PROBLEMS # 13

(1)

$$I_{11} = \rho \int_0^b dx_3 \int_0^b dx_2 \,(x_2^2 + x_3^2) \int_0^b dx_1$$

$$= \frac{2}{3} \rho b^5 = \frac{2}{3} M b^2$$

$$I_{12} = -\rho \int_0^b x_1\, dx_1 \int_0^b x_2\, dx_2 \int_0^b dx_3$$

$$= -\frac{1}{4} \rho b^5 = -\frac{1}{4} M b^2$$

It is easily seen that all the diagonal elements are equal and that all the off diagonal elements are equal.

If we define $\beta = M b^2 \Rightarrow$

$$I_{11} = I_{22} = I_{33} = \frac{2}{3} \beta$$

$$I_{12} = I_{13} = I_{23} = -1/4\, \beta$$

(2) In problem (1) we found the moment-of-inertia tensor for a cube (with origin at one corner) had non-zero off-diagonal elements. Evidently, the coordinate axes chosen for that calculation were not principal axes. If, for example, the cube rotates about the x_3-axis, then $\vec{\omega} = \omega_3 \hat{e}_3$ and the angular momentum vector $\vec{L}$ has the components

$$L_1 = -1/4\, \beta \omega_3$$

$$L_2 = -1/4\, \beta \omega_3$$

$$L_3 = 2/3\, \beta \omega_3$$

$$\Rightarrow \vec{L} = M b^2 \omega_3 \left(-\frac{1}{4} \hat{e}_1 - \frac{1}{4} \hat{e}_2 + \frac{2}{3} \hat{e}_3 \right)$$

which is not in the same direction as $\vec{\omega}$.

To find the principal moments of inertia, we must solve the secular equation

$$\begin{vmatrix} \frac{2}{3}\beta - I & -1/4\beta & -1/4\beta \\ -1/4\beta & \frac{2}{3}\beta - I & -1/4\beta \\ -1/4\beta & -1/4\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0 \qquad (1)$$

The value of the determinant is not affected by adding (or substracting) any row (or column) from any other row (or column). Eq (1) can be solved more easily if we substract the first row from the second.

$$\begin{vmatrix} \frac{2}{3}\beta - I & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -\frac{11}{12}\beta - I & \frac{11}{12}\beta - I & 0 \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0$$

We can factor $\left(\frac{11}{12}\beta - I\right)$ from the second row

$$\left(\frac{11}{12}\beta - I\right) \begin{vmatrix} \frac{2}{3}\beta - I & -\frac{1}{4}\beta & -\frac{1}{4}\beta \\ -1 & -1 & 0 \\ -\frac{1}{4}\beta & -\frac{1}{4}\beta & \frac{2}{3}\beta - I \end{vmatrix} = 0$$

Expanding we have

$$\left(\frac{11}{12}\beta - I\right)\left[\left(\frac{2}{3}\beta - I\right)^2 - \frac{1}{8}\beta^2 - \frac{1}{4}\beta\left(\frac{2}{3}\beta - I\right)\right] = 0$$

which can be factored to obtain

$$\left(\frac{1}{6}\beta - I\right)\left(\frac{11}{12}\beta - I\right)\left(\frac{11}{12}\beta - I\right) = 0$$

Thus we have the following roots, which give the principal moments of inertia

$$I_1 = 1/6\,\beta \qquad I_2 = \frac{11}{12}\beta \qquad I_3 = \frac{11}{12}\beta$$

The diagonalized moment-of-inertia tensor becomes

$$\{\mathbb{I}\} = \begin{pmatrix} 1/6\,\beta & 0 & 0 \\ 0 & \frac{11}{12}\beta & 0 \\ 0 & 0 & \frac{11}{12}\beta \end{pmatrix}$$

Because two of the roots are identical $I_2 = I_3$, the principal axis associated with I_1 must be an axis of symmetry.

To find the direction of the principal axis associated with I_1, we have

$$\left(\frac{2}{3}\beta - \frac{1}{4}\beta\right)\omega_{11} - \frac{1}{4}\beta\omega_{21} - \frac{1}{4}\beta\omega_{31} = 0$$

$$-\frac{1}{4}\beta\omega_{11} + \left(\frac{2}{3}\beta - \frac{1}{4}\beta\right)\omega_{21} - \frac{1}{4}\beta\omega_{31} = 0$$

$$-\frac{1}{4}\beta\omega_{11} - \frac{1}{4}\beta\omega_{21} + \left(\frac{2}{3}\beta - \frac{1}{4}\beta\right)\omega_{31} = 0$$

where the second subscript 1 on the ω_i signifies that we are considering the principal axis associated with I_1. Dividing the first two of this equations by $\beta/4$, we have

$$2\omega_{11} - \omega_{21} - \omega_{31} = 0$$

$$-\omega_{11} + 2\omega_{21} - \omega_{31} = 0$$

Substracting the second of these equations from the first, we find $\omega_{11} = \omega_{21}$.

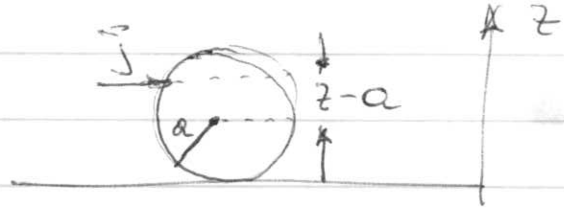
Then $\omega_{11} = \omega_{21} = \omega_{31}$, and the desired ratios are

$$\omega_{11} : \omega_{21} : \omega_{31} = 1:1:1$$

Therefore, when the cube rotates about an axis that has associated with it the moment of inertia $I_1 = \frac{1}{6}\beta = \frac{1}{6}Mb^2$, the projection of $\vec{\omega}$ on the 3 coordinate axes are all equal. Hence this principal axis corresponds to the diagonal of the cube.

Because the moments I_2 and I_3 are equal, the orientation of the principal axes associated with this moments is arbitrary, the need only to be in a plane normal to the diagonal of the cube.

(3)



The solid ball receives an impulse J, that is a force $F(t)$ is applied during a short interval of time τ so that

$$J = \int F(t')\, dt'$$

The equations of motion are

$$\frac{dp}{dt} = F$$

$$\frac{dL}{dt} = r \times F$$

which, for this case, yield

$$\Delta p = \int F(t')\, dt' = J$$

$$\Delta L = \int r \times F(t')\, dt' = r \times J$$

Since $p(t=0) = 0$ and $L(t=0) = 0$,

after the application of the impulse,

we have

$$p = M V_{CM} = J$$

$$L = I_0 \omega = r \times J = (z-a)\, J \frac{\vec{\omega}}{\omega}$$

So that

$$V_{CM} = \frac{J}{M}$$

and

$$\omega = \frac{J}{I_0} (z-a) \frac{\vec{\omega}}{\omega}$$

where $I_0 = \frac{2}{5} M a^2$

The velocity for any point α on the ball is given by

$$V_\alpha = V_{CM} + \omega \times r_\alpha$$

For the point of contact Q, this becomes

$$V_Q = V_{CM} - \omega a \quad \frac{\vec{J}}{J} = \frac{J}{M}\left[1 - \frac{5(z-a)}{2a}\right]$$

Then for rolling without slipping,

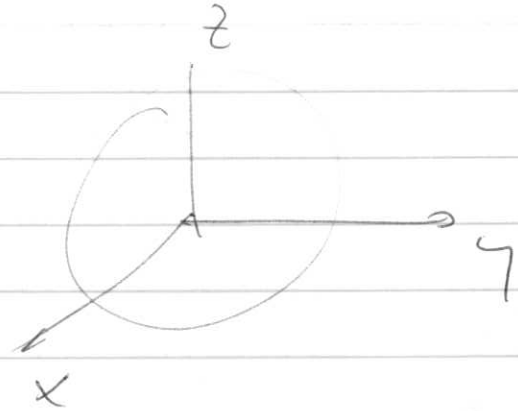
$V_Q = 0$, and we have

$$2a = 5(z-a)$$

so that

$$t = \frac{7}{5}a$$

(4) Let us compare the moments of inertia for the two spheres for axes through the center of each. For the solid sphere we have



$x = r \sin\theta \cos\phi$

$y = r \sin\theta \sin\phi$

$z = r \cos\theta$

Using the definition of moment of inertia

$$I_{ij} = \int \rho(r) \left[\delta_{ij} \sum_k x_k^2 - x_i x_j\right] dV$$

we have

$$I_{33} = \rho \int (r^2 - z^2)\, dV$$

$$= \rho \int \left(r^2 - r^2 \cos^2\theta \right) r^2 dr\, d(\cos\theta) d\phi$$

$$I_{33} = \rho \int_0^R r^4 dr \int_{-1}^{+1} (1 - \cos^2\theta)\, d(\cos\theta) \int_0^{2\pi} d\phi$$

$$= 2\pi \rho \, \frac{R^5}{5} \cdot \frac{4}{3}$$

The mass of the sphere is

$$M = \frac{4}{3}\pi \rho R^3$$

Therefore

$$I_{33} = \frac{2}{5} MR^2$$

Since the sphere is symmetric around the origin, the diagonal elements of $\{I\}$ are equal

$$I_{11} = I_{22} = I_{33} = \frac{2}{5} MR^2$$

A typical off-diagonal element is

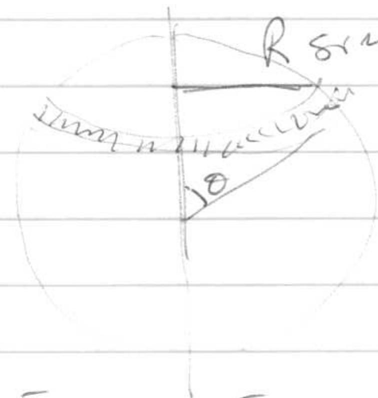
$$I_{12} = \rho \int (-xy)\, dV$$

$$= -\rho \int r^2 \sin^2\theta \sin\phi \cos\phi \, r^2 dr \, d(\cos\theta) \, d\phi$$

This vanishes because the integral with respect to ϕ is zero.

Thus $I_1 = I_2 = I_3 = \frac{2}{5} MR^2$

For the hollow sphere



$$I_4 = \sigma \int_0^{2\pi} d\phi \int_0^{\pi} (R \sin\theta)^2 R^2 \sin\theta \, d\theta$$

$$= 2\pi\sigma R^4 \int_0^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{8}{3} \pi \sigma R^4$$

Or using $4\pi\sigma R^2 = M \Rightarrow I_4 = \frac{2}{3} M R^2$

Let us roll each ball down an inclined plane.

The kinetic energy is

$$T = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} I \dot{\theta}^2$$

where y is the measure of the distance along the plane. The potential energy is

$$U = Mg(l - y) \sin\alpha$$

where l is the length of the plane and α is the angle of inclination of the plane. Now, $y = R\theta$, so that the Lagrangian can be expressed as

$$L = \frac{1}{2} M \dot{y}^2 + \frac{1}{2} \frac{I}{R^2} \dot{y}^2 + Mgy \sin\alpha$$

where the constant term in U has been suppressed. The equation of motion in y is obtained in the usual way and we find

$$\ddot{y} = \frac{g\, MR^2 \sin\alpha}{MR^2 + I}$$

Therefore, the sphere with smaller moment of inertia (the solid sphere) will have the greater acceleration down the plane.

(5) Once the mountain has been added, the Earth has only one axis of symmetry $e_{z'}$, which goes through the mountain. The angular velocity $\vec{\omega}$ now precesses about the mountain at a rate

$$\Omega = \left(\frac{I_{\parallel}}{I_{\perp}} - 1 \right) \omega_{z'} = - \frac{m R^2 \omega'_z}{\frac{2}{5} M R^2}$$

$$= - \frac{5m}{2M} \omega \cos\alpha$$

The distance $d = 100$ mi moved by the pole in time t is $d = (R \sin\alpha) |\Omega| t$.

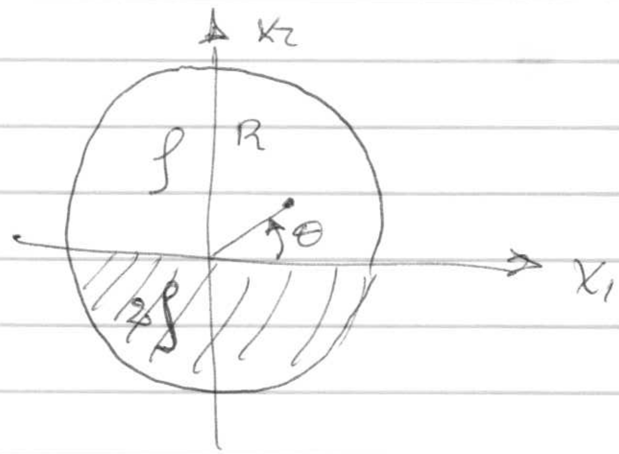
Therefore

$$t = \frac{d}{(R \sin\alpha)\Omega} = \frac{2Md}{5mR\omega \sin\alpha \cos\alpha}$$

or, putting in the numbers (including

$\omega = 2\pi/\text{day}$, $t = 3.68 \times 10^5$ days $\simeq 1000$ yr

(6)



The CM of the disk is $(0, \bar{x}_2)$ where

$$\bar{x}_2 = \frac{\rho}{M}\left[2\int_{\substack{\text{lower}\\ \text{semicircle}}} x_2\, dx_1\, dx_2 + \int_{\substack{\text{upper}\\ \text{semicircle}}} x_2\, dx_1\, dx_2\right]$$

$$= \frac{\rho}{M}\left[\int_0^R\int_0^{\pi}(r\sin\theta)\cdot r\,dr\,d\theta + 2\int_0^R\int_{\pi}^{2\pi}(r\sin\theta)\,r\,dr\,d\theta\right]$$

$$= -\frac{2}{3}\frac{\rho R^3}{M} \qquad (1)$$

Now the mass of the disk is

$$M = \rho \frac{1}{2} \pi R^2 + 2\rho \cdot \frac{1}{2} \pi R^2 = \frac{3}{2} \rho \pi R^2 \qquad (2)$$

so that $\qquad \bar{x}_2 = -\frac{4}{9\pi} R \qquad (3)$

The direct calculation of the moment of inertia with respect to an axis through the CM is tedious, so we first compute I with respect to the x_3-axis and then use Steiner's theorem

$$I_3 = \rho \left[\int_0^R \int_0^\pi r^2 \cdot r\,dr\,d\theta + 2 \int_0^R \int_\pi^{2\pi} r^2 \cdot r\,dr\,d\theta \right]$$

$$= \frac{3}{4} \pi \rho R^4 = \frac{1}{2} M R^2 \qquad (4)$$

Then

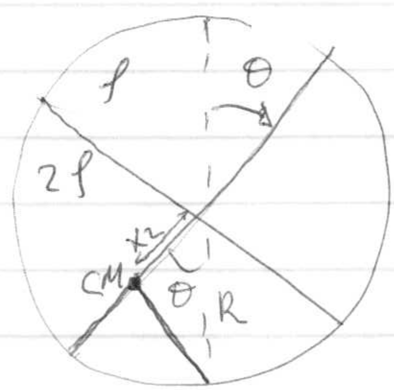
$$I_0 = I_3 - M\bar{x}_2^2$$

$$= \frac{1}{2}MR^2 - M \cdot \frac{16}{81\pi^2} R^2$$

$$= \frac{1}{2}MR^2\left[1 - \frac{32}{81\pi^2}\right] \quad (5)$$

When the disk rolls without slipping, the velocity of the CM can be gotten as follows

$$x_{CM} = R\theta - |\bar{x}_2|\sin\theta$$



$$y_{CM} = R - |\bar{x}_2|\cos\theta$$

$$\dot{x}_{CM} = R\dot{\theta} - |\bar{x}_2|\dot{\theta}\cos\theta$$

$$\dot{y}_{CM} = |\bar{x}_2|\dot{\theta}\sin\theta$$

$$\left(\dot{x}_{CM}^2 + \dot{y}_{CM}^2\right) = V^2 = R^2\dot{\theta}^2 + \bar{x}_2^2\dot{\theta}^2 - 2\dot{\theta}^2 R|\bar{x}_2|\cos\theta$$

$$V^2 = a^2 \dot{\theta}^2$$

where

$$a = \sqrt{R^2 + \bar{x}_2^2 - 2R|\bar{x}_2| \cos\theta}$$

Using (3), can be written as

$$a = R\sqrt{1 + \frac{16}{81\pi^2} - \frac{8}{9\pi}\cos\theta}$$

The kinetic energy is

$$T = T_{trans} + T_{rot}$$

$$= \frac{1}{2} M V^2 + \frac{1}{2} I_0 \dot{\theta}^2$$

Substituting and simplifying yields

$$T = \frac{1}{2} M R^2 \dot{\theta}^2 \left[\frac{3}{2} - \frac{8}{9\pi} \cos\theta \right]$$

The potential energy is

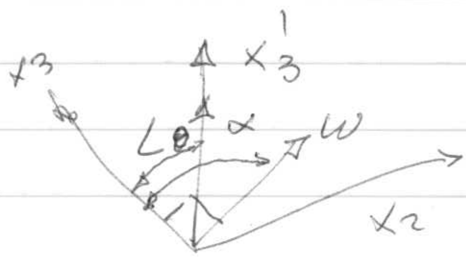
$$U = Mg\left[R + \bar{x}_2 \cos\theta\right]$$

$$\doteq MgR\left[1 - \frac{4}{9\pi}\cos\theta\right]$$

Thus the Lagrangian is

$$\mathcal{L} = \frac{1}{2}MR\left\{R\dot{\theta}^2\left[\frac{3}{2} - \frac{8}{9\pi}\cos\theta\right] - g\left[2 - \frac{8}{9\pi}\cos\theta\right]\right\}$$

(7)



Initially

$$L_1 = 0 = I_1\omega_1$$

$$L_2 = L\sin\theta = I_1\omega_2 = I_1\omega\sin\alpha$$

$$L_3 = L\cos\theta = I_3\omega_3 = I_3\omega\cos\alpha$$

Thus,

$$\tan\theta = \frac{L_2}{L_3} = \frac{I_1}{I_3}\tan\alpha \qquad (1)$$

Using eqs. (I), (7), and (I) from the lecture

$$\omega_{x'} = \sin\psi \sin\theta\, \dot\phi + \cos\psi\, \dot\theta \qquad (I)$$

$$\omega_{y'} = \cos\psi \sin\theta\, \dot\phi - \sin\psi\, \dot\theta \qquad (7)$$

$$\omega_{z'} = \cos\theta\, \dot\phi + \dot\psi \qquad (I)$$

$$\omega_3 = \dot\phi \cos\theta + \dot\psi$$

Since $\omega_3 = \omega \cos\alpha$, we have

$$\dot\phi \cos\theta = \omega \cos\alpha - \dot\psi \qquad (2)$$

Using

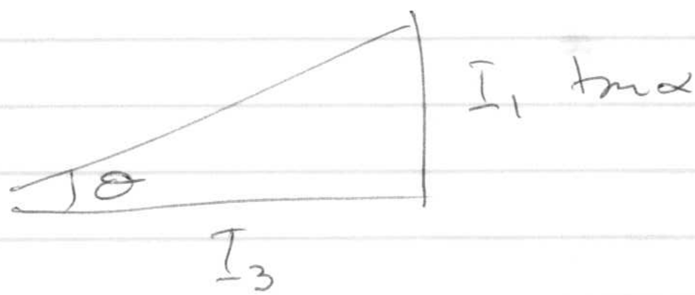
$$\Omega = \left(\frac{I_{\parallel}}{I_{\perp}} - 1\right)\omega_{z}$$

$$\dot\psi = -\Omega = -\frac{I_3 - I_1}{I_1}\,\omega_3$$

(2) becomes

$$\dot\phi \cos\theta = \frac{I_3}{I_1}\,\omega \cos\alpha \qquad (3)$$

From (1), we may construct the following triangle



from which $\cos\theta = \dfrac{I_3}{\left[I_3^2 + I_1^2 \tan^2\alpha\right]^{1/2}}$

Substituting into (3) gives

$$\dot{\phi} = \frac{\omega}{I_1} \sqrt{I_1^2 \sin^2\alpha + I_3^2 \cos^2\alpha}$$