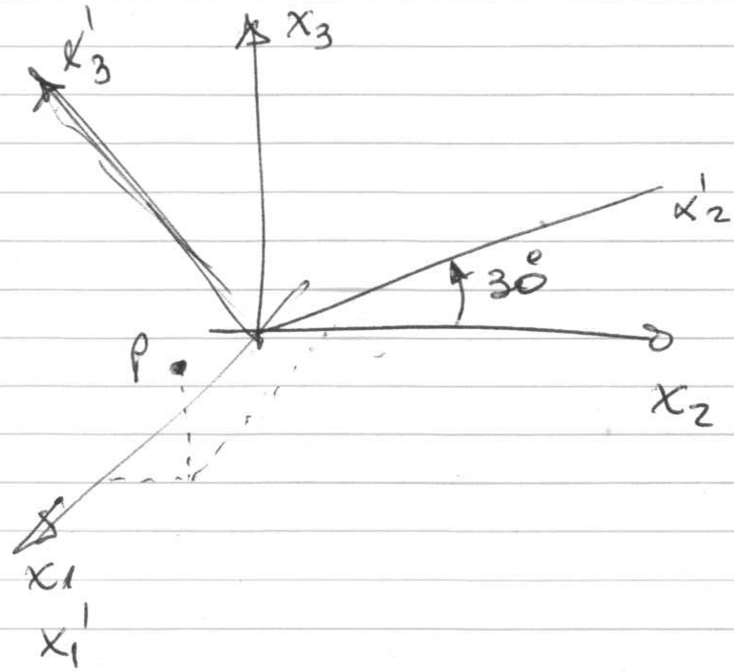
PROBLEMS 1.

(1)



The direction cosines λ_{ij} are

$$\lambda_{11} = \cos(x_1', x_1) = \cos(0°) = 1$$
$$\lambda_{12} = \cos(x_1', x_2) = \cos(90°) = 0$$
$$\lambda_{13} = \cos(x_1', x_3) = \cos(90°) = 0$$
$$\lambda_{21} = \cos(x_2', x_1) = \cos(90°) = 0$$
$$\lambda_{22} = \cos(x_2', x_2) = \cos(30°) = 0.866$$
$$\lambda_{23} = \cos(x_2', x_3) = \cos(90° - 30°) = 0.5$$
$$\lambda_{31} = \cos(x_3', x_1) = \cos(90°) = 0$$
$$\lambda_{32} = \cos(x_3', x_2) = \cos(90° + 30°) = -0.5$$
$$\lambda_{33} = \cos(x_3', x_3) = \cos(30°) = 0.866$$

$$\lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.866 & 0.5 \\ 0 & -0.5 & 0.866 \end{pmatrix}$$

$P(2,1,3)$

$$x_1' = \lambda_{11} x_1 + \lambda_{12} x_2 + \lambda_{13} x_3 = x_1 = 2$$

$$x_2' = \lambda_{21} x_1 + \lambda_{22} x_2 + \lambda_{23} x_3 = 0.866\, x_2 + 0.5\, x_3 = 2.37$$

$$x_3' = \lambda_{31} x_1 + \lambda_{32} x_2 + \lambda_{33} x_3 = -0.5\, x_2 + 0.866\, x_3 = 2.10$$

The rotation operator preserves the length of the vector.

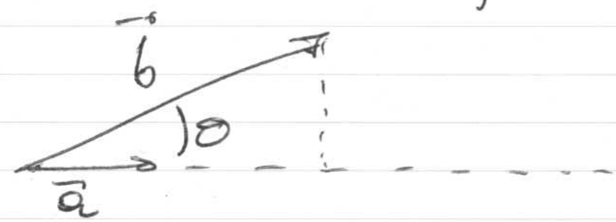
$$r = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_1'^2 + x_2'^2 + x_3'^2} = 3.74$$

(2) $\vec{a} = \hat{i} + 2\hat{j} - \hat{k}$ $\qquad$ $\vec{b} = -2\hat{i} + 3\hat{j} + \hat{k}$

(i) $\vec{a} - \vec{b} = 3\hat{i} - \hat{j} - 2k$

(ii) $|\vec{a} - \vec{b}| = \sqrt{3^2 + (-1)^2 + (-2)^2} = \sqrt{14}$

(iii) component of $\vec{b}$ along $\vec{a}$



The length of the component of $\vec{b}$ along $\vec{a}$ is $b\cos\theta$

$$\vec{a}\cdot\vec{b} = ab\cos\theta$$

$$b\cos\theta = \frac{\vec{a}\cdot\vec{b}}{a} = \frac{-2+6-1}{\sqrt{6}} = \frac{\sqrt{6}}{2}$$

The direction is of course, along $\vec{a}$. A unit vector in the $\vec{a}$ direction is

$$\frac{1}{\sqrt{6}}\left(\hat{i}+2\hat{j}-\hat{k}\right)$$

So the component of $\vec{b}$ along $\vec{a}$ is

$$\frac{1}{2}\left(\hat{i}+2\hat{j}-\hat{k}\right)$$

(iv) $$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{ab} = \frac{3}{\sqrt{6}\sqrt{14}} = \frac{\sqrt{3}}{2\sqrt{7}}$$

$$\theta = \arccos \frac{\sqrt{3}}{2\sqrt{7}} \Rightarrow \theta = 71^\circ$$

(v) $$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 2 & -1 \\ -2 & 3 & 1 \end{vmatrix} = i \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} - j \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} + k \begin{vmatrix} 1 & 2 \\ -2 & 3 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = 5i + j + 7k$$

(vi) $$\vec{a} - \vec{b} = 3i - j - 2k$$

$$\vec{a} + \vec{b} = -i + 5j$$

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \begin{vmatrix} i & j & k \\ 3 & -1 & -2 \\ -1 & 5 & 0 \end{vmatrix}$$

$$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 10\hat{i} + 2\hat{j} + 14\hat{k}$$

3 (i)

$$AB = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix}$$

Expand by the first row

$$|AB| = 1 \begin{vmatrix} -2 & 9 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} 1 & 9 \\ 5 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix}$$

$$|AB| = -104$$

(ii)

$$AC = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 13 & 9 \\ 5 & 2 \end{pmatrix}$$

(iii) $$ABC = A(BC) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & 5 \\ -2 & -3 \\ 9 & 4 \end{pmatrix}$$

$$ABC = \begin{pmatrix} -5 & -5 \\ 3 & -5 \\ 25 & 14 \end{pmatrix}$$

(IV) $AB - B^T A^T$

$$AB = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix}$$

$$B^T A^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \\ -2 & -2 & 3 \\ 1 & 9 & 3 \end{pmatrix}$$

$$AB - B^T A^T = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & 6 \\ 4 & -6 & 0 \end{pmatrix}$$

(4) The product of two orthogonal matrices AB will be an orthogonal matrix $C \iff C^T = C^{-1}$

$$C_{ij} = \sum_k A_{ik} B_{kj}$$

$$(C^T)_{ij} = C_{ji} = \sum_k A_{jk} B_{ki} = \sum_k B_{ki} A_{jk}$$

Identifying $B_{ki} = (B^t)_{ik}$ and $A_{jk} = (A^T)_{kj}$

$$(C^T)_{ij} = \sum_k (B^T)_{ik} (A^T)_{kj}$$

$$C^T = (AB)^T = B^T A^T$$

$\Rightarrow (AB)^T = B^T A^T$ ⊛, but because A and B are orthogonals $A^T = A^{-1}$ and $B^T = B^{-1}$

Multiplying the expression ⊛ by AB from the righ

$$(AB)^T AB = B^T A^T A B$$
$$= B^T B = \underline{\mathbb{1}}$$

$\Rightarrow$

$$\mathbb{1} = (AB)^{-1} AB$$

$\Rightarrow$

$$(AB)^T = (AB)^{-1}$$

Then the matrix C is orthogonal

(5) (ii) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b})$

$$(b \times c)_i = \sum_{j,k} \varepsilon_{ijk}\, b_j\, c_k$$

using the definition of the scalar product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \sum_{ijk} \varepsilon_{ijk}\, a_i\, b_j\, c_k$$

Similarly for the right hand side

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \sum_{ijk} \varepsilon_{ijk}\, c_i\, a_j\, b_k$$

From the definition of ε_{ijk}, we can

interchange two adjacent indices of ε_{ijk}, which changes the sign

$$\vec{c}\cdot(\vec{a}\times\vec{b}) = \sum_{ijk} -\varepsilon_{jik}\, c_i a_j b_k$$

$$= \sum_{ijk} \varepsilon_{jki}\, a_j b_k c_i$$

Indices $i, j k$ are dummy and can be renamed to obtain the identity.

(iii) $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) =$

$$(a\times b)_i = \sum_{jk} \varepsilon_{ijk}\, a_j b_k$$

$$(c\times d)_i = \sum_{lm} \varepsilon_{ilm}\, c_l d_m$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \sum_i \left(\sum_{jk} \varepsilon_{ijk}\, a_j b_k\right)\left(\sum_{lm} \varepsilon_{ilm}\, c_l d_m\right)$$

rearranging the summations we have

$$(\vec{a}\times\vec{b})\cdot(c\times d) = \sum_{\substack{lm\\jk}} \left(\sum_i \varepsilon_{jki}\,\varepsilon_{lmi}\right) a_j b_k c_l d_m$$

where the indices of the ε's have been permuted (twice each so that no sign change occurs) to place in the third position the index over which the sum is carried out.

We now use

$$\sum_k \varepsilon_{ijk}\,\varepsilon_{lmk} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$$

$\Longrightarrow$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \sum_{\substack{j,k\\ l,m}} \left(\delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}\right) a_j\, b_k\, c_l\, d_m$$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \sum_{lm} \left(a_l\, b_m\, c_l\, d_m - a_m\, b_l\, c_l\, d_m\right)$$

This equation can be re-arranged to obtain

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \left(\sum_l a_l c_l\right)\left(\sum_m b_m d_m\right) - \left(\sum_l b_l c_l\right)\left(\sum_m a_m d_m\right)$$

Because each term in parenthesis on the right hand side is just a scalar product we have

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{b} \cdot \vec{c})(\vec{a} \cdot \vec{d})$$

(iii) Using the result in (i)

$$\vec{a} \cdot (\vec{a} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{a}) = 0$$

(6) $\vec{\nabla}(\ln|\vec{r}|) = \dfrac{\vec{r}}{r^2}$

$$\vec{\nabla}(\ln|\vec{r}|) = \sum_i \frac{\partial}{\partial x_i}(\ln|\vec{r}|)\, e_i$$

where $|\vec{r}| = \sqrt{\sum_i x_i^2}$

$$\frac{\partial}{\partial x_i}(\ln|\vec{r}|) = \frac{1}{|r|} \frac{x_i}{\sqrt{\sum_i x_i^2}} = \frac{x_i}{|r|^2}$$

$$\nabla(\ln|\vec{r}|) = \frac{1}{r^2}\left(\sum_i x_i e_i\right) = \frac{\vec{r}}{|r|^2}$$

(7) $\bar{\nabla}(\phi\psi) = \sum_{i=1}^{3} e_i \frac{\partial(\phi\psi)}{\partial x_i} = \sum_i e_i \left(\phi \frac{\partial \psi}{\partial x_i} + \frac{\partial \phi}{\partial x_i}\psi\right)$

$$= \sum_i e_i \phi \frac{\partial \psi}{\partial x_i} + \sum_i e_i \frac{\partial \phi}{\partial x_i} \psi$$

$$\bar{\nabla}(\phi\psi) = \phi \bar{\nabla}\psi + \psi \bar{\nabla}\phi$$

(8) First note that

$$\frac{d}{dt}(\vec{a} \times \dot{\vec{a}}) = \dot{a} \times \dot{a} + a \times \ddot{a}$$

But since the first term in the right hand side vanishes

$$\int (a \times \ddot{a})\, dt = \int \frac{d}{dt}(\vec{a} \times \dot{\vec{a}})\, dt$$

$$\Rightarrow \int (a \times \ddot{a})\, dt = \vec{a} \times \dot{\vec{a}} + C$$

where C is a constant.