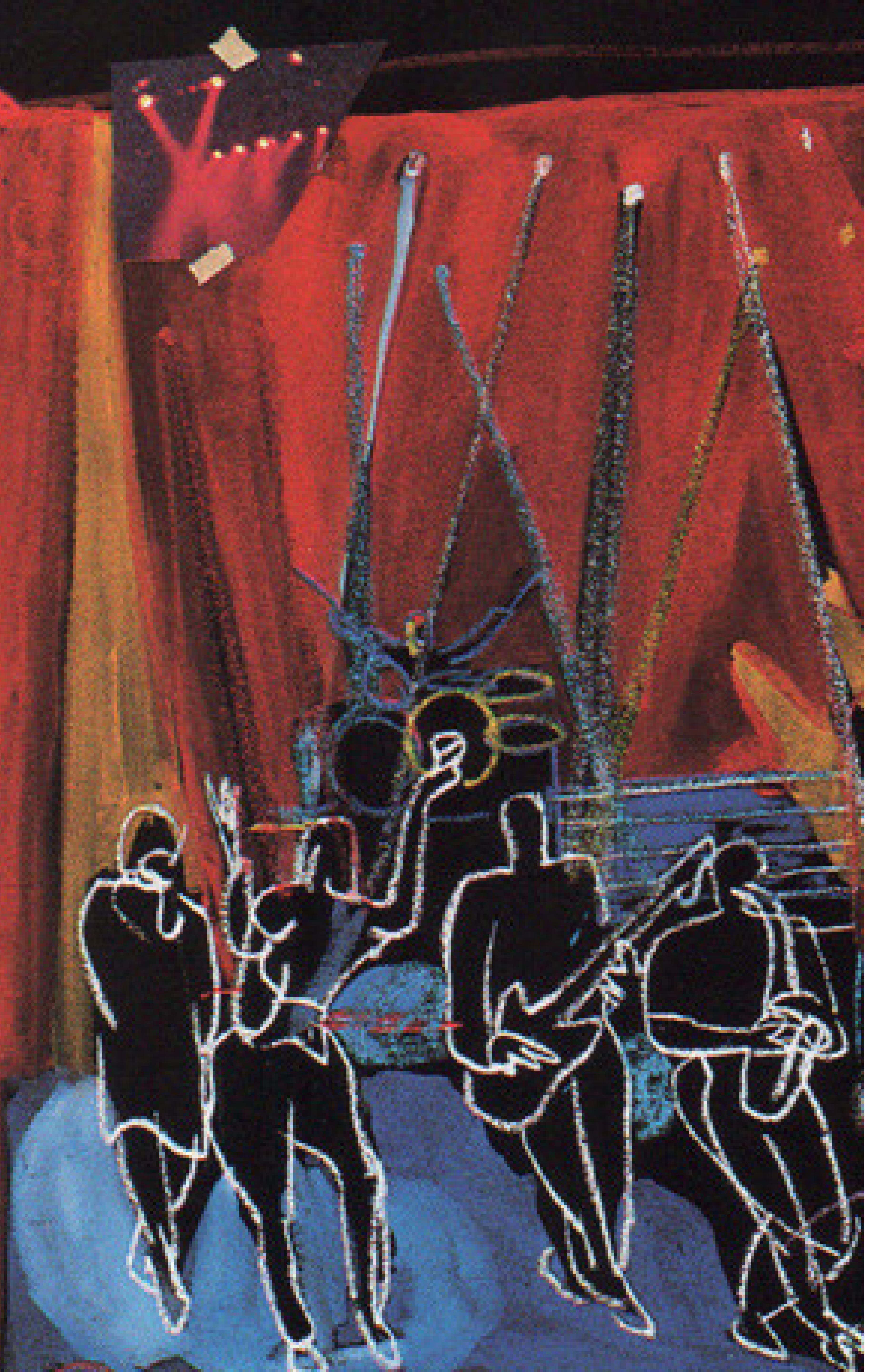


PHY-141

From Wave to Note

Luis Anchordoqui

Outline

- The octave serves as the fundamental building block of music, representing a unique, almost universal perceptual boundary where a note and its higher/lower counterpart sound functionally identical, yet exist at different pitch registers

- To wrap up our exploration of sound in music, today's class focuses on the octave and its constituent intervals

The Harmony of Numbers: Pythagoras

- Pythagoras was the first to scientifically define the major scale by discovering that harmonious musical intervals are rooted in simple mathematical ratios

$$1:1 \quad 2:1 \quad 3:1$$

- In what follows, we concentrate on the $1:1$ and $2:1$ ratio

Pythagorean Harmonic Ratios

- To ideally reconstruct Pythagoras's experiment, we visualize a single-string guitar (monochord) by stretching a string between two bridges separated by a distance L, as shown in the figure

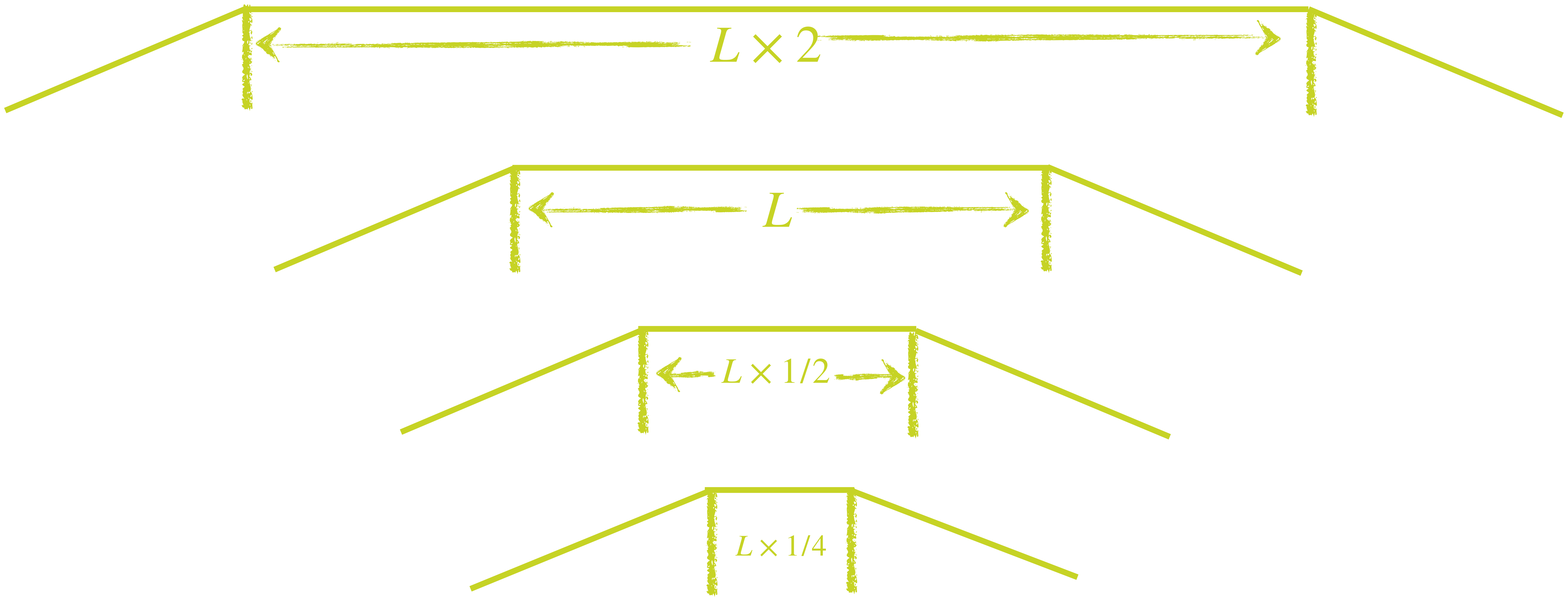
- Plucking the string now produces a specific pitch

Pythagorean Harmonic Ratios (cont'd)

- Next, we build an identical instrument, but with the bridge distance reduced to $L/2$ as shown in figure
- The key to achieving harmonic resonance lies in maintaining uniform tension across strings of different lengths
- According to modern physical principles, a string half the length of another will vibrate at twice the frequency, creating a $2:1$ ratio
- Because this relationship is fundamental to pitch recognition, both tones are perceived as the same musical note (e.g., $440\ \mathrm{Hz}$ and $880\ \mathrm{Hz}$)
- This principle of equivalence is what enables collective singing, allowing individuals with different vocal registers to sing the same melody in unison

Pythagorean Harmonic Ratios (cont'd)

➢ This procedure is reversible and repeatable, allowing for the construction of the following sequence of single-string guitars



Natural Ordering

- Given the right conditions, these strings generate this ratio of frequencies

$$1:2 \quad 1:1 \quad 2:1 \quad 4:1$$

- These ratios follow a natural, ascending order, establishing a natural ordinal scale

- This scale is part of the following infinite, ordered sequence

$$\cdots \quad \boxed{\frac{1}{4}} < \boxed{\frac{1}{2}} < \boxed{\frac{1}{1}} < \boxed{\frac{2}{1}} < \boxed{\frac{4}{1}} < \boxed{\frac{8}{1}} \quad \cdots$$

Frequency Scale

- By assigning the ratio $1/1$ to a reference frequency of 220Hz, the following derivative scale is generated

$$\cdots \boxed{55 \text{ Hz}} \cdots \boxed{110 \text{ Hz}} < \boxed{220 \text{ Hz}} < \boxed{440 \text{ Hz}} < \boxed{880 \text{ Hz}} < \boxed{1760 \text{ Hz}} \cdots$$

- Note that this scale is theoretically infinite, even though it rapidly extends beyond the range of human hearing in both directions

The Octave: Music's Fundamental Interval

- So far, our scale has an ordered ranking without defined start or end points
- We can turn this into an interval scale by assuming the distance between each consecutive ratio or frequency is the same

$$\frac{1}{4} - \frac{1}{2} = \frac{1}{2} - \frac{1}{1} = \frac{1}{1} - \frac{2}{1} = 1 \text{ octave}$$

$$55 \text{ Hz} - 110 \text{ Hz} = 110 \text{ Hz} - 220 \text{ Hz} = 220 \text{ Hz} - 440 \text{ Hz} = 1 \text{ octave}$$

- The resulting interval-unit is musically fundamental, and is called an *octave*
- It is defined by the gap between a pitch and the next one sharing the same note name
- In terms of frequency, this happens whenever a pitch's vibration rate doubles
- For instance, the note A at 220 Hz and the next A at 440 Hz are exactly one octave apart; likewise, the jump from 440 Hz to 880 Hz forms another octave

The Logarithmic Scale of Pitch: Frequency Ratios in Octaves

- It's important to recognize that the specific numerical difference between frequencies doesn't actually matter in music; what counts is the ratio between them
- We could think of a musical scale as a logarithmic scale
- For instance, if we convert those frequency ratios into base-2 logarithms, we get a scale that looks much more familiar

$$\ldots \quad -2 \quad < \quad -1 \quad < \quad 0 \quad < \quad +1 \quad < \quad +2 \quad \ldots$$

- Specifically, values are spaced apart by simple subtraction; for instance, a difference of 1 represents a distance of exactly one octave

Making Sense of Cents in Musical Scales

- We have seen that an octave is the interval between two pitches where the higher frequency is double the lower ($2:1$ ratio)

- In Western music, an octave consists of 12 semitones (or half-steps), each 100 cents apart, totaling 1200 cents

- A cent is a logarithmic unit for measuring intervals, where 100 cents make 1 semitone, and $5-10$ cents is the minimum difference people usually detect

Musical Intervals

- In the context of measuring musical intervals, one must bear in mind the following *equations*

$$\boxed{\text{intervals} \;=\; \text{ratios}}$$

and

$$\boxed{\text{adding intervals} \;=\; \text{multiplying ratios}}$$

- For example

$$\begin{aligned} 3 \text{ octaves} &= 1 \text{ octave} + 1 \text{ octave} + 1 \text{ octave} \\ &= 2/1 \times 2/1 \times 2/1 \\ &= 8/1 \end{aligned}$$

Musical Intervals (cont'd)

- Likewise, when we define a semitone as a twelfth of an octave, we are referring to the following relationship ☛

$$s+s+s+s+s+s+s+s+s+s+s+s = \text{octave}$$

$$r \times r \times r \times r \times r \times r \times r \times r \times r \times r \times r \times r = 2/1$$

where s is the semitone and r is the corresponding ratio

- To put it concisely ☛ $r^{12} = 2$

  which implies ☛ $r = 2^{1/12}$

- Therefore, the frequency ratio r for a semitone is an irrational number, approximately 1.05946309

- Using the same logic, we can determine that a single cent corresponds to a ratio of roughly 1.0005777895

Short List of Interval Units

Unit Name	Relative Value	Pitch Ratio
octave	fundamental	$2/1$
semi-tone	$1/12$ of an octave	$2^{1/12}$
cent	$1/100$ of a semi-tone	$2^{1/1200}$

Units of Musical Intervals: Cents, Semi-tones, and Octave

- When working with musical intervals, it is important to remember that units like cents, semitones, and octaves represent ratios rather than fixed frequencies

- To apply these intervals to a specific frequency (such as $440\ \mathrm{Hz}$), you use multiplication or division rather than simple addition

- For example, moving up or down an octave means multiplying or dividing the frequency by 2

- In the same way, adjusting a frequency by one cent involves multiplying or dividing it by the ratio $2^{1/1200}$

The 3:2 Ratio Scale

- While contemporary music divides the octave into semi-tones and cents, this is a relatively modern development

- Specifically, Pythagorean tuning is derived by placin the $3:2$ ratio between $1:1$ and $2:1$, resulting in an original, ordered scale

$$\boxed{\frac{1}{1}} < \boxed{\frac{3}{2}} < \boxed{\frac{2}{1}}$$

Stacking Perfect Fifths

- Then, we take the ratio $(2/1)/(3/2) = 4/3$, which can be depicted as

$$\boxed{\frac{1}{1}} \underbrace{\qquad\qquad\qquad}_{3/2} \boxed{\frac{3}{2}} \underbrace{\qquad\qquad\qquad}_{4/3} \boxed{\frac{2}{1}}$$

or a microtonal scale defined by cents (cent)

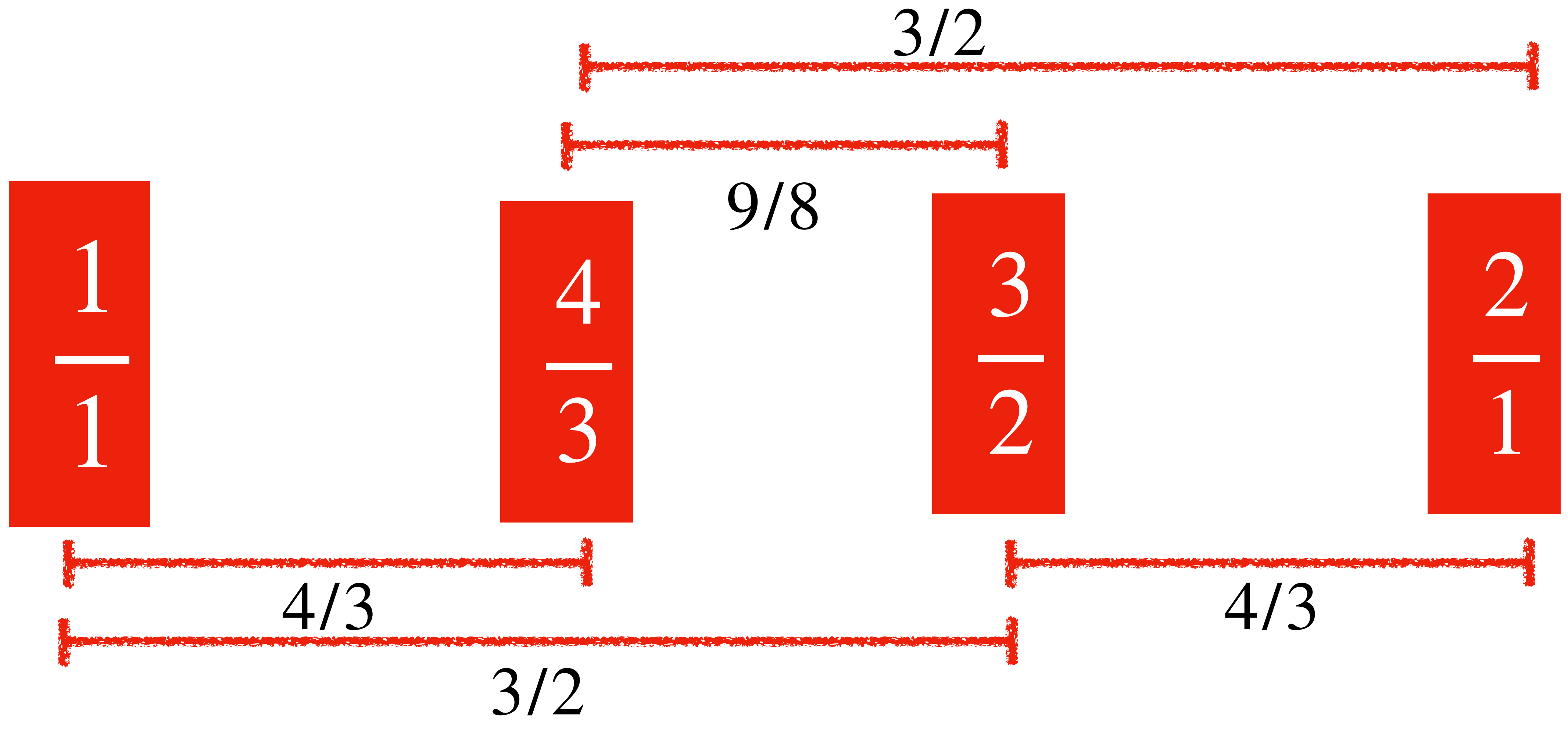
$$\boxed{\frac{1}{1}} \underbrace{\qquad\qquad\qquad}_{702\ \text{cent}} \boxed{\frac{3}{2}} \underbrace{\qquad\qquad\qquad}_{498\ \text{cent}} \boxed{\frac{2}{1}}$$

- We take our new ratio of 4/3 and place it between 1/1 and 3/2 to create this ordinal scale

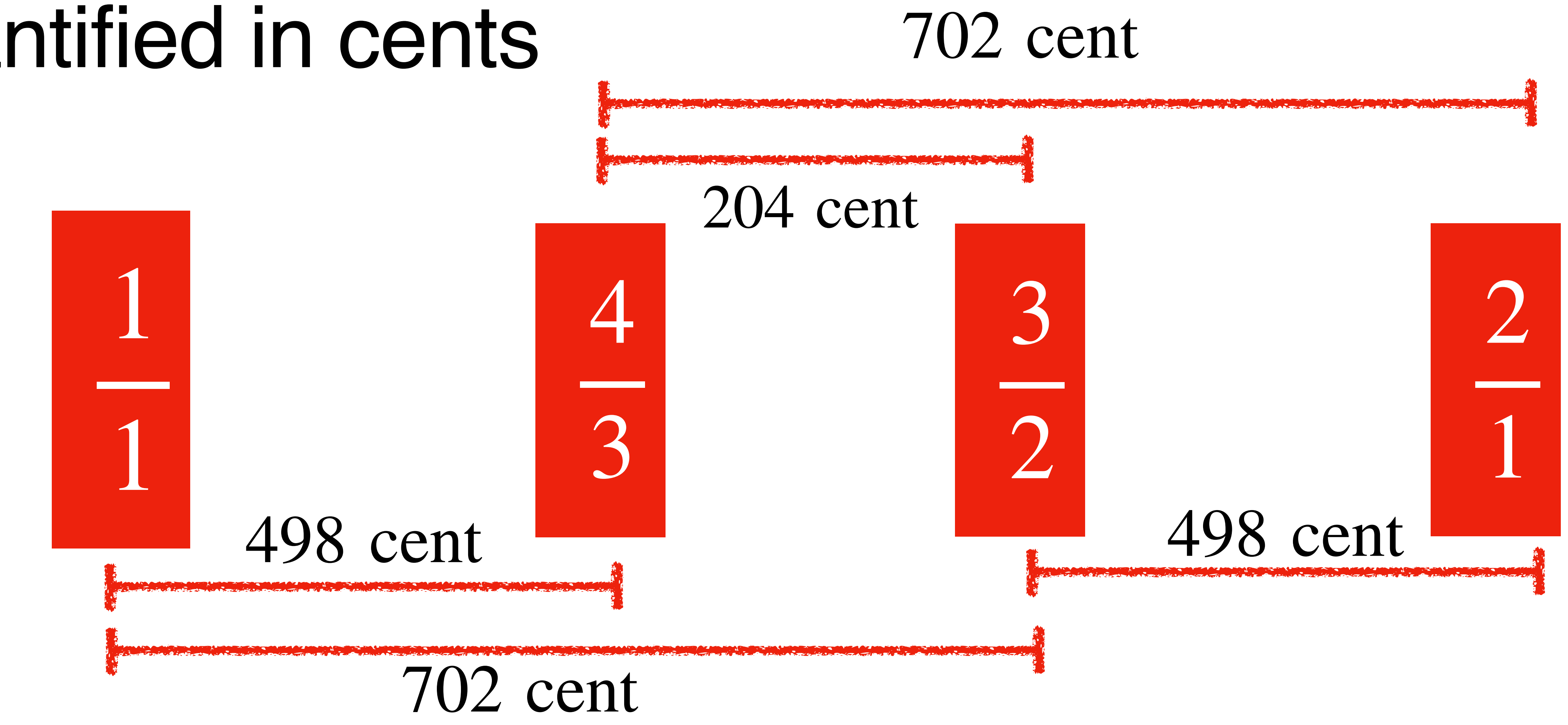
$$\boxed{\frac{1}{1}} < \boxed{\frac{4}{3}} < \boxed{\frac{3}{2}} < \boxed{\frac{2}{1}}$$

Perfect Fifth, Perfect Fourth, and Perfect Second

- We next measure the interval between $4/3$and $3/2$, which is calculated as $(3/2)/(4/3) = 9/8$ and leads to



or a scale with intervals quantified in cents



- These intervals have special names ☛ *perfect fifth*, *perfect fourth*, and *perfect second*

Extended List of Interval Units

1/12

Unit Name	Pitch Ratio	Cent Value
octave	2 : 1	1200
perfect fifth	3 : 2	702
perfect fourth	4 : 3	498
perfect second	9 : 8	204
semi-tone	$2^{1/12}$	100
cent	$2^{1/1200}$	1

Pythagorean Interval Ratios

- For the next step of the Pythagorean construction, we use the $9/8$ ratio (204 cent)as our base increment to fill in four more ratios

$\frac{1}{1}$	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	$\frac{2}{1}$

$9/8$ $9/8$ $9/8$ $9/8$ $9/8$

- Calculating the final two increments is simple using cents: both are exactly 90

- While the resulting ratio $(256/243)$ is cumbersome, we will label this 90 cent interval a *half-step* and the standard 204 cent interval a *whole-step*, despite 90 being slightly less than half of 204

0	204	408	498	702	906	1100	1200

204 204 204 204 204

The Math Behind the Major Musical Scale

- Finally, we arrive at the Pythagorean construction of the major scale, represented in its familiar form

do $\ll$ re $\ll$ mi $<$ fa $\ll$ sol $\ll$ la $\ll$ ti $<$ do

204   204   90   204   204   204   90

- In musical terms, this corresponds to the white keys of a piano, beginning with C

C $\ll$ D $\ll$ E $<$ F $\ll$ G $\ll$ A $\ll$ B $<$ C’

and it is known as the C-major scale, where $\ll$ indicates a whole-step and $<$ indicates a half-step

What is the definition of an octave in music?

A. Two notes played simultaneously

B. The distance between one note and the next note with double or half the frequency

C. A scale consisting of only four notes

D. Three notes played simultaneously

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How many semitones are contained in a standard octave?

A. 8

B. 10

C. 12

D. 16

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B. 10

C. 12

D. 16

Which mathematical ratio represents the Perfect Fifth in Pythagorean harmony?

A. 2 : 1

B. 4 : 3

C. 3 : 2

D. 5 : 4

Which mathematical ratio represents the Perfect Fifth in Pythagorean harmony?

A. 2 : 1

B. 4 : 3

C. 3 : 2

D. 5 : 4

Which of the following pairs represents an octave?

A. C to G

B. C to C

C. C to B

D. C to E

Which of the following pairs represents an octave?

A. C to G
B. C to C
C. C to B
D. C to E

Bonus Challenge (not on the final exam)

A typical marching cadence is $120 - 135$ steps per minute, with a standard step size of 22.5 inches $(57\ \mathrm{cm})$

This results in a speed of roughly $0.5\ \mathrm{m/s}$ to $1.5\ \mathrm{m/s}$ directly toward or away from a listener

Show that for a marching band moving at typical speeds, the doppler shift is usually right at the threshold of human detection (typically 3 cents)

Bonus Challenge (not on the final exam)

If the source (band) is moving and the observer is stationary the Dopple shift

is estimated to be $f_O = f_S \dfrac{u}{u - v}$

and so the ratio of the frequencies is $f_O/f_S \approx 1.00439$ where we have taken

the maximum speed of a typical marcher $(v \approx .1.5 \ \text{m/s})$, the speed of sound

$u = 343 \ \text{m/s}$ at 20°C

Bonus Challenge (not on the final exam)

The human ear perceives pitch changes in cents

$$n = 1200 \cdot \log_2\left(\frac{f_O}{f_S}\right) \approx 7.6 \text{ cents}$$

However, this assumes maximum speed

Duplicating the calculation assuming the marcher is moving at 0.5 m/s (a more relaxed pace), the shift is roughly 2.5 cents flat (moving away) or sharp (moving toward)