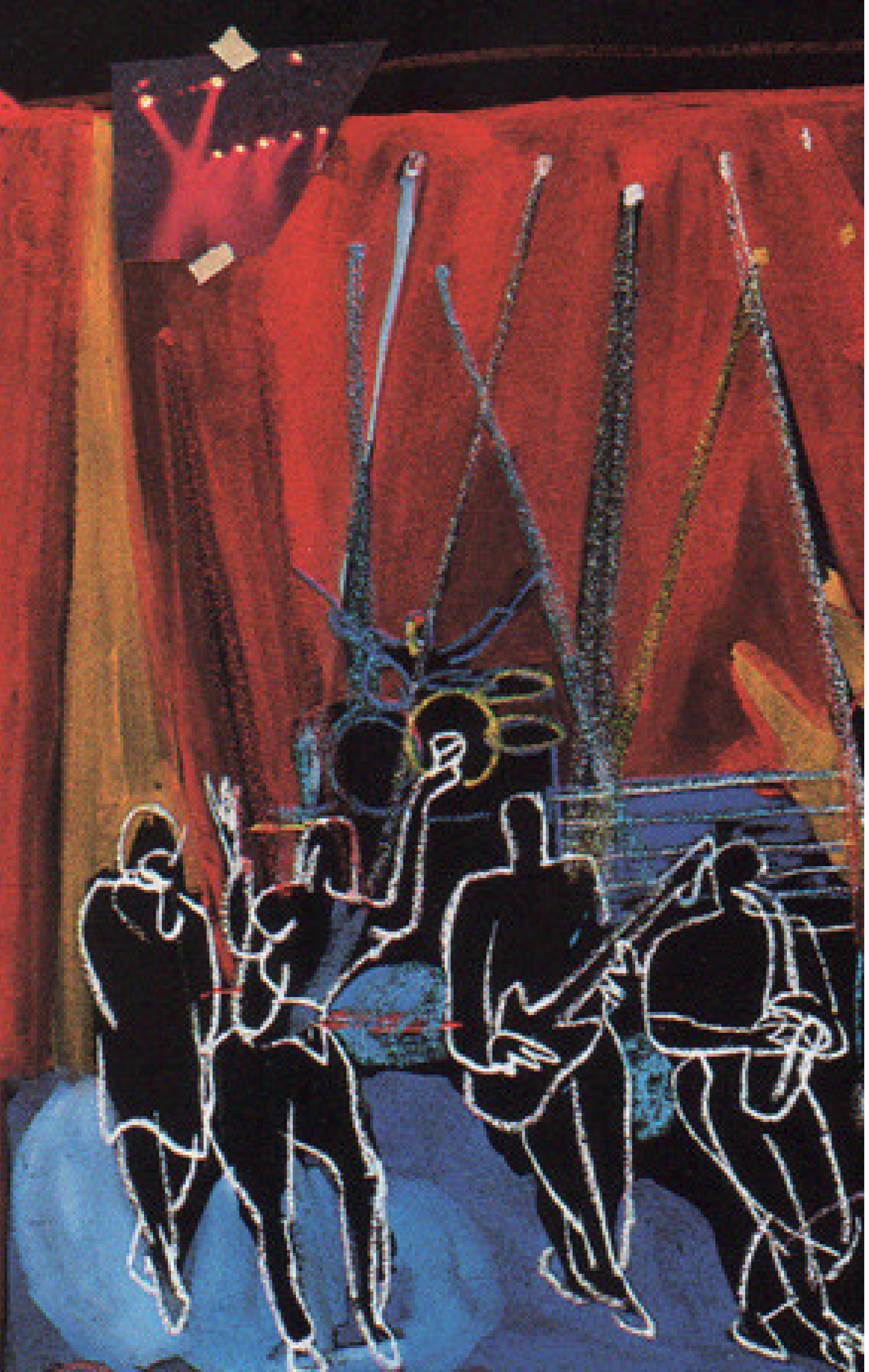


PHY-141

Periodic Motion

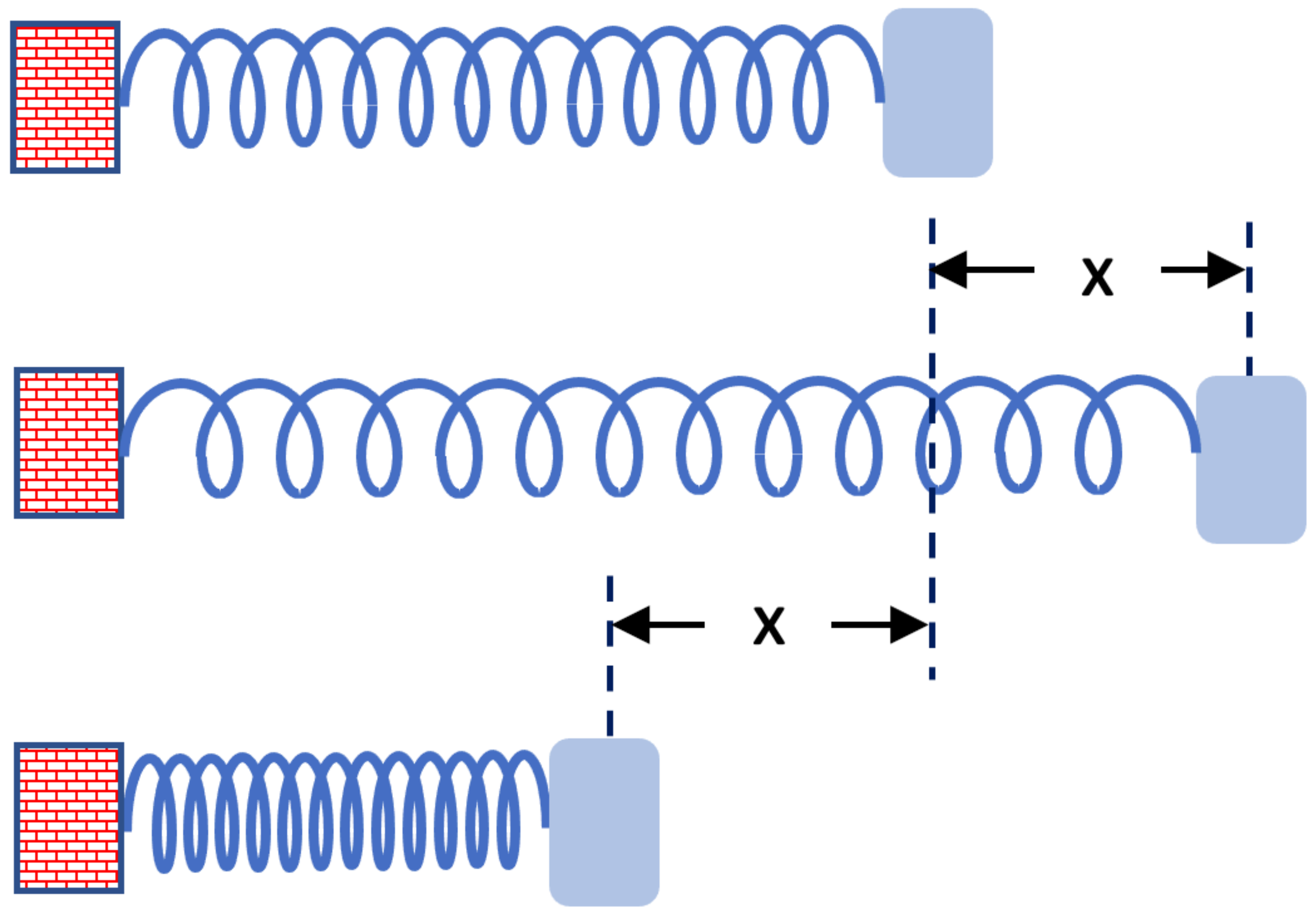
Luis Anchordoqui

Outline

- The motion that repeats itself after fixed intervals of time is called periodic motion
- For example, the oscillatory motion is a type of periodic motion
- Oscillatory motion is defined as the to and from motion of an object from its mean position
- For instance, the oscillation of simple pendulum, vibrating strings of musical instruments, vibration prongs of tuning fork, etc
- A simple form of oscillatory motion is the simple harmonic motion (SHM)
- In this motion, the restoring force is directly proportional to its displacement from its equilibrium position
- The study of oscillatory motion is basic to physics; its concepts are required for the understanding of many physical phenomena
- A wave is an example of oscillatory motion

Simple Harmonic Motion (SHM)

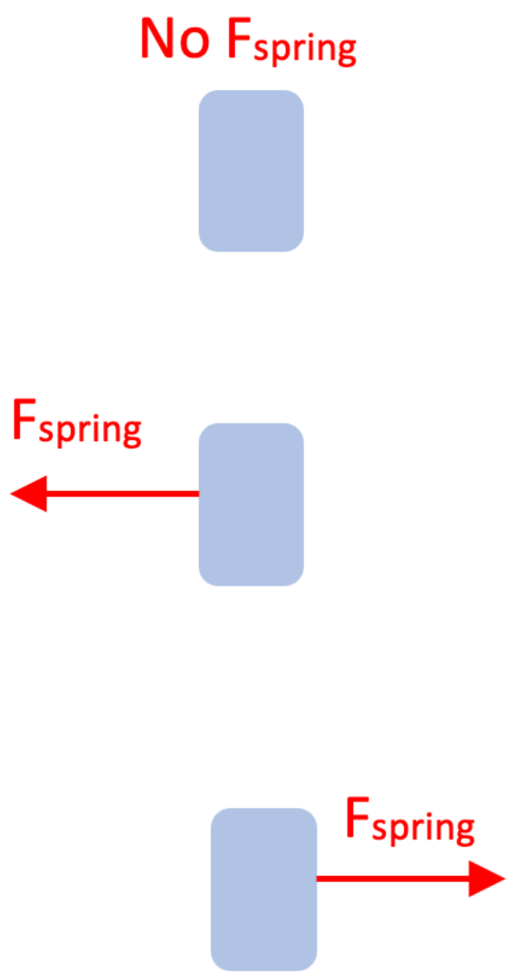
- Two conditions must be met to produce SHM in a mechanical system
- First ☛ an equilibrium position must exist, *viz* there must be one position at which the object executing the motion would be at rest if place there and to which it returns if displaced and release
- Second ☛ the force tending to pull the object back to its equilibrium position must be a *linear restoring force*, *viz* the force must be linearly proportional to the distance of the object from the equilibrium position



Relaxed spring position

Spring Force pulls mass to the left when mass is displaced to the right of the relaxed position

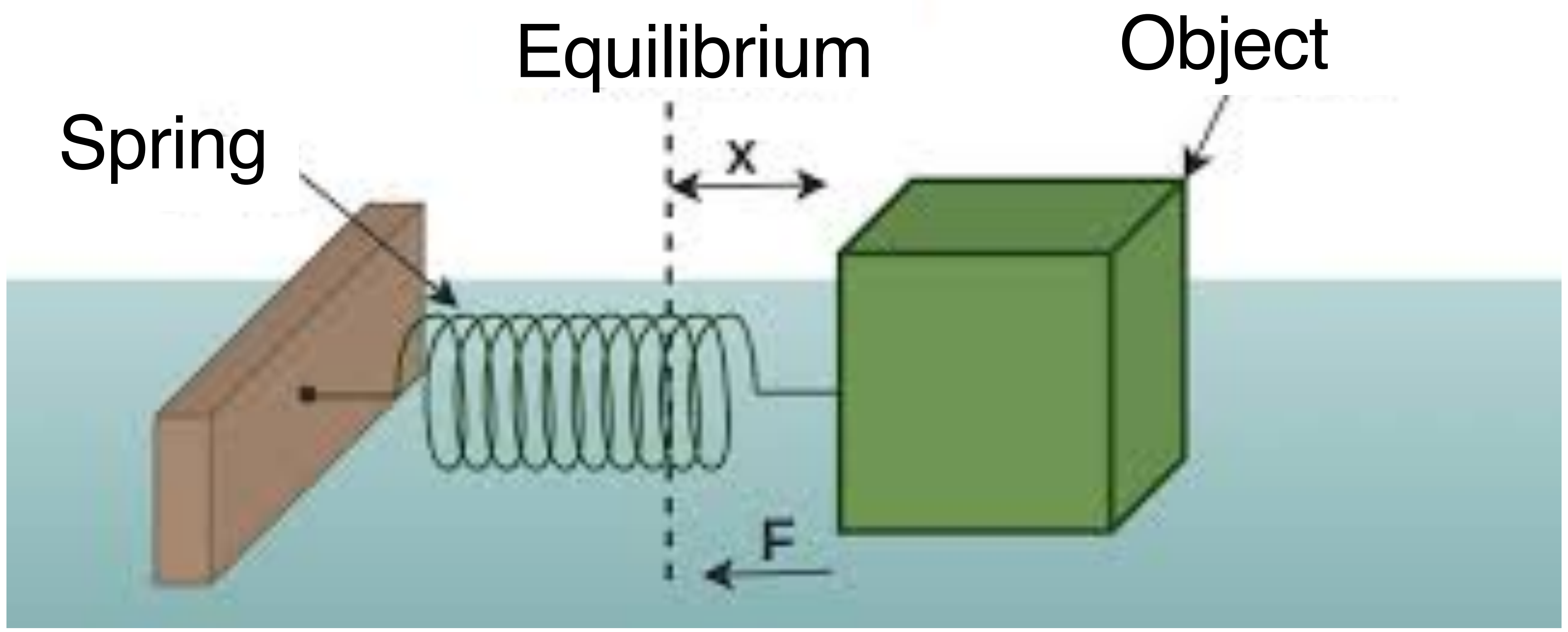
Spring Force pushes mass to the right when displaced to the left of the relaxed position



SHM

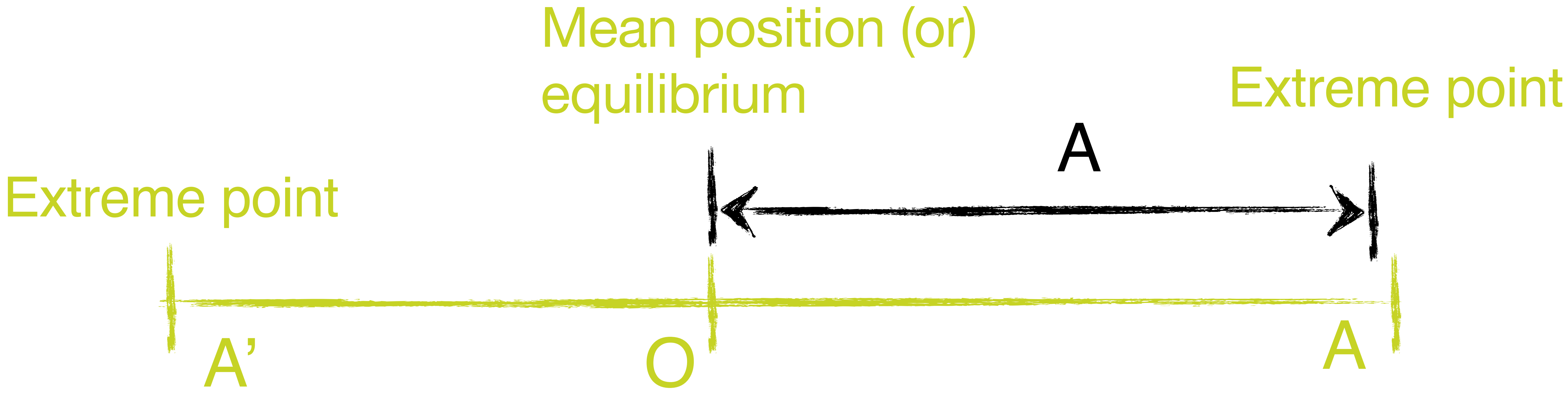
- Consider the motion of a block of mass m that can slide without friction along a horizontal surface
- The mass is attached to a spring of constant k, which is anchored to a wall on the other end
- We introduce a one-dimensional coordinate system to describe the position of the mass, such that the axis is co-linear with the motion, the origin is located where the spring is at rest, and the positive direction corresponds to the spring being extended

Linear Simple Harmonic Motion



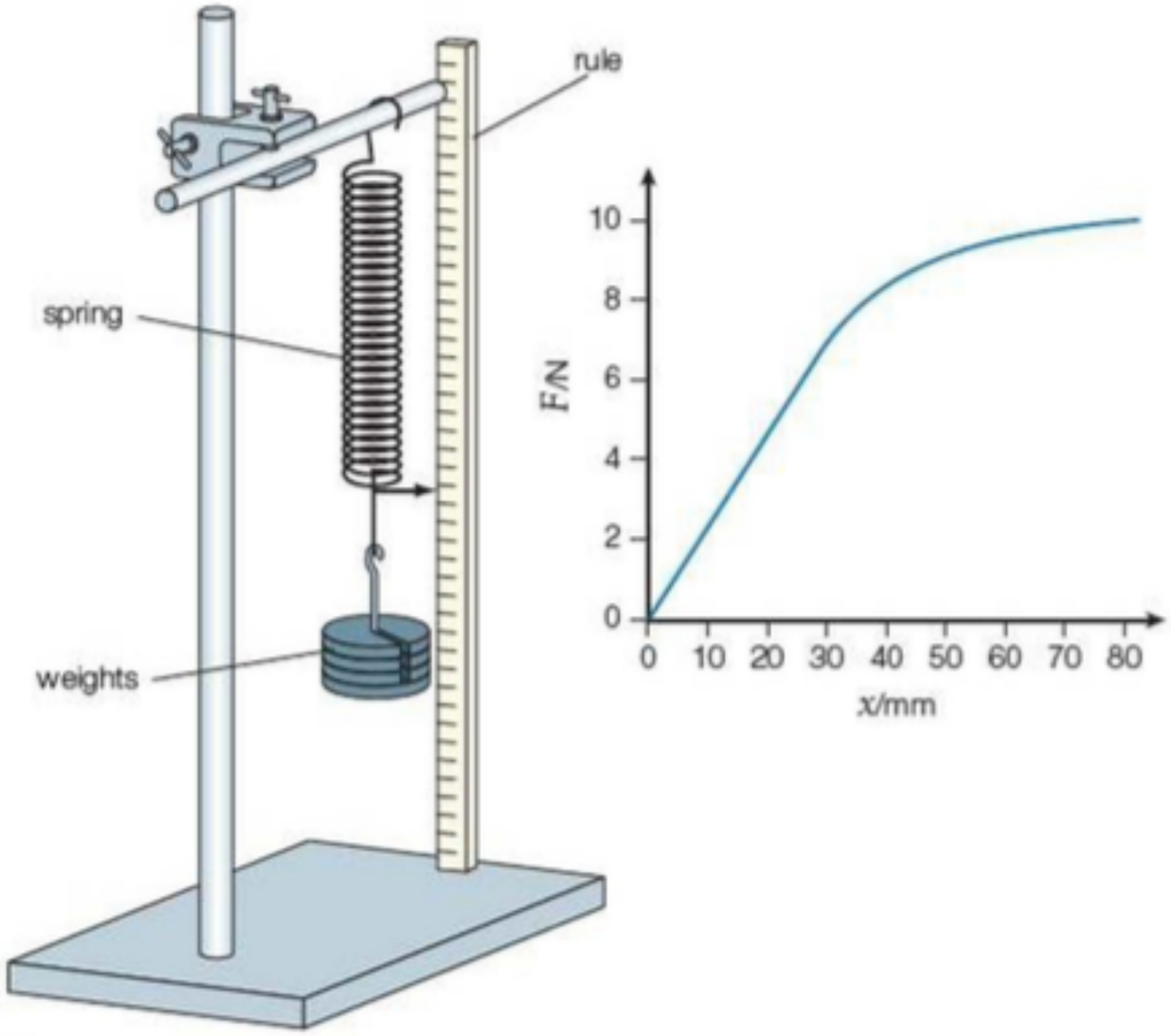
SHM

- If the spring is compressed (or extended) by a distance A relative to the rest position, and the mass is then released, the mass will oscillate back and forth between $x = \pm A$
- When the mass is at $x = \pm A$, its speed is zero $(v = 0)$, as these points correspond to the location where the mass turns around
- If it requires $1\ \mathrm{N}$ of force to compress the spring $1\ \mathrm{cm}$, it will require $2\ \mathrm{N}$ for a $2\ \mathrm{cm}$ displacement, $3\ \mathrm{N}$ for a $3\ \mathrm{cm}$ displacement, and so forth, because for this system the restoring force is linear
- Likewise, if the restoring force is linear it will require $1\ \mathrm{N}$ to stretch the spring $1\ \mathrm{cm}$ from the equilibrium position as well



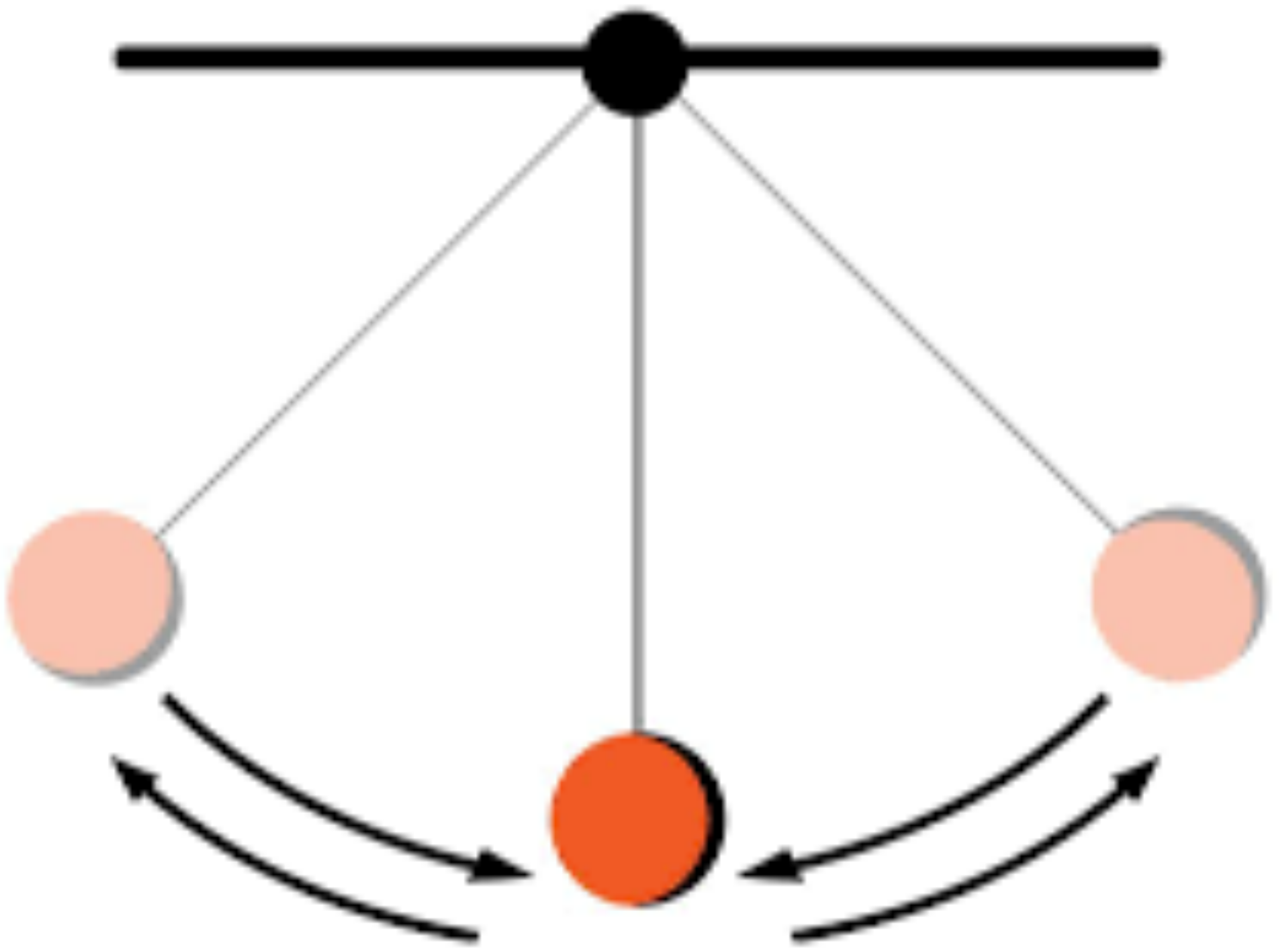
Hooke's Law

- This is an example of a linear restoring force, with a force per unit of stretch of $1\ \mathrm{N/cm}$
- Such a linear system is said to obey Hooke's law $\vec{F} = -k\vec{x}$, where x is the displacement and k the spring constant
- Hooke's law applies only within the elastic limit, where the spring return to its original shape once the force is removed
- If this limit is exceeded, the spring undergoes permanent deformation



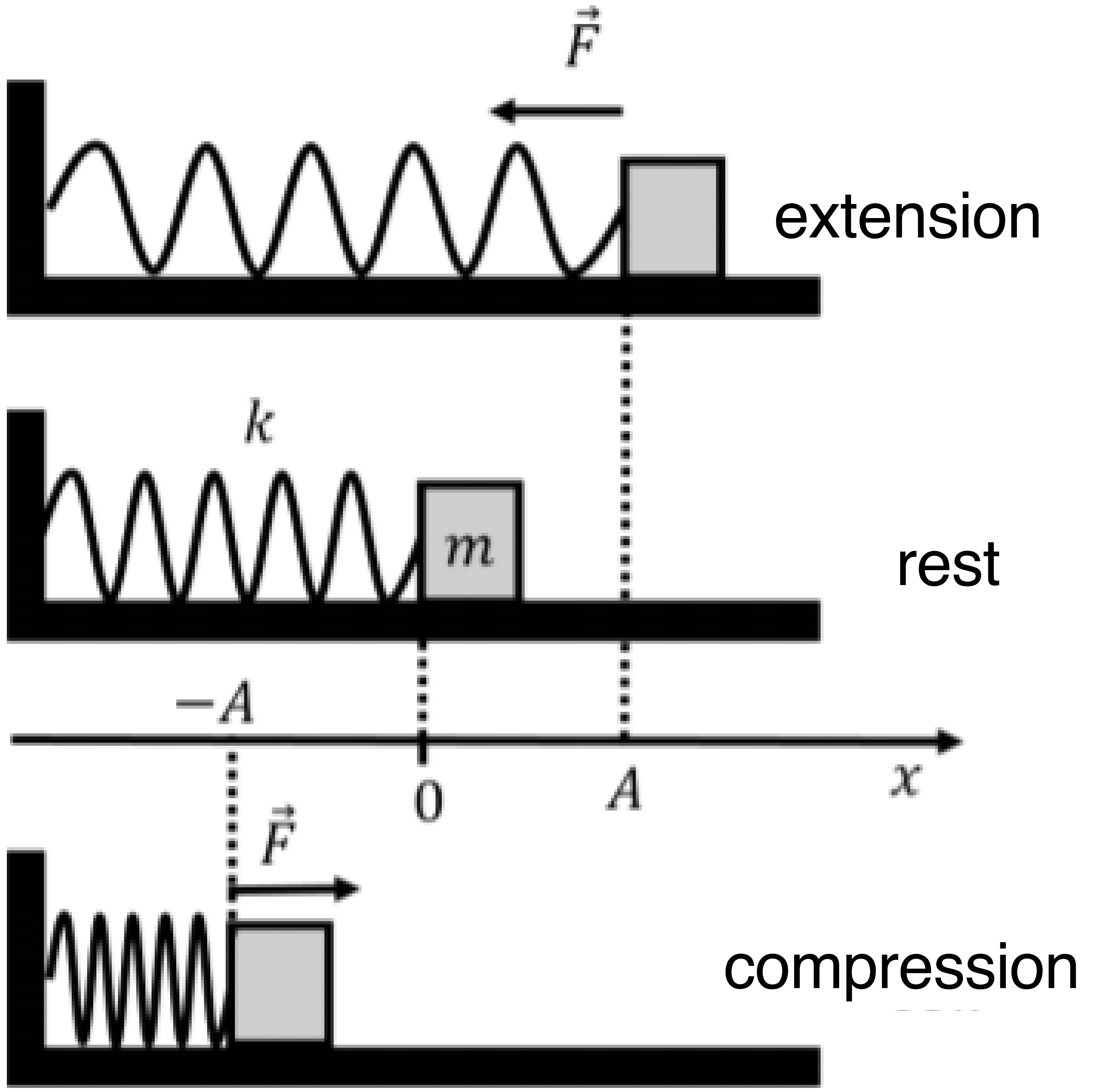
Pendulum

- Another example of SHM is a pendulum consisting of a mass or bob, hanging by a string from a fixed point and free to oscillate back and forth in a plane
- The equilibrium point is the point directly below the suspension point, where the pendulum will hang motionless
- The force required to pull the pendulum away from its equilibrium position is linearly proportional to the displacement of the bob from equilibrium, so long as the amplitude is small
- When displaced through some small angle and released from rest, the bob executes SHM in the plane of the paper



Example

- For example, say the mass on the spring is moved $3\ \mathrm{cm}$ to the right of its equilibrium position and released, it begins to accelerate leftward, passing through its equilibrium position, and executes SHM
- At time $t = 0$ the mass is released from rest, $3\ \mathrm{cm}$ away $(x = +3\ \mathrm{cm})$ the equilibrium position $x = 0$
- The mass starts moving, builds up speed as it passes through the equilibrium point, then slows down to zero velocity by the time it reaches the point $x = -3\ \mathrm{cm}$ to the left of the equilibrium point

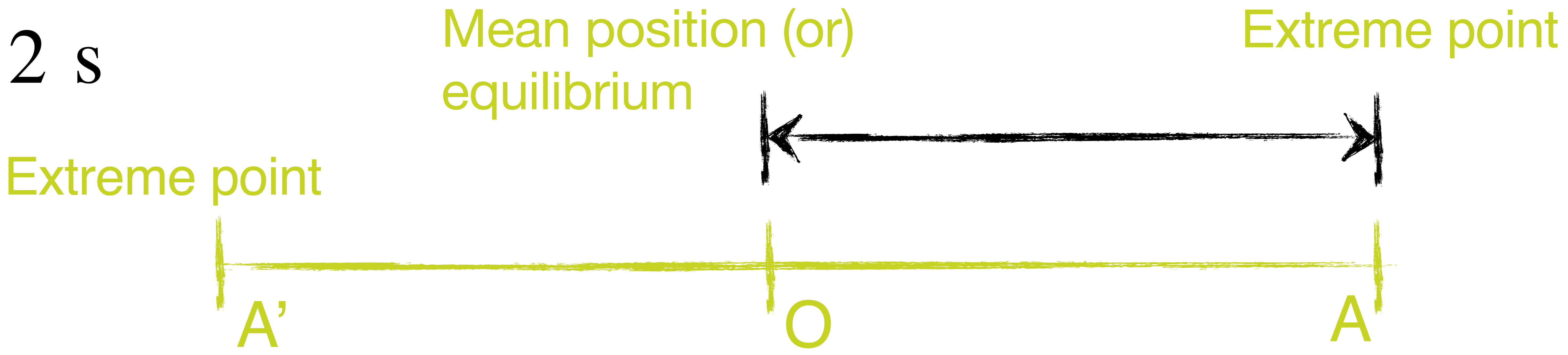


- The mass then starts moving faster to the right until it goes through the equilibrium point, then slows down to zero velocity by the time it reaches its original position at $x = +3\ \mathrm{cm}$
- The motion then repeats every $2\ \mathrm{s}$

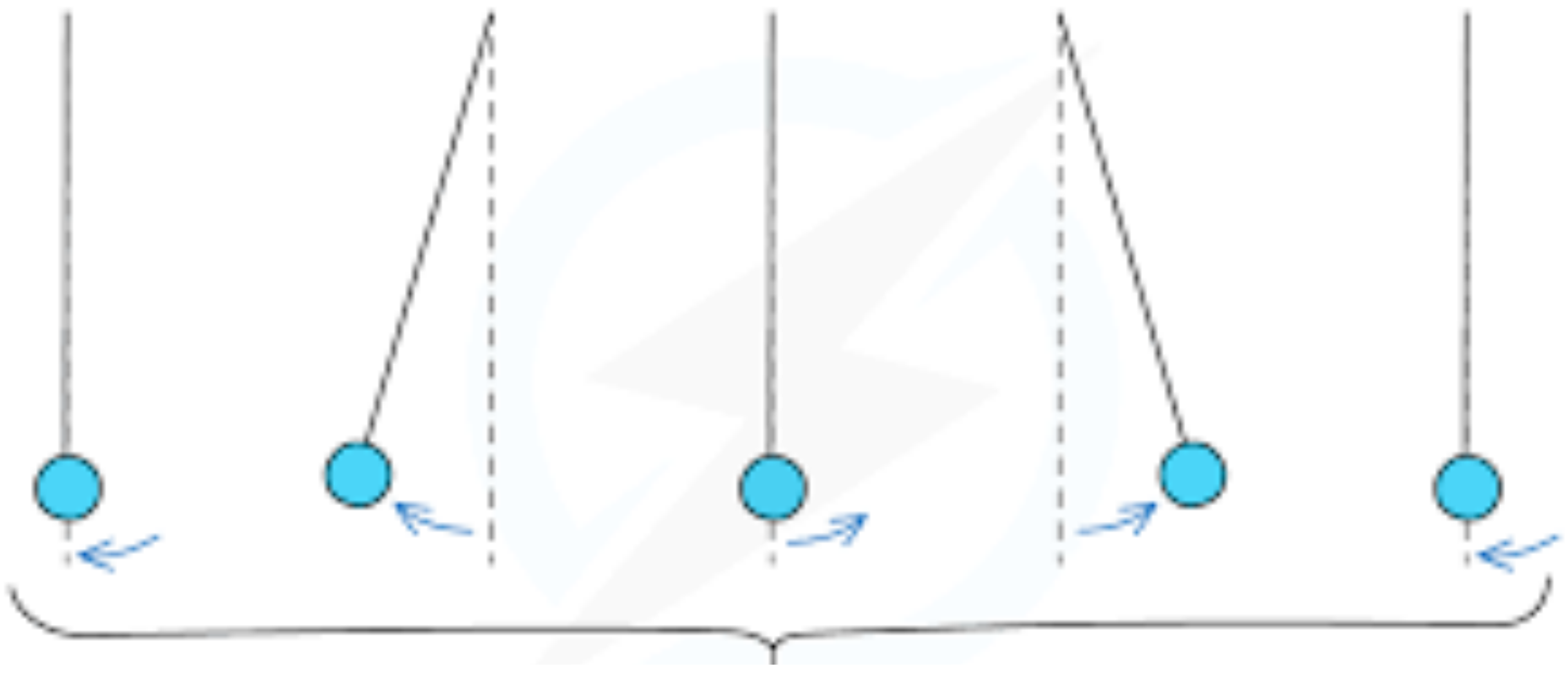
Amplitude and Period

➢ The maximum displacement A of the mass from the equilibrium position (in either direction) is called the *amplitude*, and the repetition time τ is called the *period* of the SHM

For our example, $A = 3 \text{ cm}$ and $\tau = 2 \text{ s}$



➢ The pendulum period is determined by its length (L) and the acceleration of gravity at the Earth surface (g) via $\tau = 2\pi\sqrt{L/g}$ and is independent of pendulum mass or amplitude



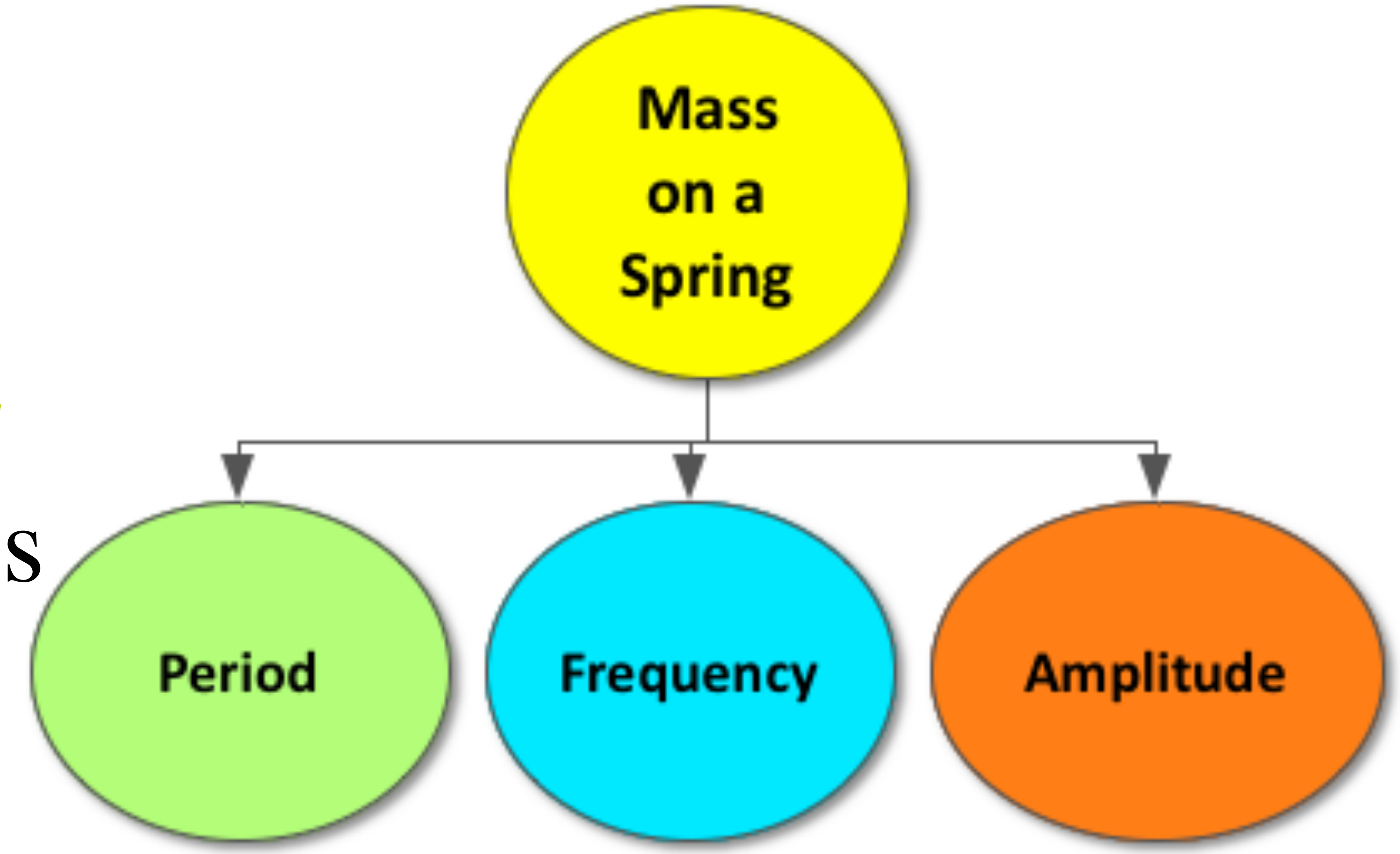
1 Oscillation (τ)

Frequency

- Now, the period is related to the frequency f, which is the number of periods (or cycles) per second $f = 1/\tau$
- A high-frequency oscillation has many vibrations per second; to fit many vibations into a 1-second time interval, the period must be short
- Hence, a high-frequency oscillation has a short period, whereas a low-frequency oscillation has a long period
- In our example the frequency is found to be

$$f = \frac{1}{\tau} = \frac{1 \text{ cycle}}{2 \text{ s}} = 1/2 \text{ cycle per second} = 0.5 \text{ Hz}$$

- The unit of frequency is the hertz (Hz), which is 1 cycle/s

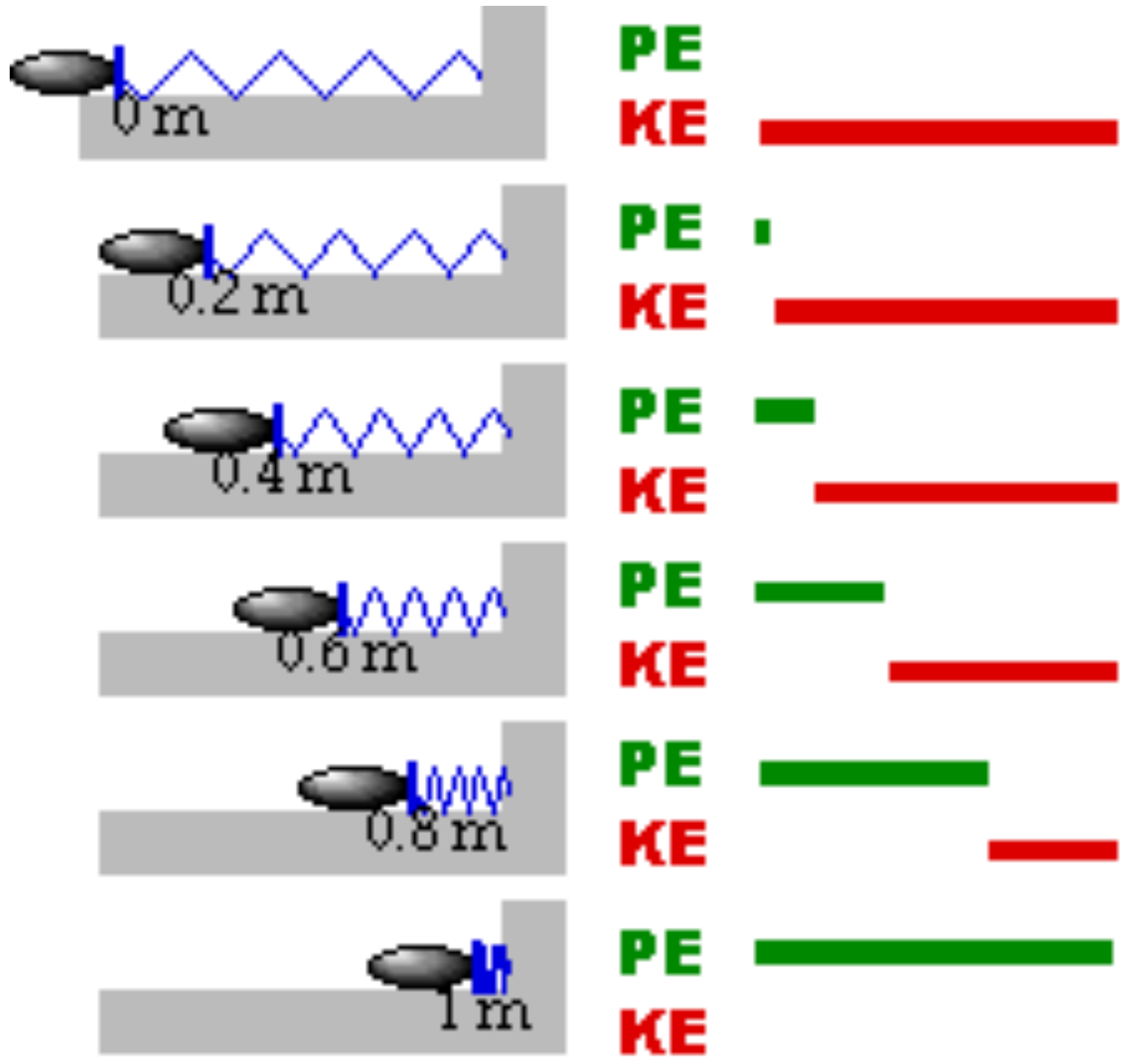


Conservation of Energy

- We can describe the motion of the mass using energy, since the mechanical energy of the mass is conserved

- Recall that at any position x the mechanical energy $E_{\text{mechanical}}$ of the mass will have a term from the potential energy E_{elastic}, associated with the spring force, and kinetic energy E_{kinetic},

$E_{\text{mechanical}} = E_{\text{elastic}} + E_{\text{kinetic}}$ or equivalently $E_{\text{mechanical}} = \dfrac{1}{2}kx^2 + \dfrac{1}{2}mv^2$



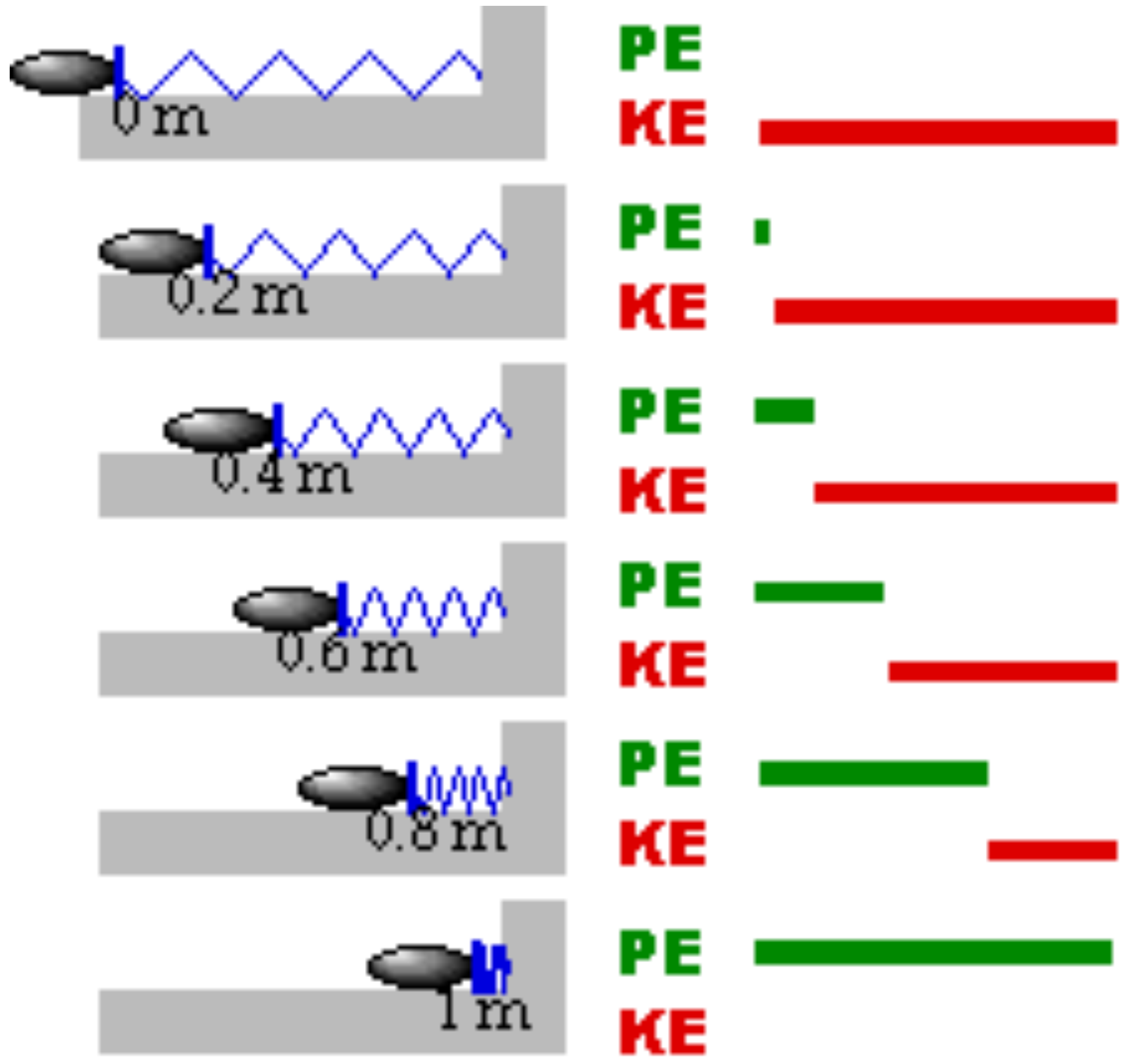
Conservation of Mechanical Energy

- We can find the mechanical energy by evaluating the energy at one of the turning points
- At these points, the kinetic energy of the mass is zero, so $E_{\text{mechanical}} = E_{\text{elastic}}(x = A) = \frac{1}{2}kA^2$
- We can then write the expression for mechanical energy as:

$$E_{\text{mechanical}} = \frac{1}{2}kA^2 = \frac{1}{2}kx^2 + \frac{1}{2}mv^2$$

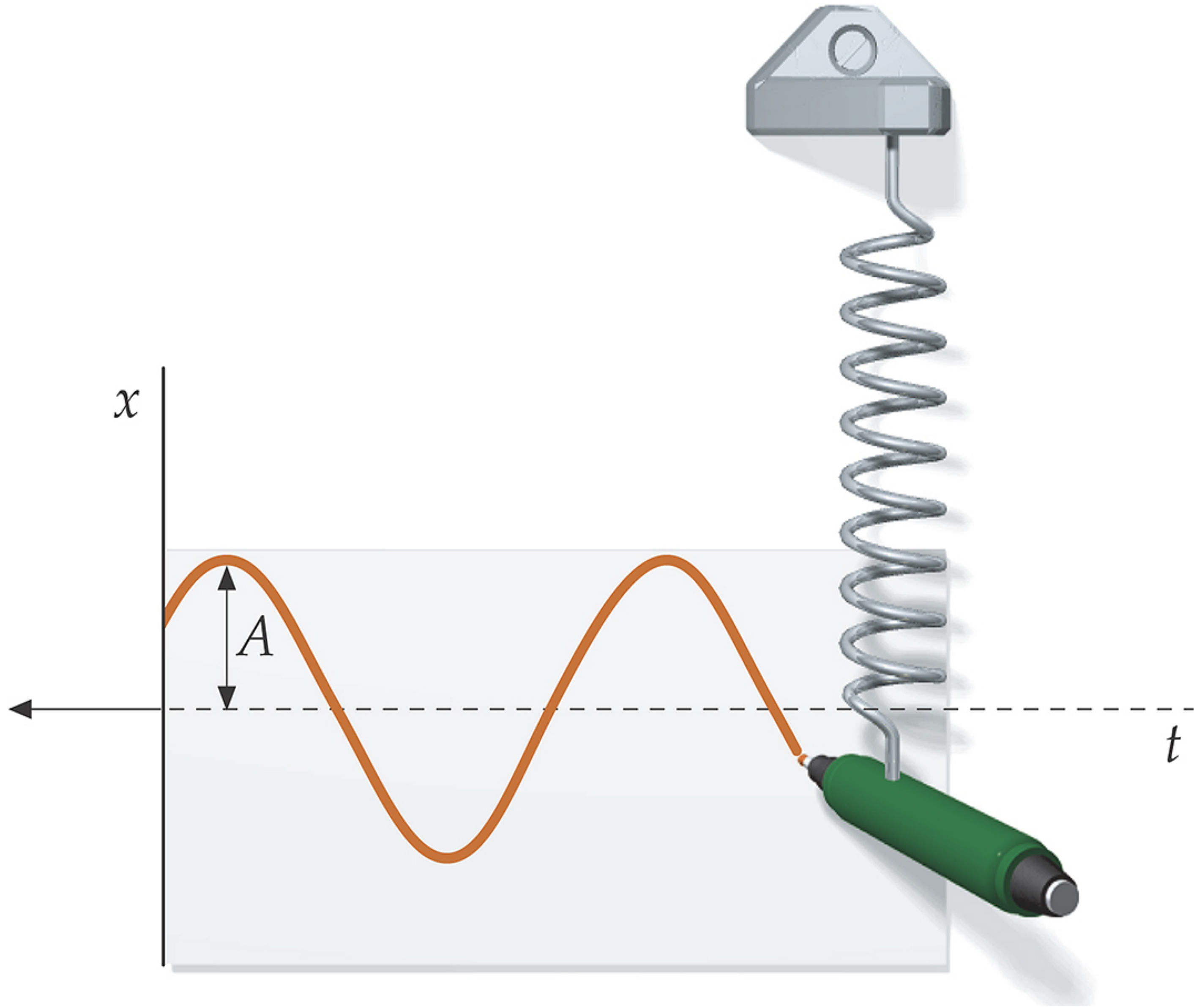
- With a little bit of algebra, we can thus always know the speed of the mass at any position if we know the amplitude:

$$v = \sqrt{k(A^2 - x^2)/m}$$



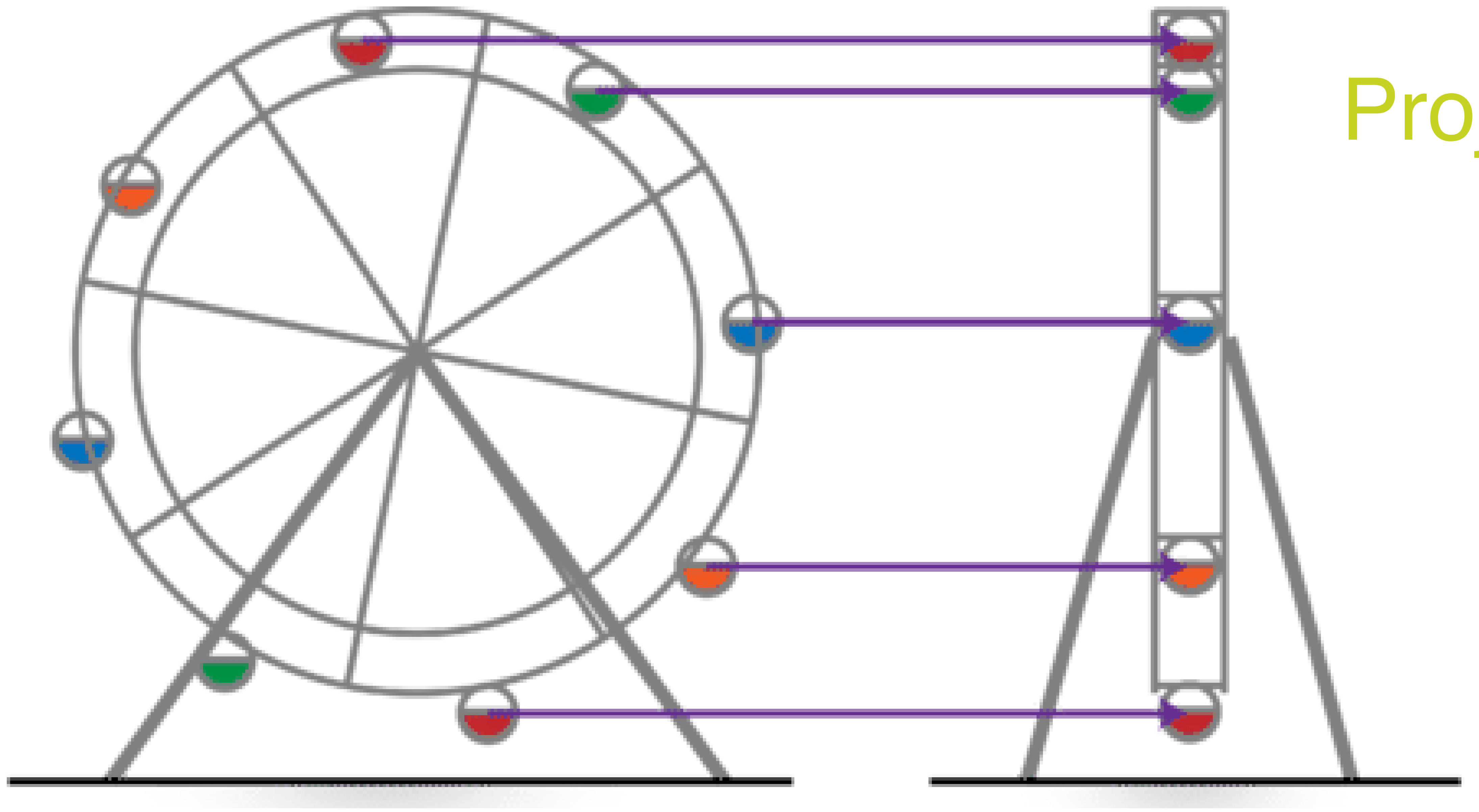
Experimental Determination of x versus t

- A marking pen is attached to a mass on a spring and the paper is pulled to the left
- As the paper moves with constant speed the pen traces out the displacement x as a function of time
- In this example we have chosen x to be positive when the spring is compressed



SHM and uniform circular motion

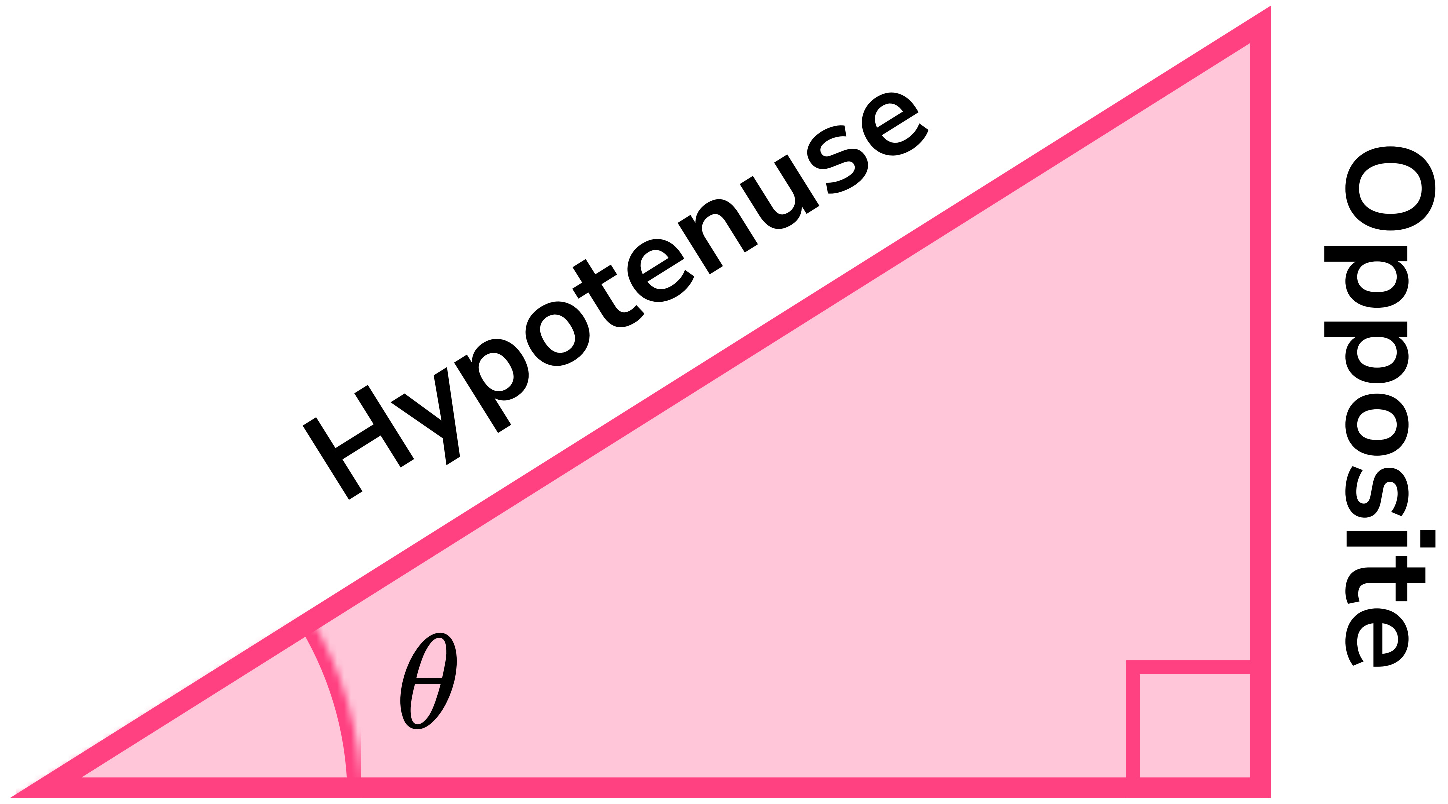
- Some important details of SHM can be illustrated by comparison of circular harmonic motion, which is the motion of an object around a circle with constant speed
- Uniform circular motion, view from a distant point in the plane of the motion is SHM
- Indeed, SHM is the projection of uniform circular motion on a line in the plane of the motion



Project Motion

What are trigonometric functions?

- Trigonometric functions are functions that relate an angle in a right angled triangle to the ratio of two of its sides



- Sine ☛ $\sin\theta$

- When we input a given angle into $\sin\theta$ it gives us the ratio between the opposite length and the hypotenuse $\sin\theta = \dfrac{\text{opposite}}{\text{hypothenuse}}$

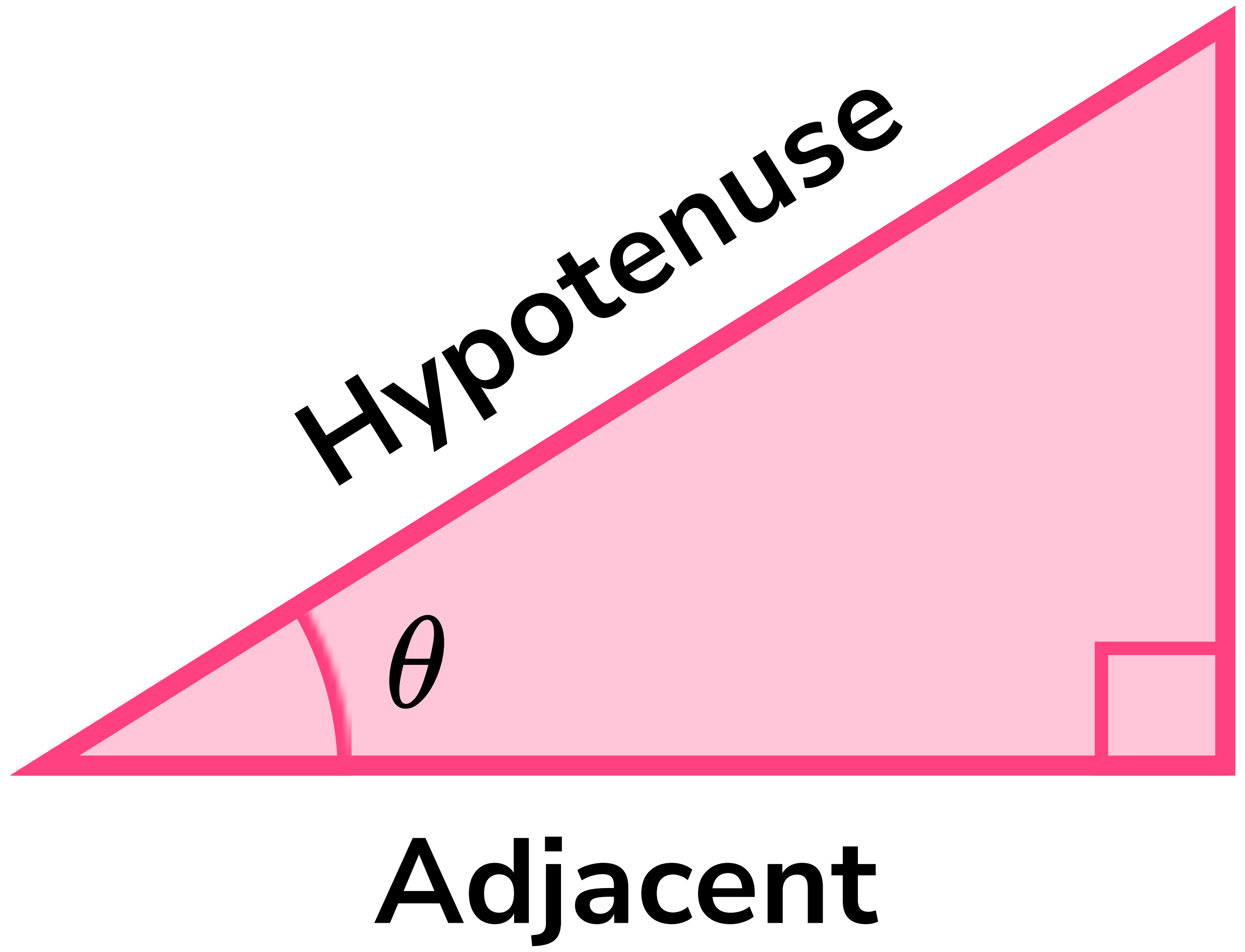
- It tells us how many times bigger the opposite is than the hypotenuse

What are trigonometric functions?

- Cosine ☛ $\cos\theta$

- When we input a given angle into $\cos\theta$ it gives us the ratio between the adjacent length and the hypotenuse

$$\cos\theta = \frac{\text{adjacent}}{\text{hypothenuse}}$$

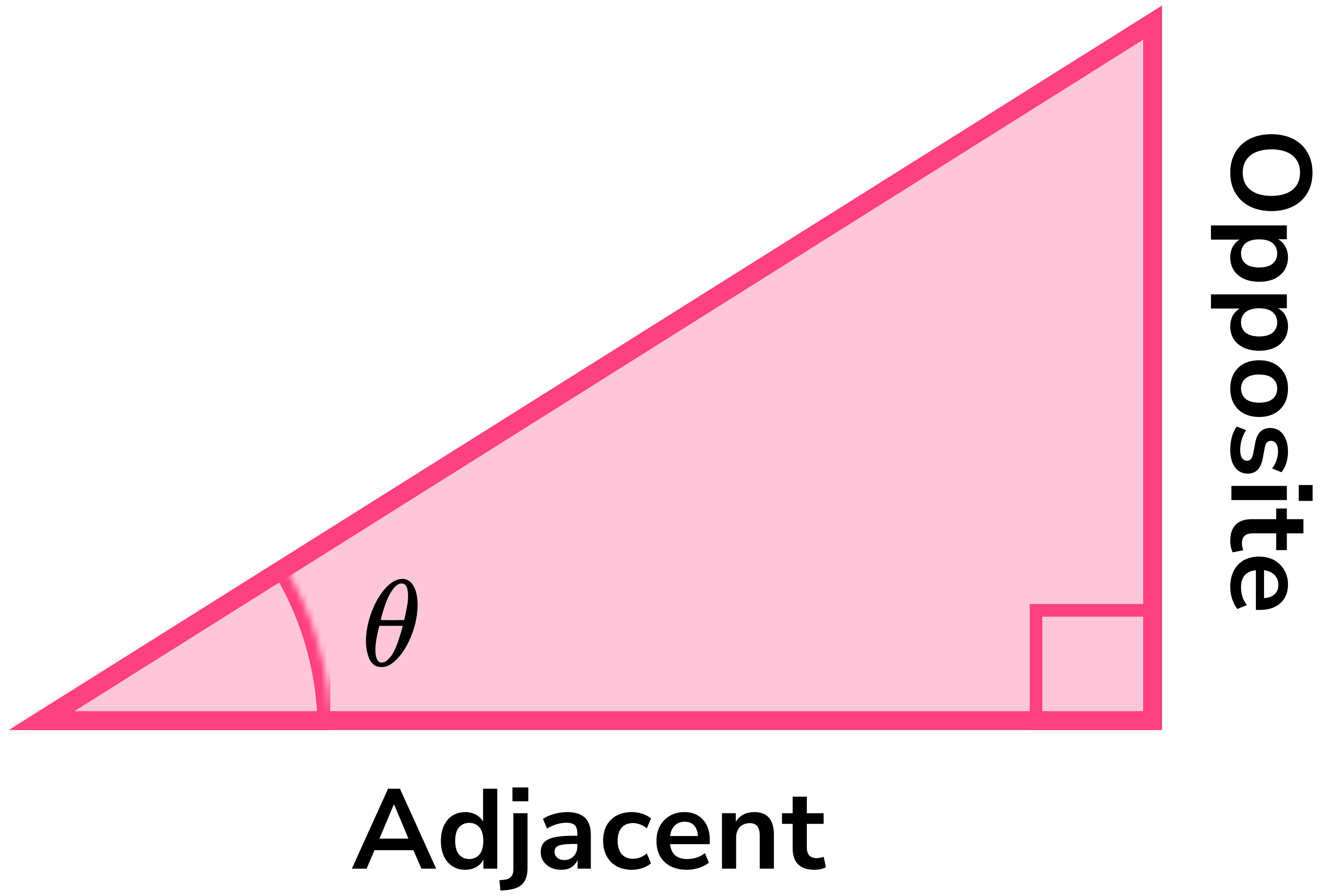


- It tells us how many times bigger the adjacent is than the hypotenuse

What are trigonometric functions?

- Tangent ☛ $\tan\theta$

- When we input a given angle into $\tan\theta$ it gives us the ratio between the adjacent length and the opposite length



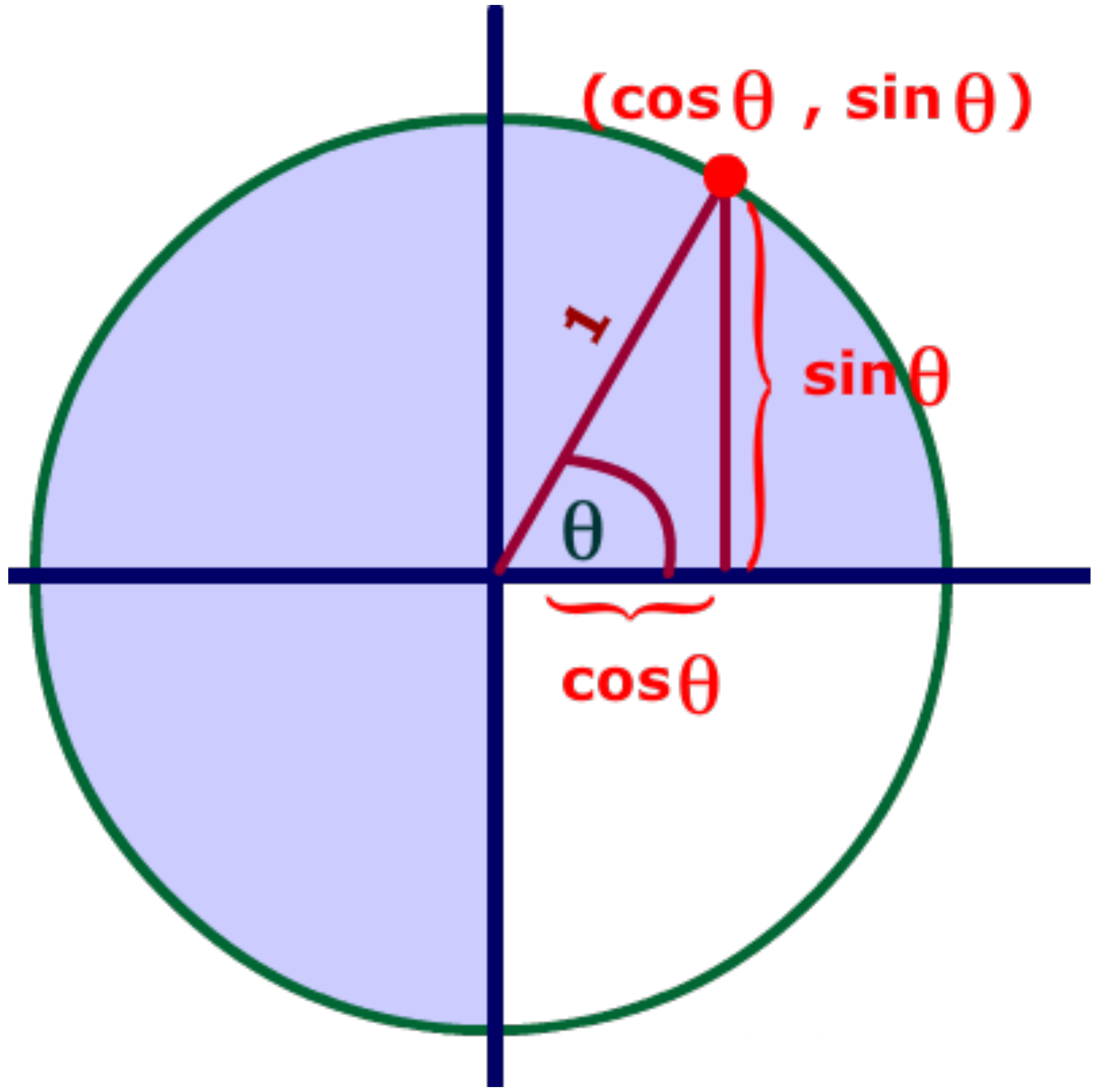
$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

- It tells us how many times biger the opposite is than the adjacent

Trigonometric circle

- Trigonometric functions are visualized on a unit circle (radius $r = 1$) by tracking the (x, y) coordinates of a point as it rotates counterclockwise
- The y-coordinate represents the height and is associated to the sine, $\sin\theta = \dfrac{\text{opposite}}{\text{hypothenuse}}$
- The x-coordinate represents the horizontal position and is associated to the cosine

$$\cos\theta = \frac{\text{adjacent}}{\text{hypothenuse}}$$



- θ ☛ angular position measured from the positive horizontal x-axis
- As the point moves, sin and cos oscillate between -1 and $+1$, repeating every $360°$(or 2π radians)

The Sinusoidal Description of SHM

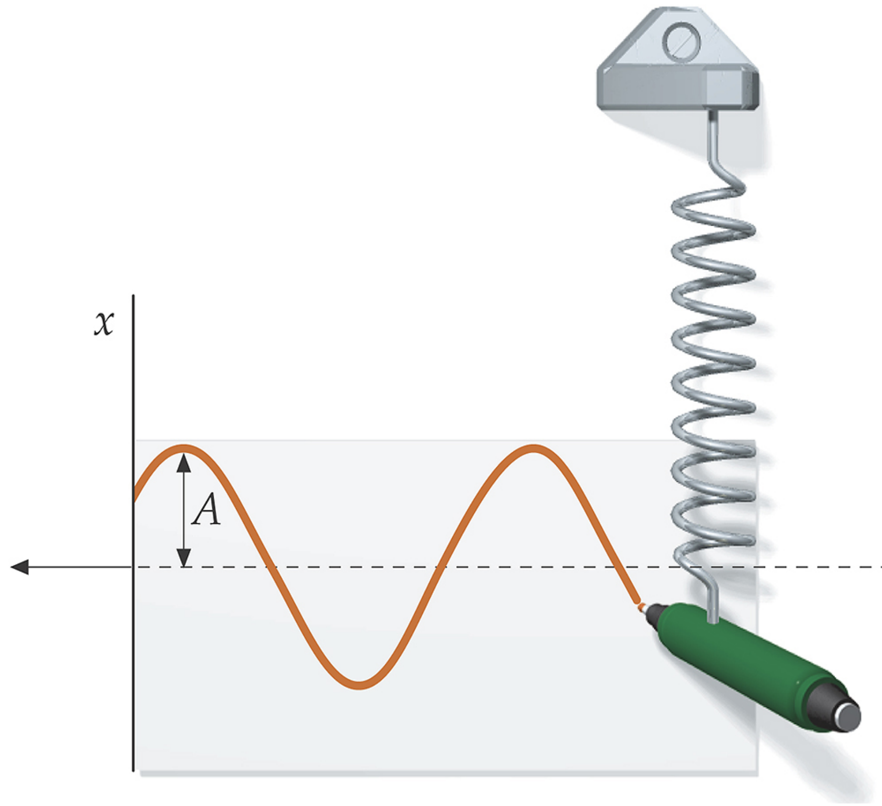
- Displacement in a SHM can be mathematically expressed as a sinusoidal function of time
- Indeed, general equation to describe sinuosidal curve

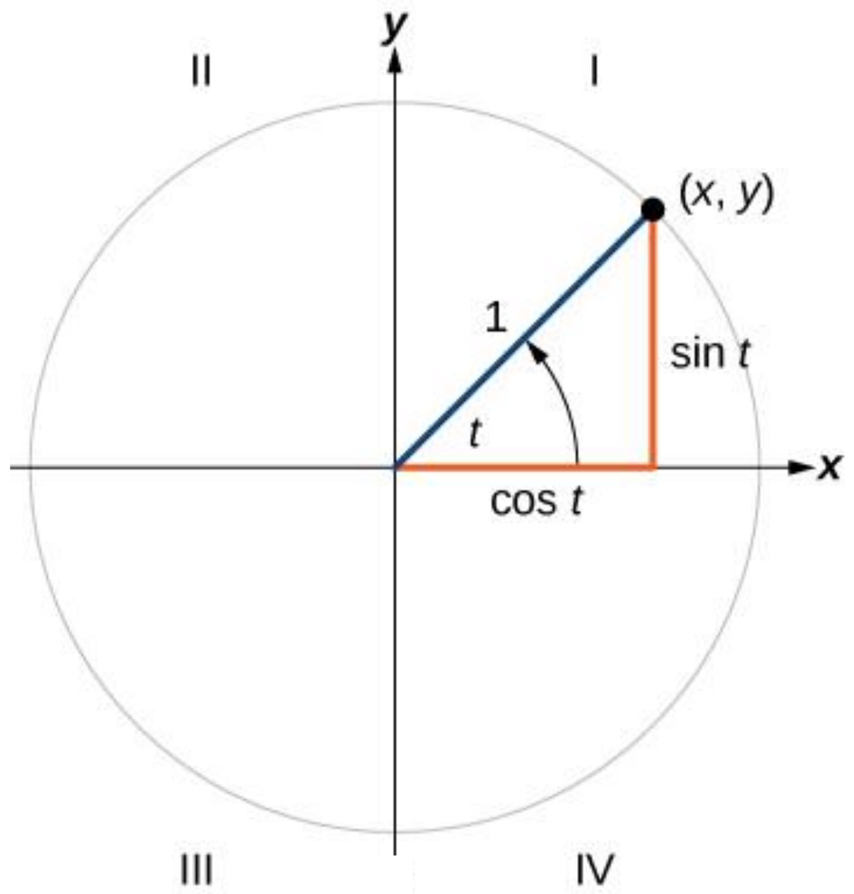
$$x = A\cos(\omega t + \delta)$$

$\omega = 2\pi f$ ☛ angular frequency

δ ☛ phase angle $\equiv$ parameter in SHM that defines the initial state (position and velocity) of the oscillating mass at $t = 0$

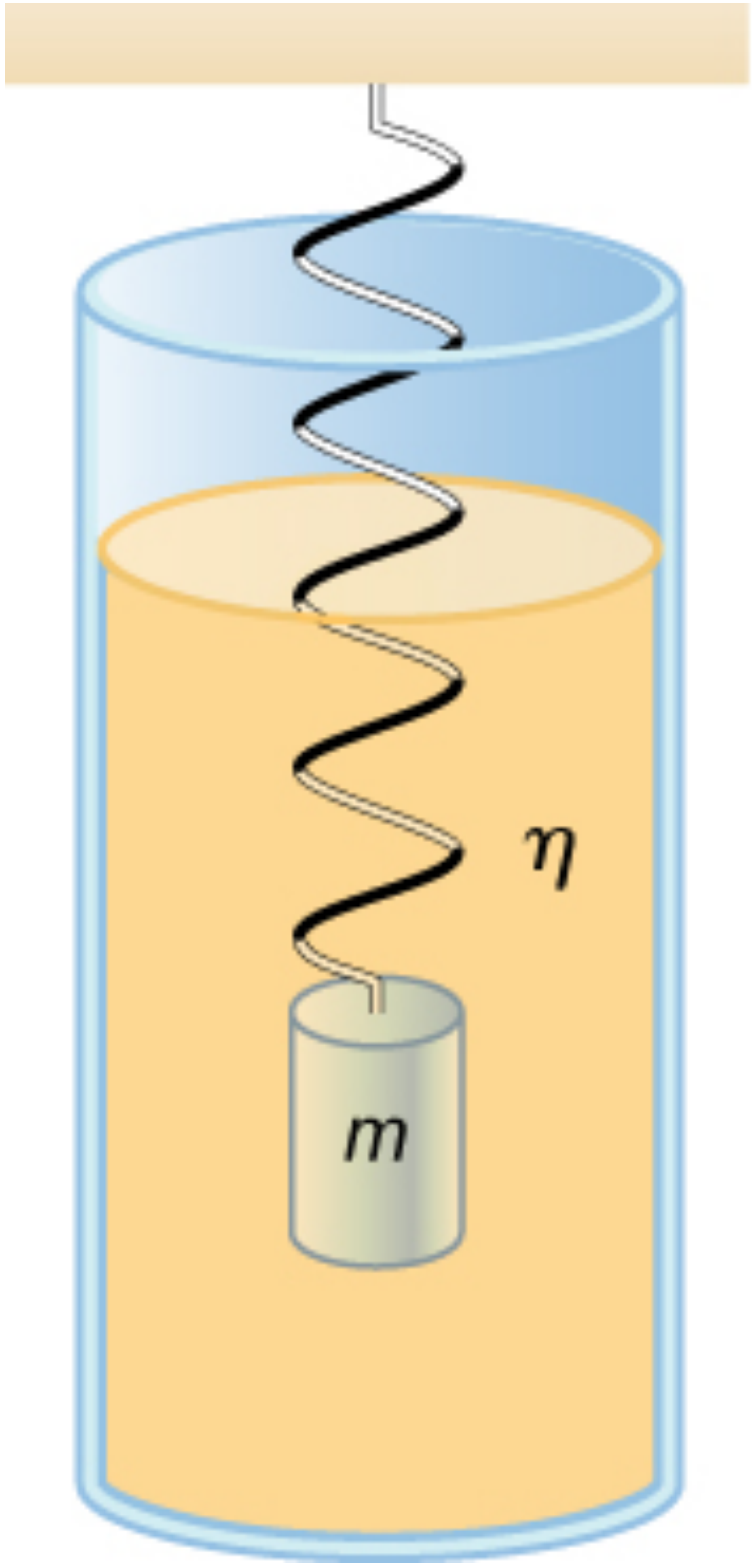
- δ ☛ shifts the SHM cosine or sine behavior along the horizontal axis, acting as a timing offset

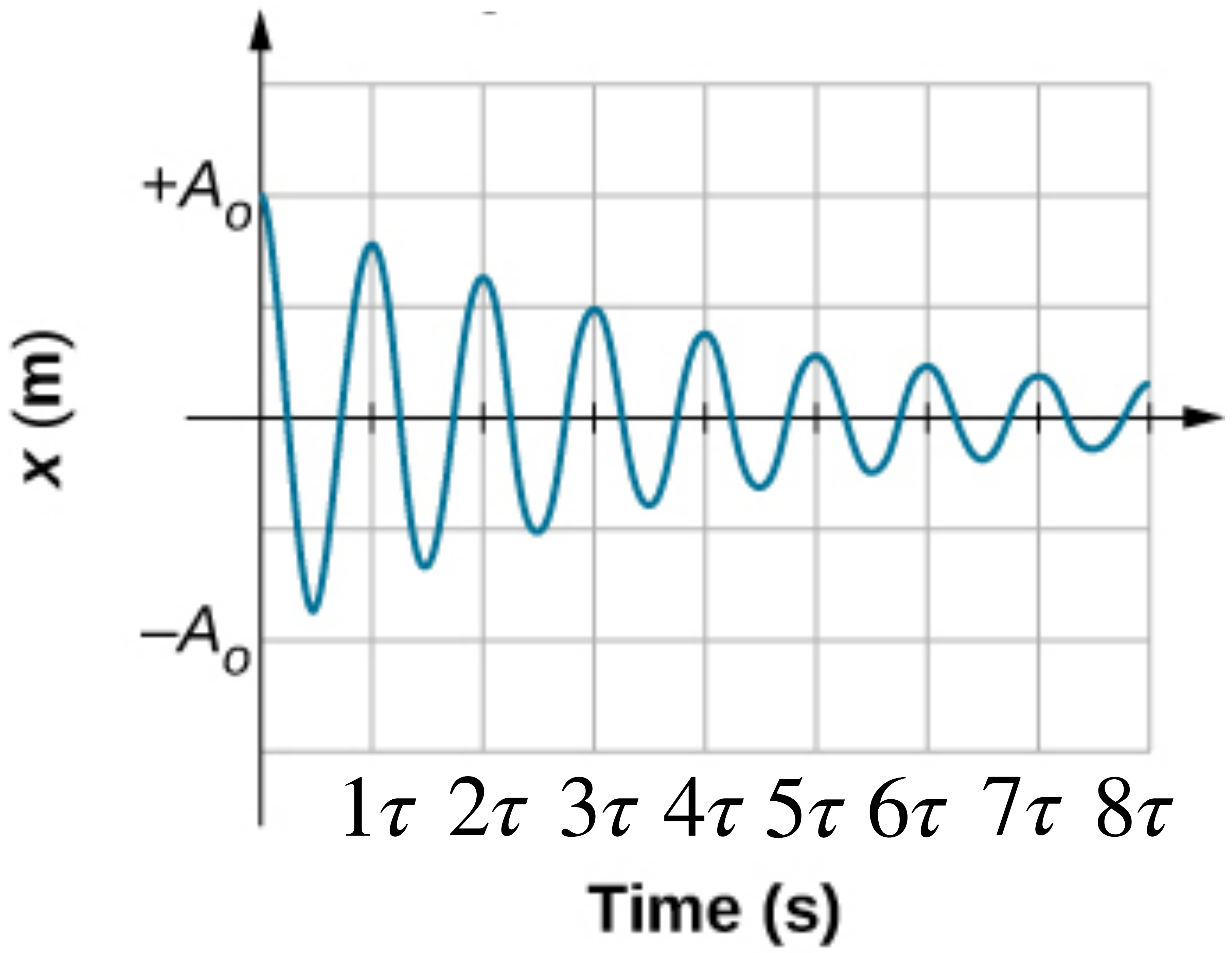




Damped Oscillations

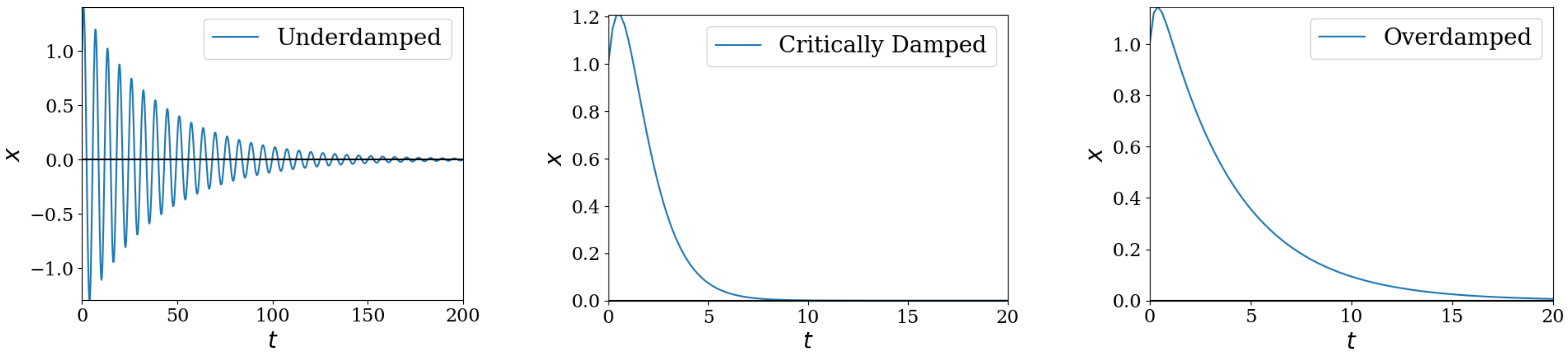
- Damping occurs when resistive forces (like friction or air resistance) dissipate a system's energy
- The amplitude of the oscillation decreases over time until the system eventually stops





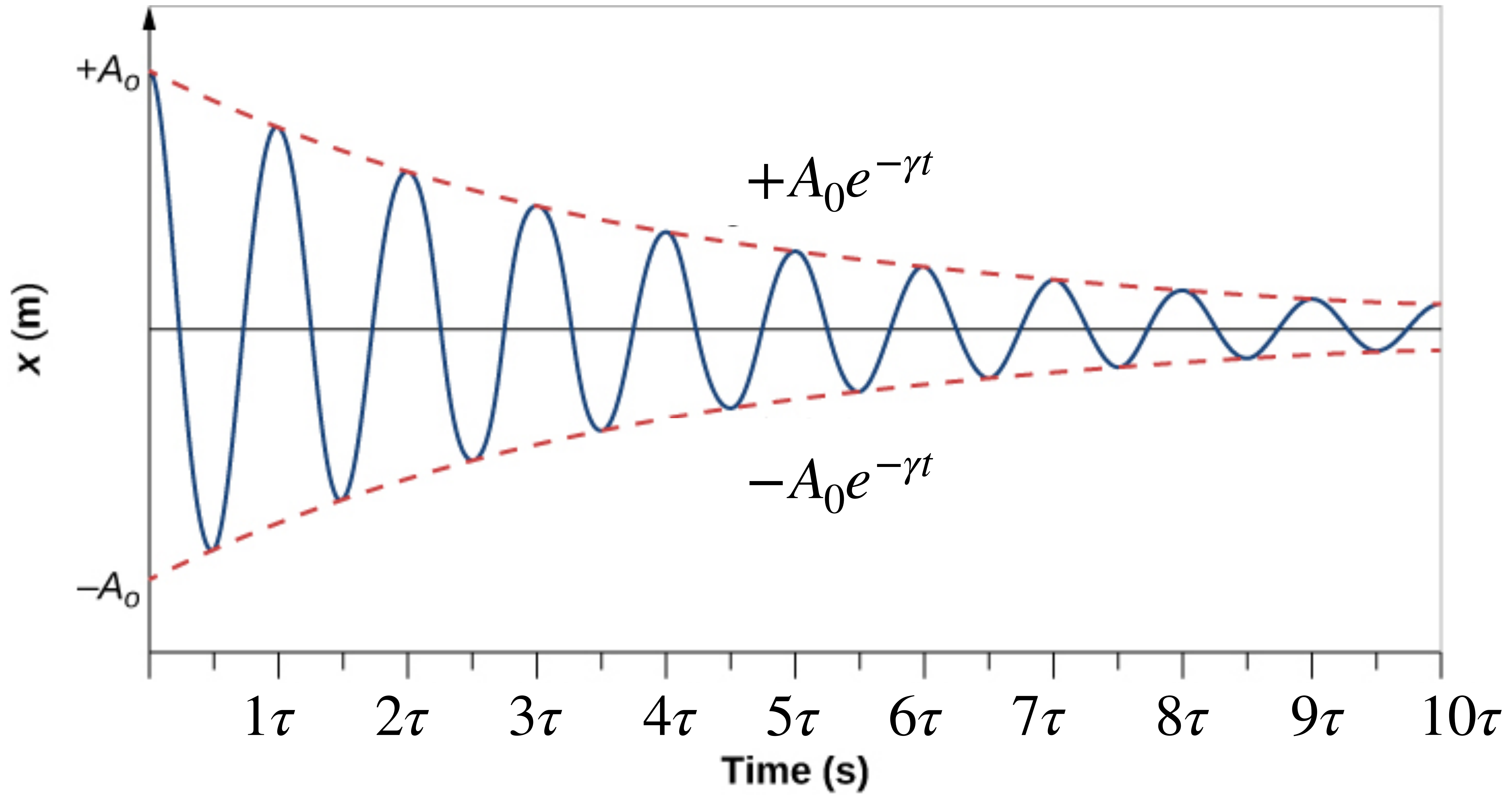
There are three main types of damping

- *Underdamped*, in which the system oscillates with a gradually decreasing amplitude
- *Critically damped*, in which the system returns to equilibrium as quickly as possible without oscillating
- *Overdamped*, in which the system returns to equilibrium slowly without oscillating



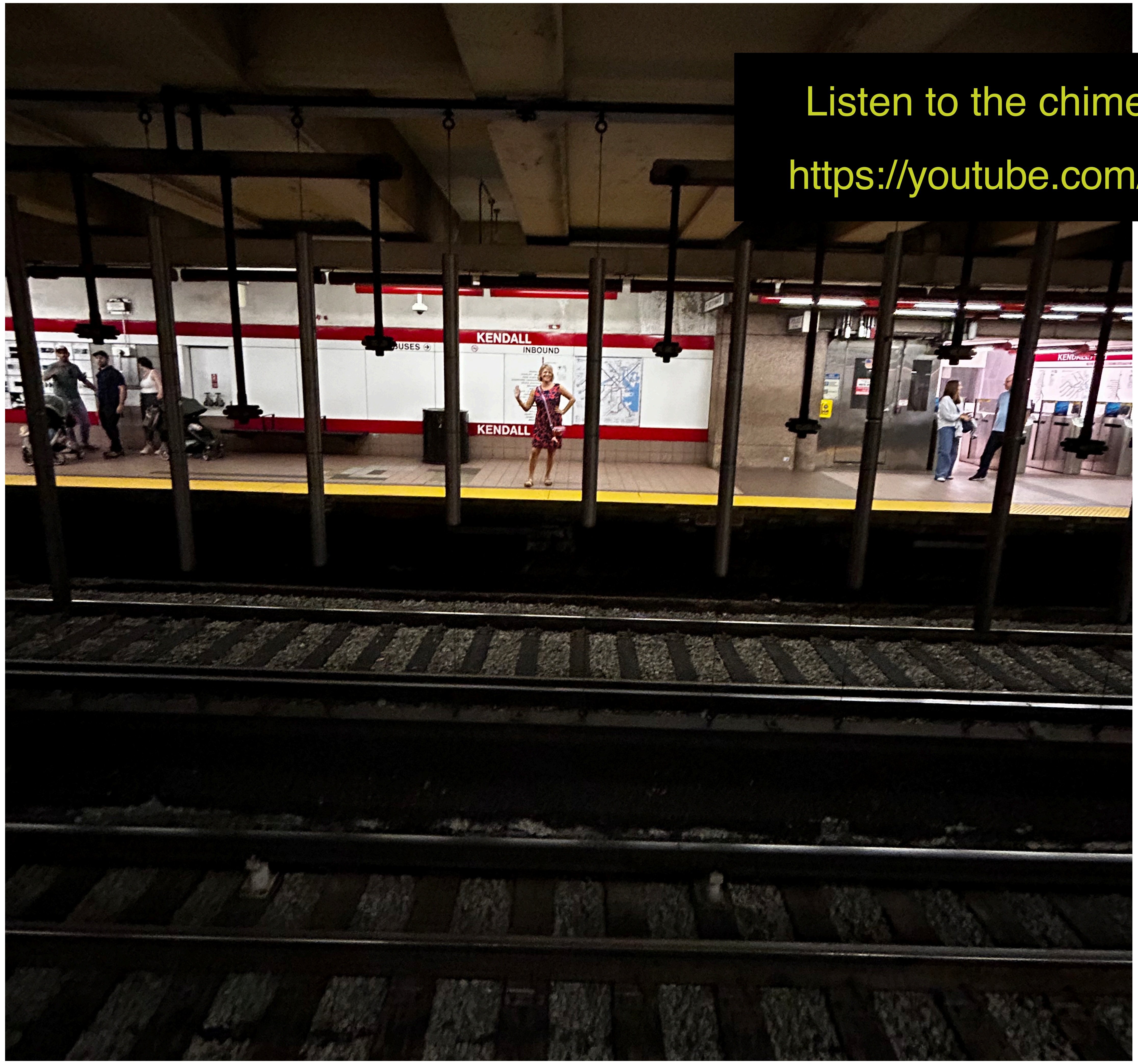
Decay Rates of Damped Oscillatory Systems

➢ For a linearly damped harmonic oscillator, the amplitude A decays exponentially over time according to a decay constant γ, $A(t) = A_0 e^{-\gamma t}$ with $e \simeq 2.72$



Driven Oscillations

A *driven* (or forced) oscillation occurs when a periodic external force is applied to a damped system to maintain its motion ☛ causing it to vibrate at the driving frequency rather thanits natural frequency



Listen to the chimes of the Kendall Band

https://youtube.com/shorts/ROg-NCxyIZM?

Resonance

- The phenomenon known as resonance occurs when a driven oscillator is made to vibrate at the same frequency as the driver oscillator
- Resonance effects are characterized by vibrations with large amplitudes
- Think about a child being pushed on a swing
- The pusher (driver) pushes at the same frequency as the natural frequency of the oscillation of the swing (driven)
- The result is that the swing oscillates with large amplitude, since the energy it needs to gain altitude is being supplied at exactly the right time in the cycle

A mass on a spring undergoes SHM.

The maximum displacement from the equilibrium is called?

A. Period

B. Frequency

C. Amplitude

D. Wavelength

E. Speed

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In a periodic process, the number of cycles per unit of time is called?

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In a periodic process, the number of cycles per unit of time is called?

A. Period
B. Frequency
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In a periodic process, the time required to complete one cycle is called?

A. Period

B. Frequency

C. Amplitude

D. Wavelength

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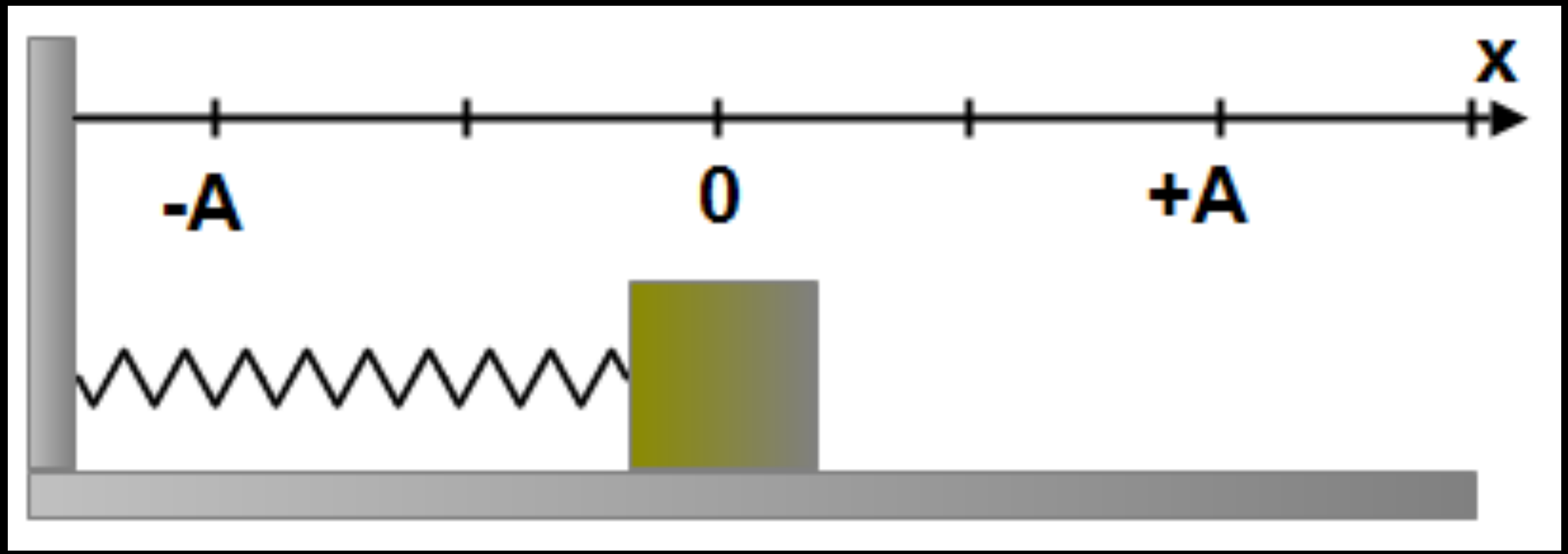
A mass in the diagram to the right undergoes simple harmonic motion



When the mass reaches point $x = +A$ its instantaneous velocity is?

A. Maximum and positive

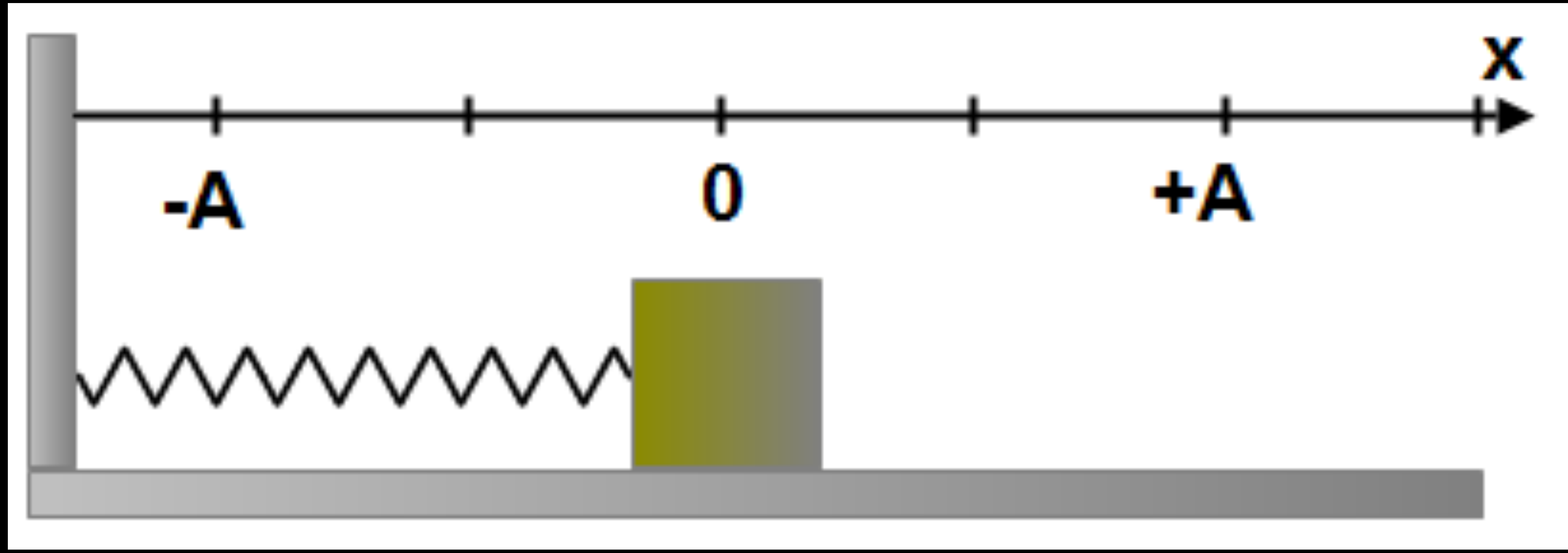
B. Maximum and negative

C. Zero

D. Less than maximum and positive

E. Less than maximum and negative

A mass in the diagram to the right undergoes simple harmonic motion



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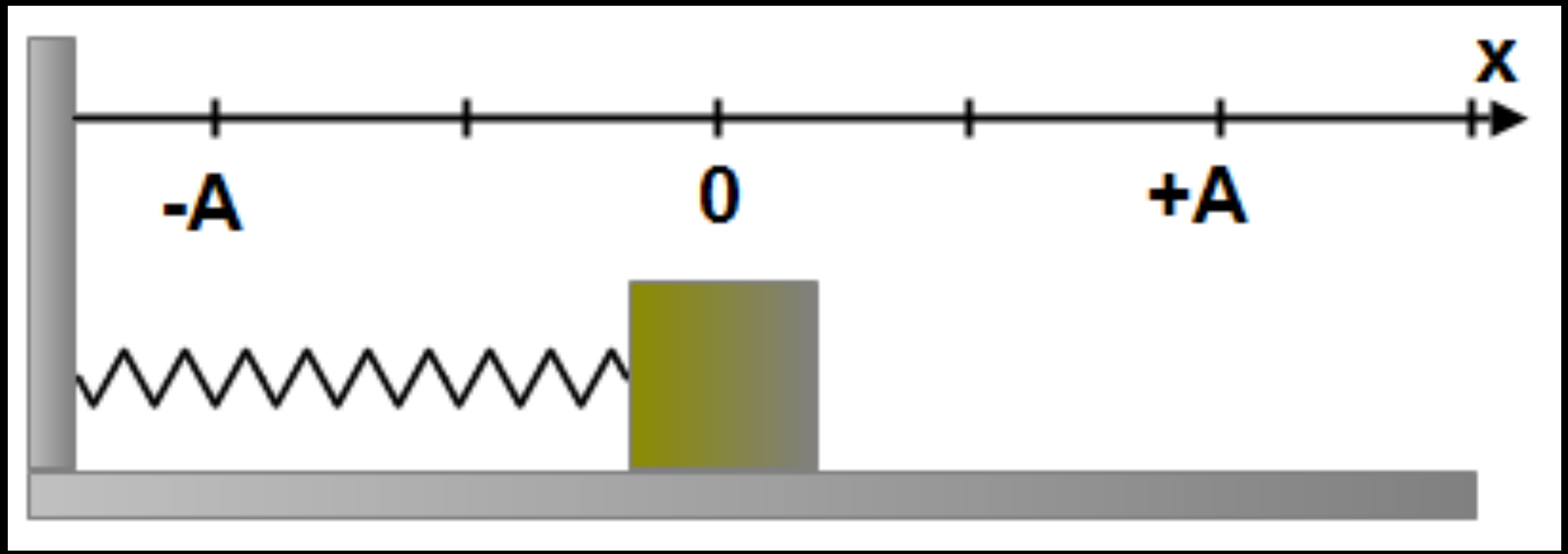

When the mass reaches point $x = 0$ its instantaneous velocity is?

A. Maximum and can be positive or negative

B. Constant and doesn't depend on the location

C. Zero

D. Slightly less than maximum and positive

E. Slightly less than maximum and negative

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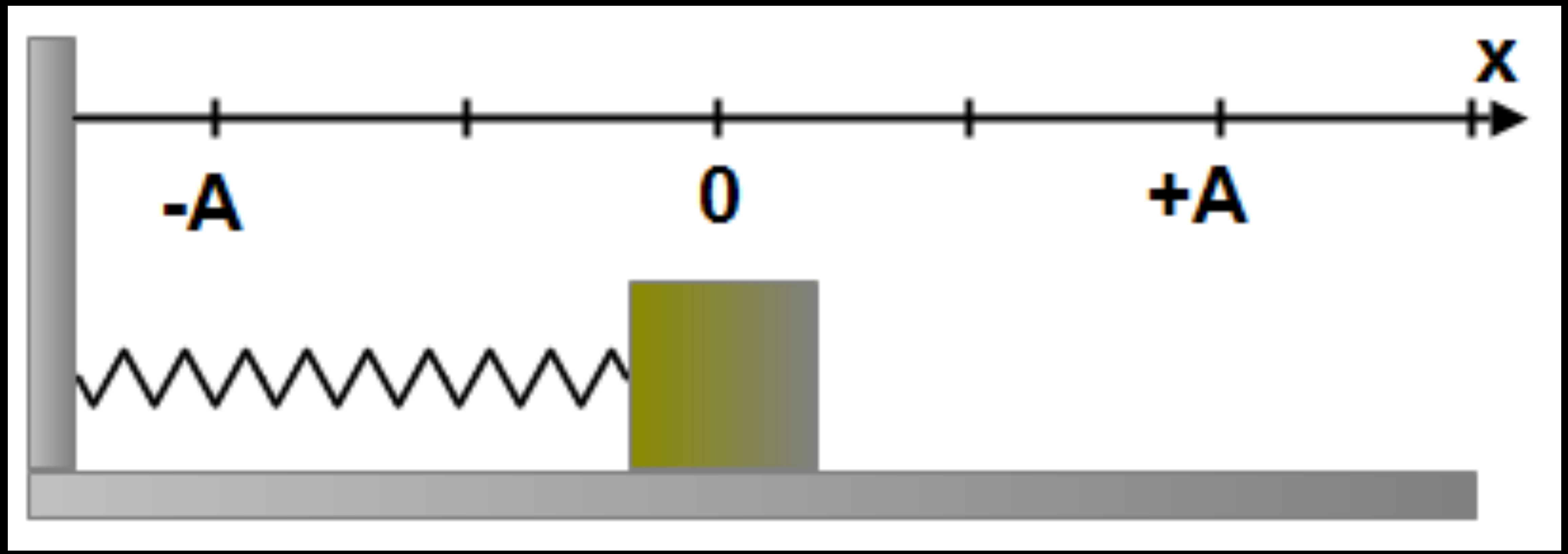

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A mass-spring oscillating system undergoes SHM with maximum amplitude A

If the amplitude is doubled what effect will it produce on the mechanical energy of the system?

A. The energy is increased by factor two
B. The energy is increased by factor four
C. The energy is decreased by factor two
D. The energy is decreased by factor four
E. It doesn't affect the energy

[hint: $E_{\text{mechanical}} = \frac{1}{2}kA^2$]

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A mass M suspended from a string L undergoes SHM

Which of the following is true about the period of oscillations?

A. The period increases with increasing amplitude

B. The period increases with increasing mass

C. The period increases with decreasing lenght

D. The period increases with increasing lenght

E. The period doesn't depend on acceleration due to gravity

[hint: $\tau = 2\pi\sqrt{L/g}$]

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[hint: $\tau = 2\pi\sqrt{L/g}$]

A simple pendulum is moved from the Earth to the Moon. How does it change the period of oscillations?

Acceleration due to gravity on moon= 1.6 m/s^2

A. The period is increased by factor $\sqrt{6}$

B. The period is increased by factor four

C. The period is decreased by factor $\sqrt{6}$

D. The period is decreased by factor four

E. The period remains the same

[hint: $\tau = 2\pi\sqrt{L/g}$]

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[hint: $\tau = 2\pi\sqrt{L/g}$]

The length of a simple pendulum oscillating with a period τ is quadrupled, what is the new period of oscillations in terms of τ ?

A. 2τ

B. 4τ

C. τ

D. $\frac{1}{2}\tau$

E. $\frac{1}{4}\tau$

[hint: $\tau = 2\pi\sqrt{L/g}$]

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[hint: $\tau = 2\pi\sqrt{L/g}$]

A simple pendulum has a period of 1 s. What is the length of the string?

A. 1 m

B. 2 m

C. 4 m

D. $\frac{1}{2}$ m

E. $\frac{1}{4}$ m

[hint: $\tau = 2\pi\sqrt{\frac{L}{g}} \Rightarrow L = g\left(\frac{\tau}{2\pi}\right)^2$]

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A simple pendulum with a length of 1 m oscillates on the surface of a hypothetical planet X

What is the surface gravity on the planet if the period of oscillations is 4 s?

A. $1.6 \ \frac{\mathrm{m}}{\mathrm{s}^2}$

B. $3.7 \ \frac{\mathrm{m}}{\mathrm{s}^2}$

C. $11.2 \ \frac{\mathrm{m}}{\mathrm{s}^2}$

D. $2.5 \ \frac{\mathrm{m}}{\mathrm{s}^2}$

E. $9.8 \ \frac{\mathrm{m}}{\mathrm{s}^2}$

[hint: $\tau = 2\pi\sqrt{\frac{L}{g}} \Rightarrow g_X = L\left(\frac{2\pi}{\tau}\right)^2$]

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D. $2.5\ \dfrac{\mathrm{m}}{\mathrm{s}^2}$

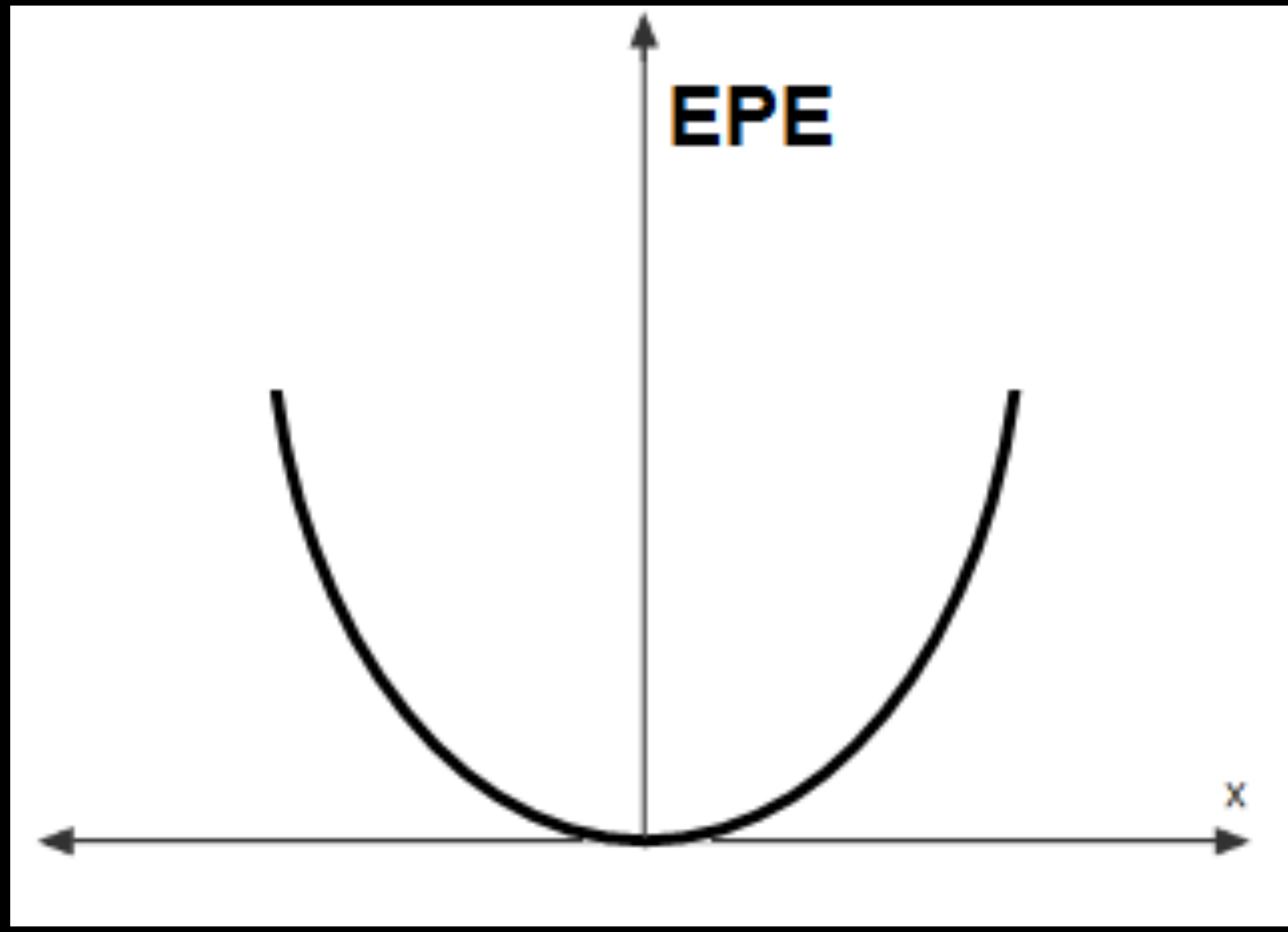
E. $9.8\ \dfrac{\mathrm{m}}{\mathrm{s}^2}$

[hint: $\tau = 2\pi\sqrt{\dfrac{L}{g}} \Rightarrow g_X = L\left(\dfrac{2\pi}{\tau}\right)^2$]

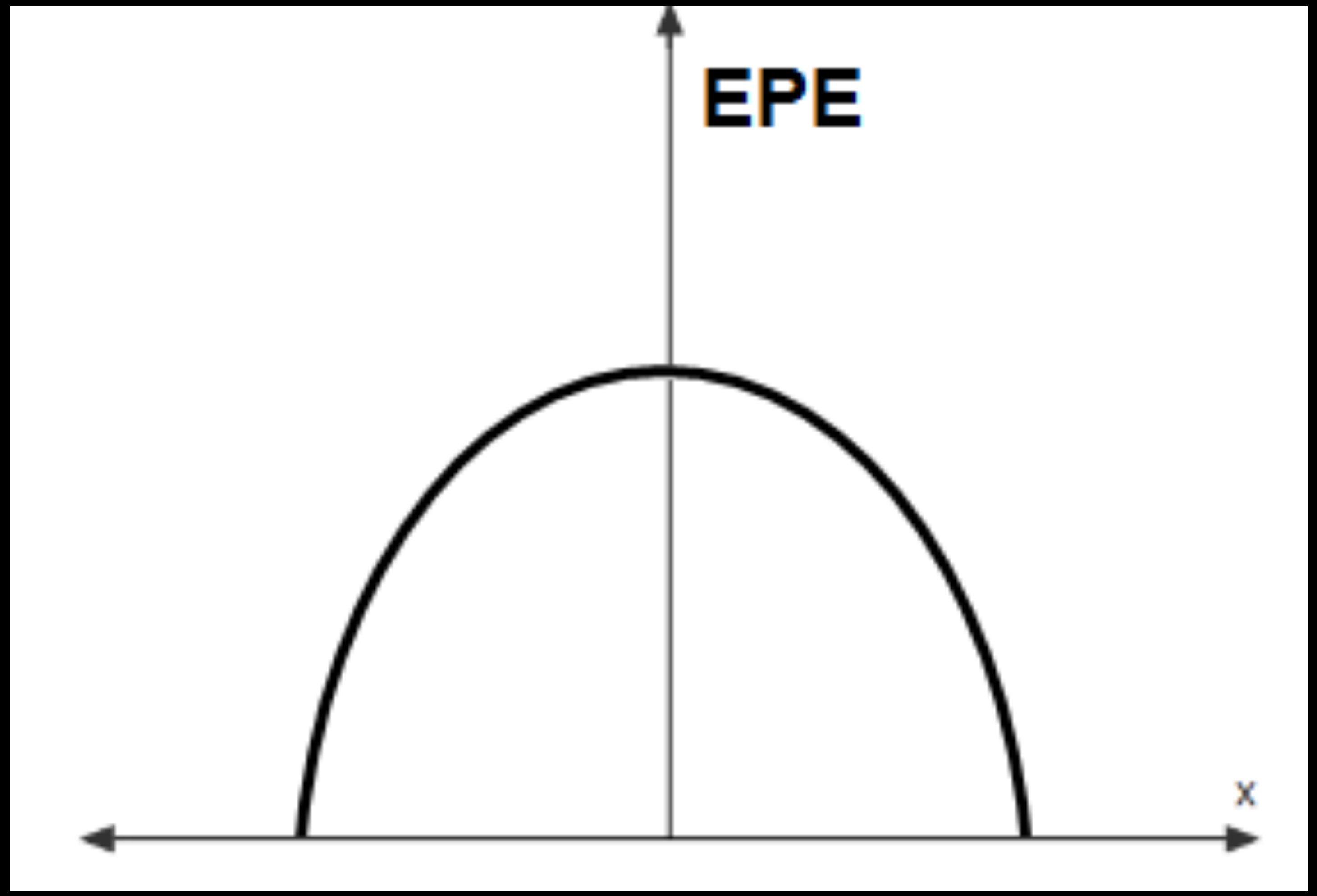
A mass-spring oscillating system undergoes SHM

Which of the following graphs represents the elastic potential energy as a function of position?

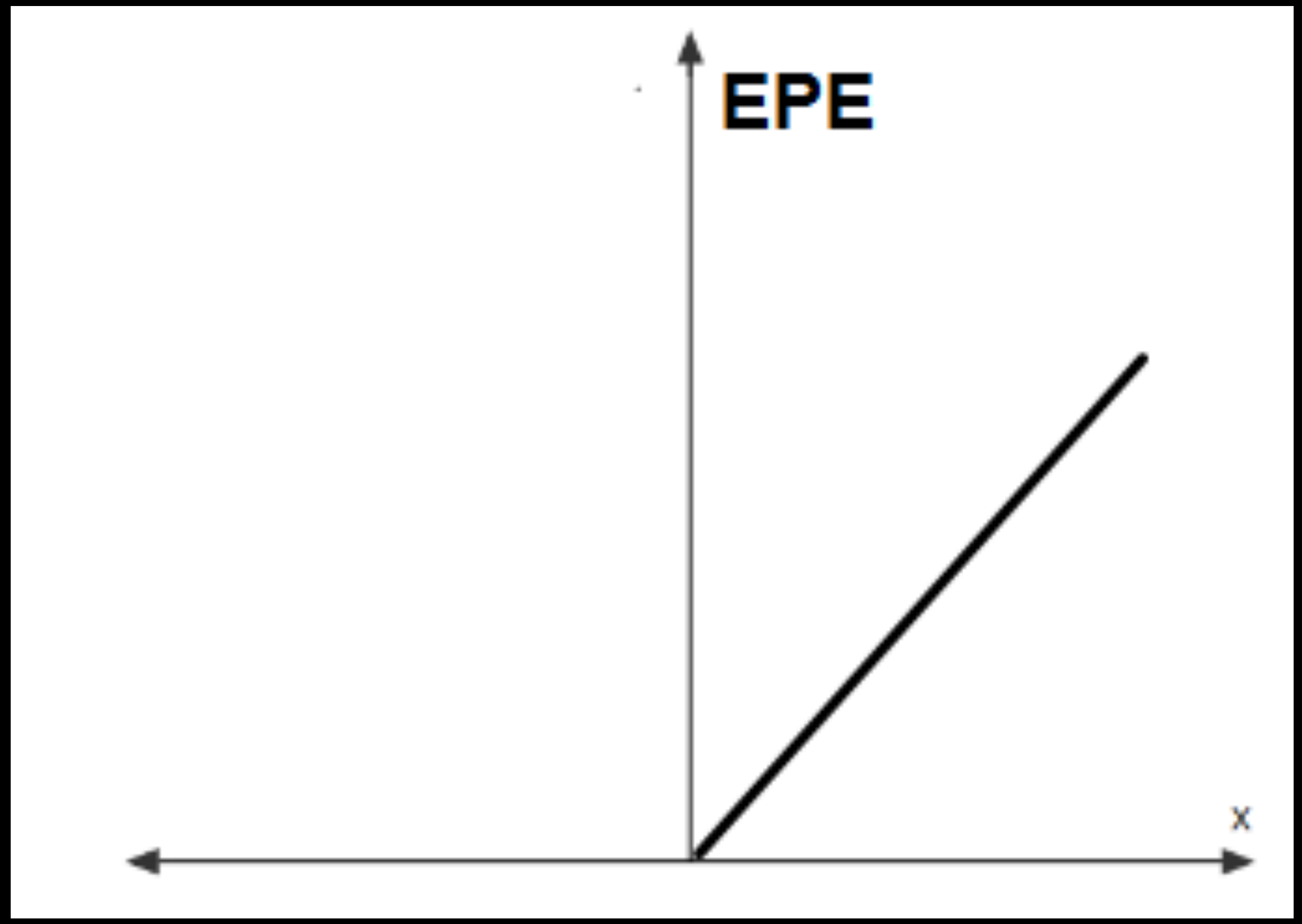
A.



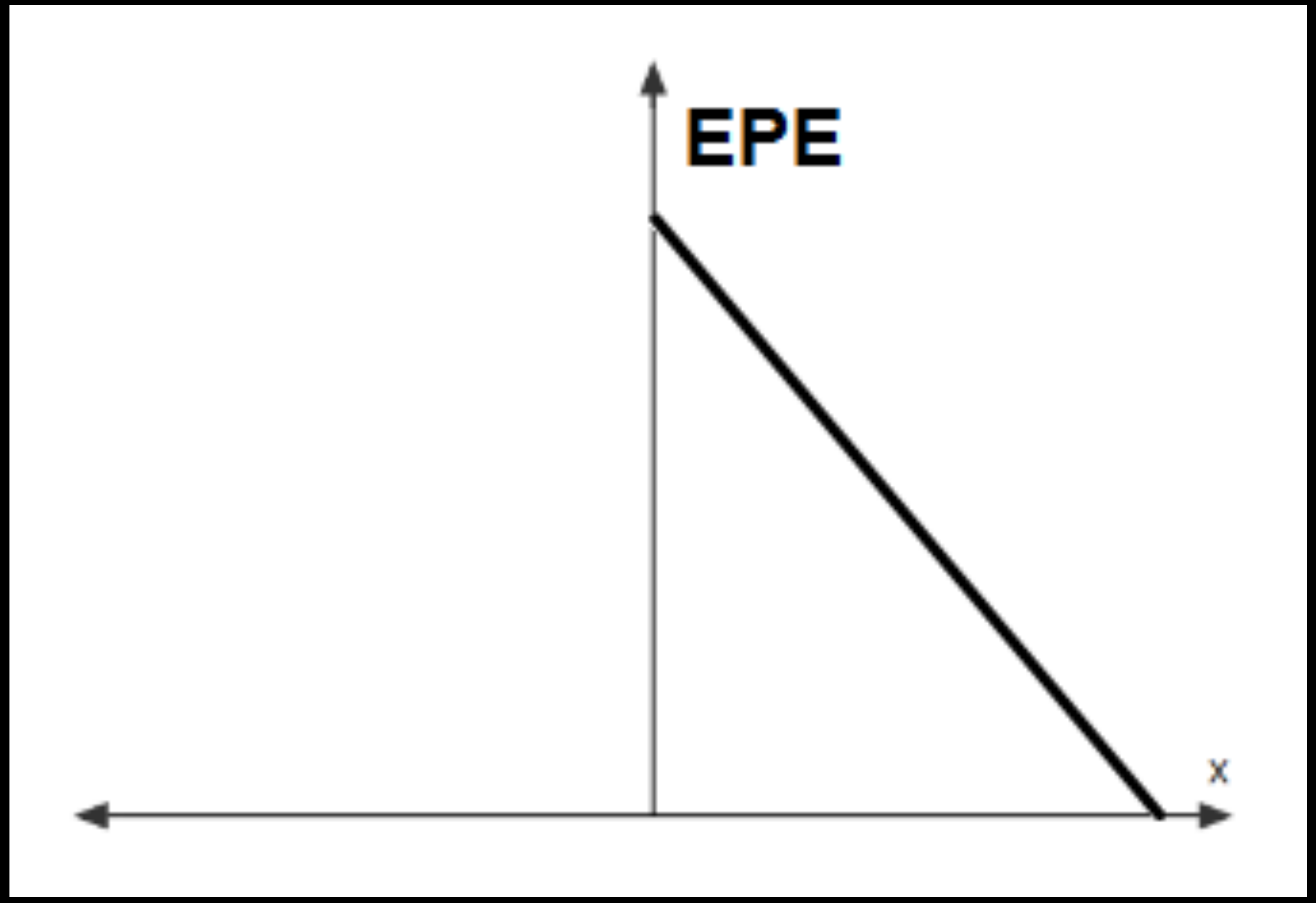
B.



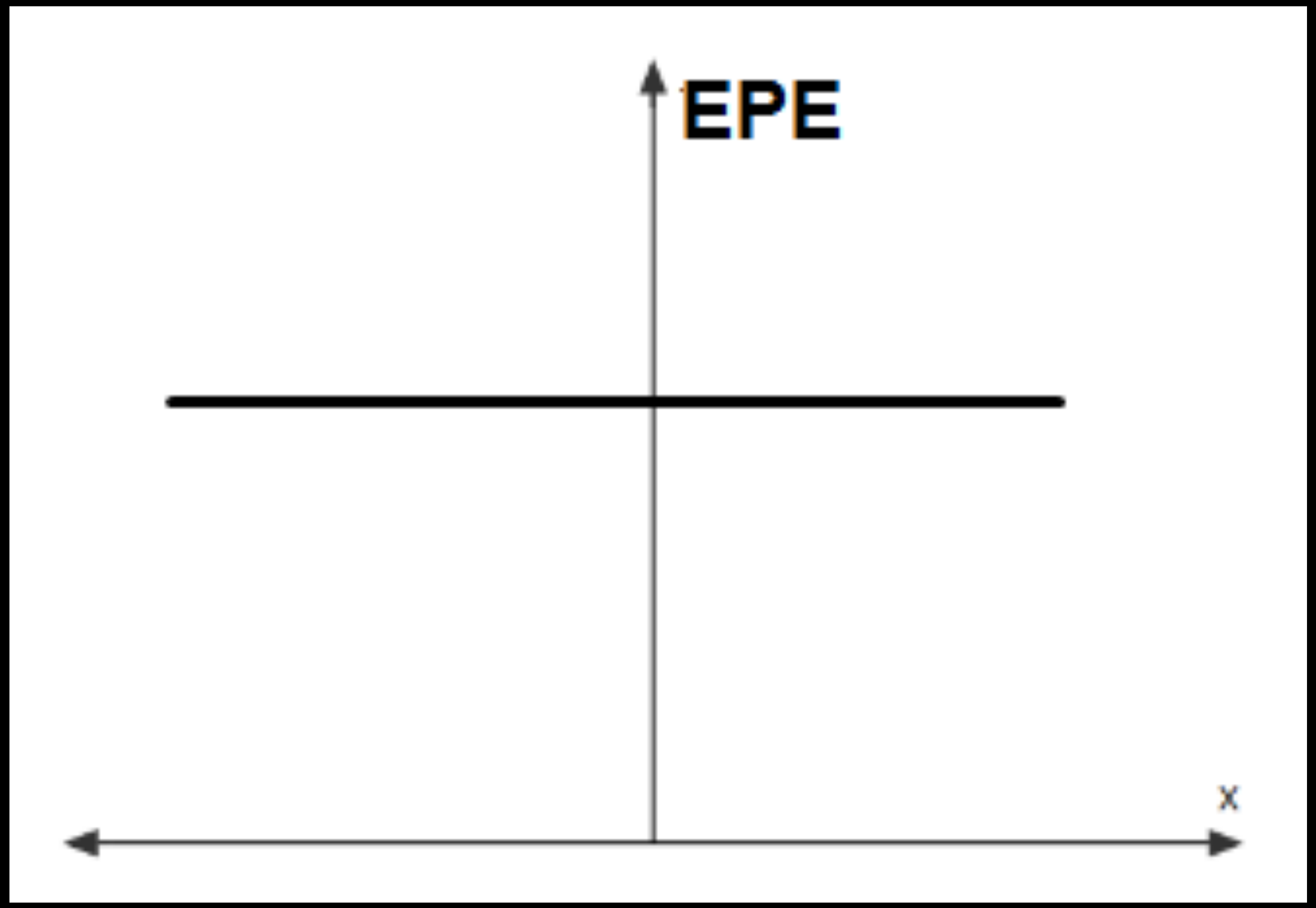
C.



D.



E.



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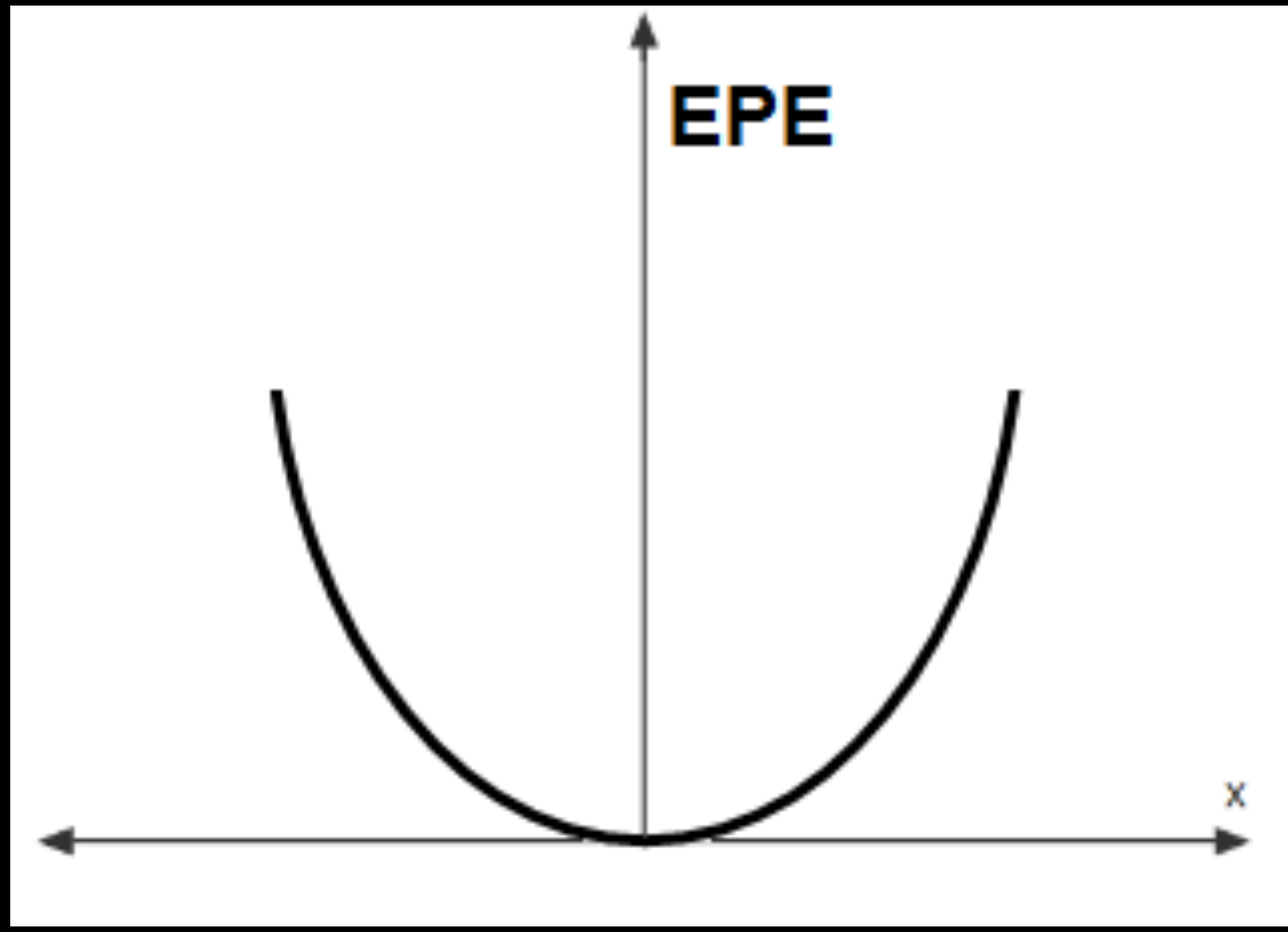
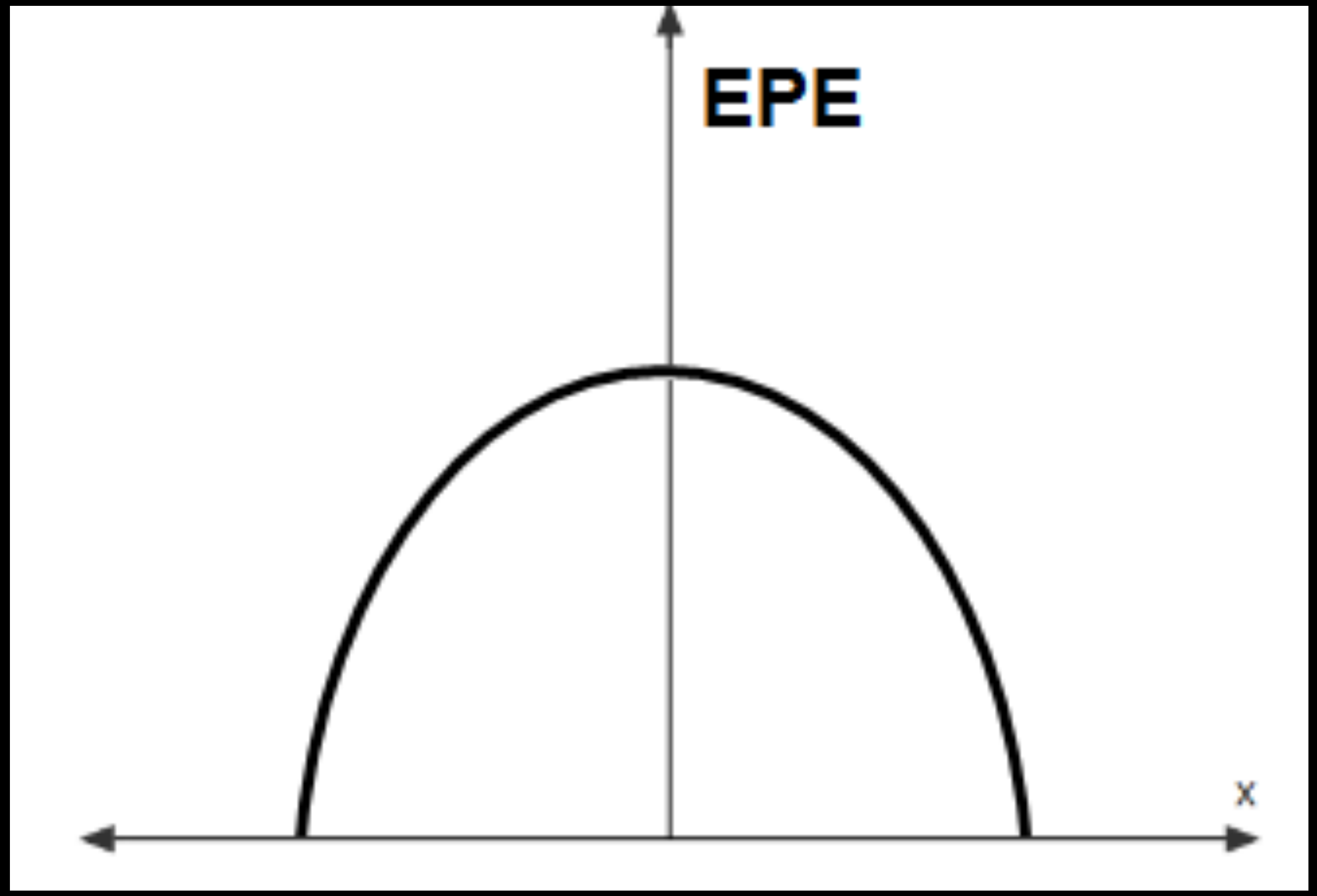
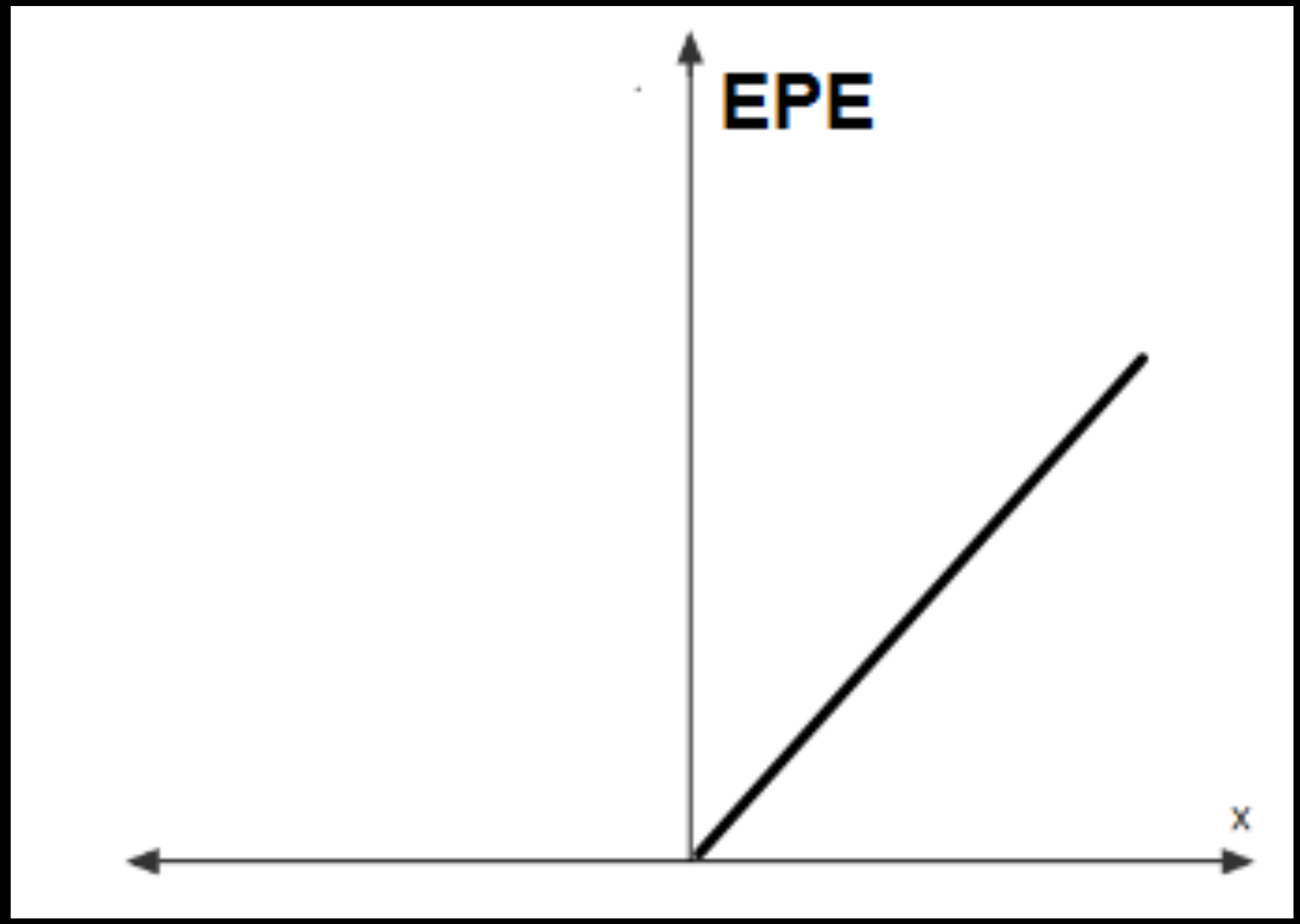


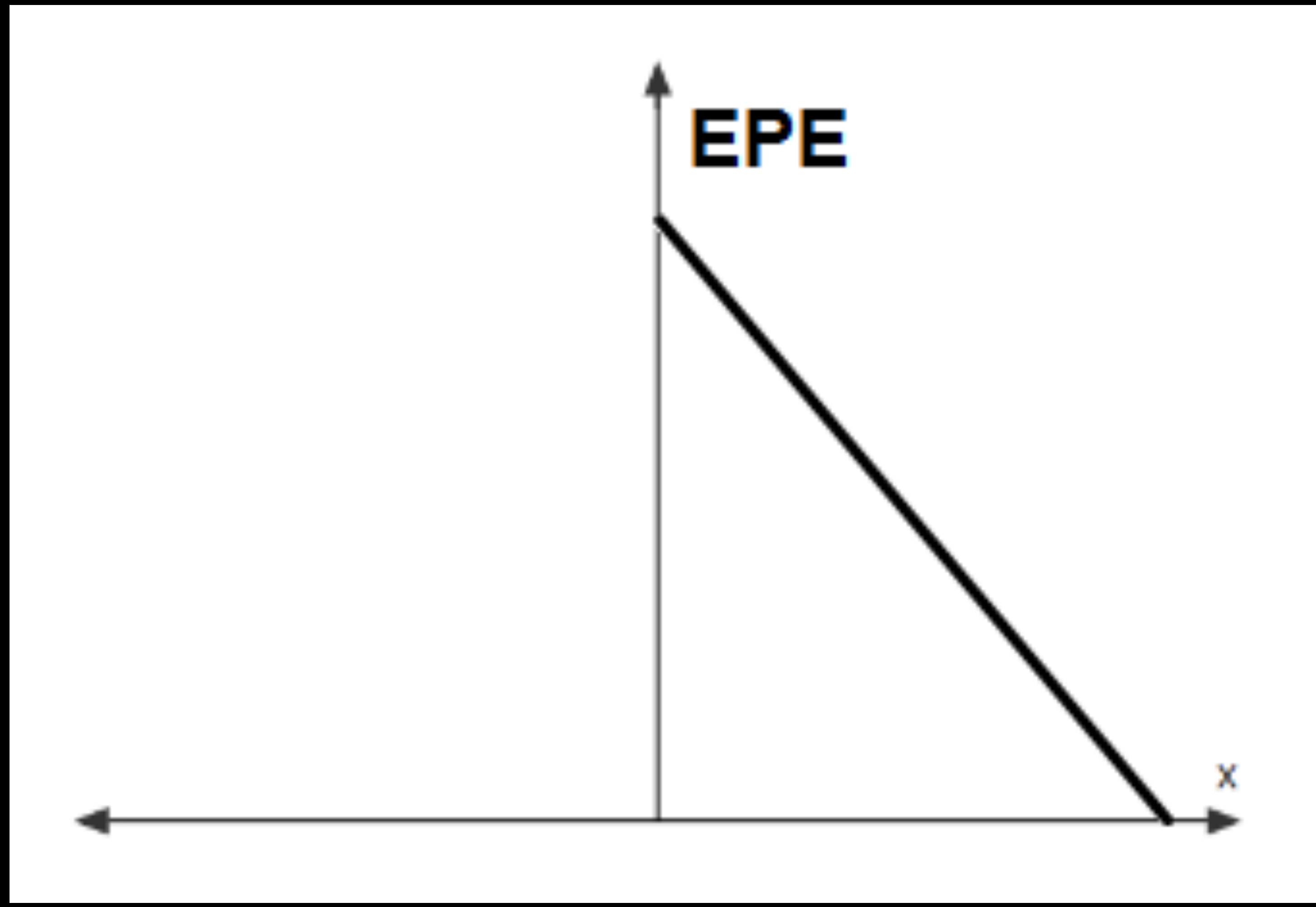
B.



C.



D.



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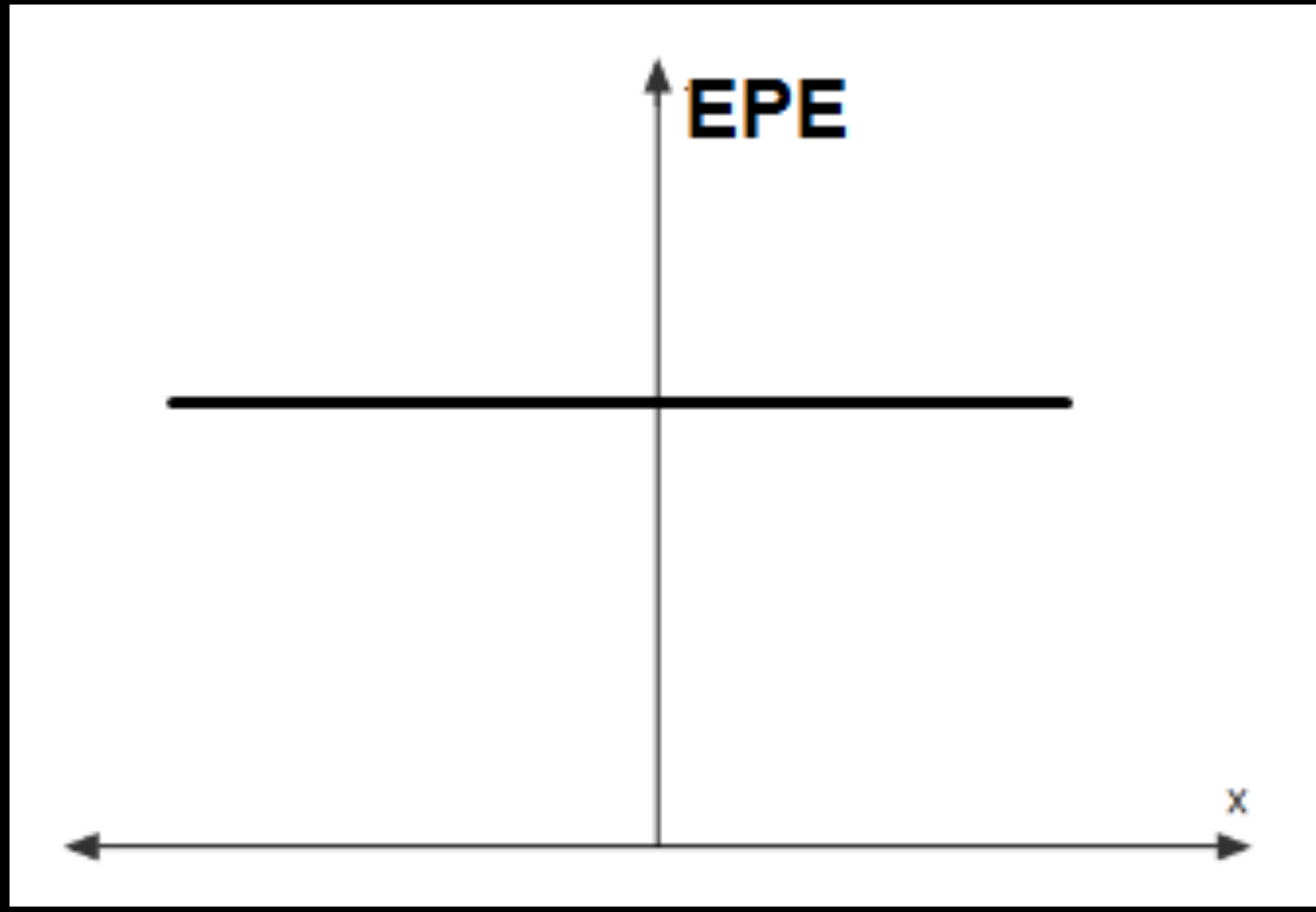

In damped oscillations, the amplitude of the oscillations:

A. Remains constant over time
B. Increases linearly with time
C. Decays exponentially over time
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A. Its natural frequency f_0

B. The frequency of the driving force f_d

C. The difference between the two frequencies $(f_0 - f_d)$

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What is the effect of increased damping on a forced oscillator?

A. The maximum amplitude increases
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Bonus Challenge (not on the final exam)

In a lightly damped system, if the amplitude of the oscillator takes time $t_{A_{1/2}}$ to fall to half its initial value, the total mechanical energy of the system falls to half its initial value in:

A. $t_{A_{1/2}}$

B. $2t_{A_{1/2}}$

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Explanation of the Bonus Challenge

- For a damped harmonic oscillator, the amplitude A decays exponentially over time according to a decay constant γ

$$A(t) = A_0 e^{-\gamma t} \qquad \text{with } e \simeq 2.72$$

- Substituting this into the mechanical energy relation $E_{\rm mechanical} = \frac{1}{2} k A^2$

  we find that the mechanical energy decays as fast as $E_{\rm mechanical}(t) = \frac{1}{2} k A_0^2 e^{-2\gamma t}$

- This gives us the energy decay equation ☛ $E_{\rm mechanical}(t) = E_0 e^{-2\gamma t}$

Compare the "*Half-Life*" Times

- Let $t_{A_{1/2}}$ be the time where $A(t_{A_{1/2}}) = A_0/2$ and $t_{E_{1/2}}$ be the time where $E_{\mathrm{mechanical}}(t_{E_{1/2}}) = E_0/2$
- For amplitude ☛ $e^{-\gamma t_{A_{1/2}}} = 1/2$
- For energy ☛ $e^{-2\gamma t_{E_{1/2}}} = 1/2$
- Since both expressions equal $1/2$ ☛ their exponents must be equal $\gamma t_{A_{1/2}} = 2\gamma t_{E_{1/2}}$
- Dividing both sides by γ $t_{A_{1/2}} = 2t_{E_{1/2}} \Rightarrow t_{E_{1/2}} = t_{A_{1/2}}/2$

Conclusion of Bonus Challenge

- The energy reduces to half its value in $t_{A_{1/2}}/2$ because energy depends on the square of the amplitude
- When the amplitude is squared, the exponential decay rate effectively doubles, meaning any percentage reduction in energy happens in half the time required for the same percentage reduction in amplitude