0$, $T = mg$ *(check!)*



Car on a banked road

Idea: make F_N contribute to F_c and increase F_{fr}

Both friction force and normal force contribute to $\mathbf{F}_c$:



"z" $$-mg\cos\theta + F_N = m\frac{v^2}{R}\sin\theta$$

"x" $$mg\sin\theta + F_{fr} = m\frac{v^2}{R}\cos\theta$$

$$F_N = mg\cos\theta + m\frac{v^2}{R}\sin\theta$$

$$F_{fr} = -mg\sin\theta + m\frac{v^2}{R}\cos\theta$$

Optimal angle: $F_{fr} = 0 \Longrightarrow \tan\theta = \frac{v^2}{Rg}$

Traction condition: $F_{fr} \le \mu_s F_N$

$$-g\sin\theta + \frac{v^2}{R}\cos\theta \le \mu_s\left(g\cos\theta + \frac{v^2}{R}\sin\theta\right)$$

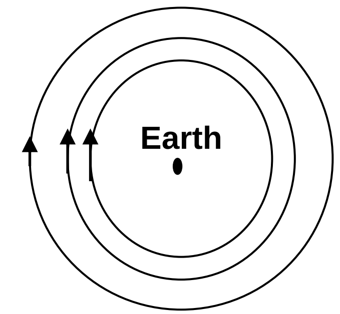
$$v^2(\cos\theta - \mu_s\sin\theta) \le gR(\sin\theta + \mu_s\cos\theta)$$

$$v^2(1 - \mu_s\tan\theta) \le gR(\tan\theta + \mu_s)$$

$$v^2 \le gR\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta} \quad \text{for} \quad \mu_s\tan\theta < 1$$

For $\mu_s\tan\theta > 1$ the traction condition is satisfied for any speed!

Evolution of views on planetary motion

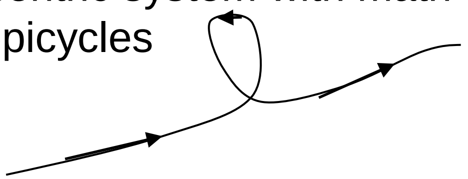


- Ancient Greeks

Naiv geocentric system
Also heliocentric system!

- Claudius Ptolemaeus (87-150, Egypt)

Elaborate geocentric system with math methods and epicycles



- Nicolaus Copernicus (1473-1543, Poland)

Revived the heliocentric system

- Galileo Galilei (1564-1642, Italy)

Championed heliocentric system by Copernicus, built telescopes

- Tycho Brahe (1546-1601, Danmark)

Collected lots of high-accuracy data
On planet motion (without telescopes)

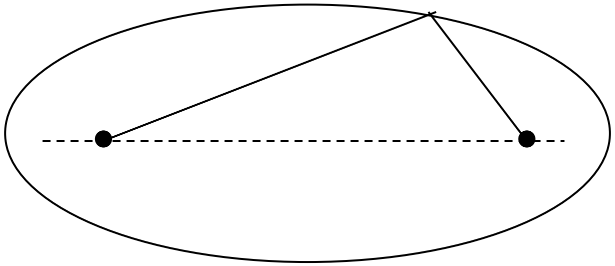
- Johannes Kepler (1571-1630, Germany)

Obtained 3 laws of planetary motion from analysis of Tycho‘s data

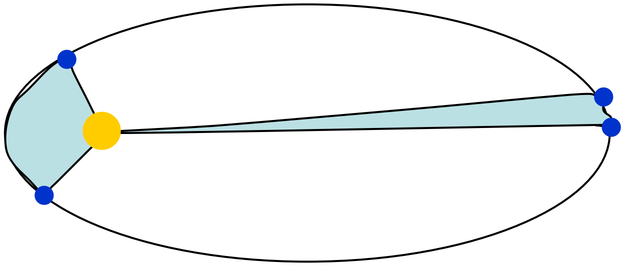
- Isaac Newton (1642-1727, England)

Obtained the law of gravitation from Kepler‘s laws

Planetary motion: Kepler‘s laws



1. The orbits of planets are ellipses
   with the sun in one of the focuses



2. The radius vector sweeps out
   equal areas in equal time

(particular case: circle)

3. $T^2/R^3 = \text{const}$ for all planets of our solar system.
   T – period of the motion
   R – average distance from the sun

Newton‘s law of universal gravitation

F=m**g** → Large distances → ?

Planet motion - projectile motion!

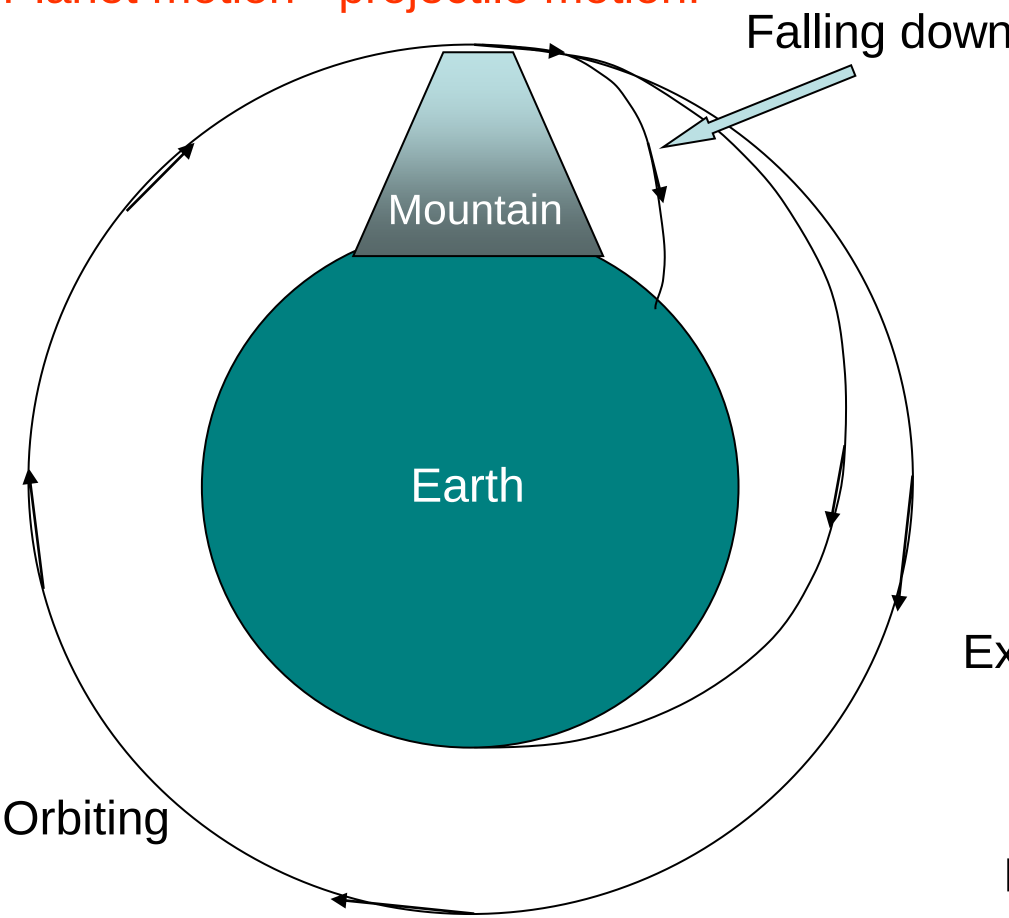
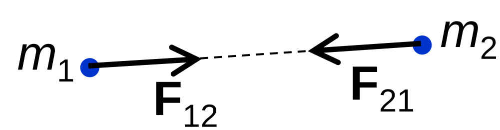


Illustration from Newton´s „Principia“

Gravitational force should decrease with distance

Newton showed mathematically that Kepler‘s laws follow from the second Newton´s law with

$$F = \frac{G m_1 m_2}{r^2}$$



Attraction

Experiments by Cavendish (1731-1810)

$$G = 0.667 \times 10^{-10}\ \mathrm{N} \bullet \mathrm{m}^2 / \mathrm{kg}^2$$

Mass of the Earth:

$$\cancel{m} g = \frac{G \cancel{m} m_E}{R_E^2} \Rightarrow m_E = \frac{g R_E^2}{G} \approx 0.6 \times 10^{25}\ \mathrm{kg}$$

with $R_E = 6400$ km

Special case of Kepler´s third law for circular orbits

m – mass of the Earth, M – mass of the Sun

$$G\frac{mM}{R^2} = m\frac{v^2}{R} \qquad (m << M) \Rightarrow v^2 = G\frac{M}{R}$$

Also, $v = \frac{2\pi R}{T}$

Eliminating v from these two equations, one obtains $G\frac{M}{R} = \left(\frac{2\pi R}{T}\right)^2 \Rightarrow \boxed{\frac{T^2}{R^3} = \frac{4\pi^2}{GM}}$

($T = 1$ year, R = 1.5×10^{11} m => M = 2.0×10^{30} kg)

Then for two objects rotating around the same center $\frac{T_1^2}{R_1^3} = \frac{T_2^2}{R_2^3}$ - the third Kepler´s law

It is more convenient to derive the 3-rd Kepler's law using the angular velocity

$$G\frac{mM}{R^2} = m\omega^2 R \Rightarrow \omega^2 R^3 = GM$$

- this is already the third Kepler´s law!

Using $\omega = \frac{2\pi}{T}$, one rewrites it as $\boxed{\frac{R^3}{T^2} = \frac{GM}{4\pi^2}}$

Geosynchronous satellites

A satellite can have the period of its orbiting equal to that of the rotation of the Earth over its axis. If the satellite is orbiting in the equatorial plane, that it will have the same position on the sky. For this, the orbit radius should have a particular value

$$R = \left(\frac{T^2}{4\pi^2} GM\right)^{1/3}$$

With $T = 1\ day = 24 \times 3600 = 86400\ s$ and $M = 0.6 \times 10^{25}\ kg$ one obtains

$$R = \left(\frac{86400^2}{4\pi^2} \times 0.667 \times 10^{-10} \times 0.6 \times 10^{25}\right)^{1/3} = 4.23\ \times 10^7\ m = 42300\ km.$$