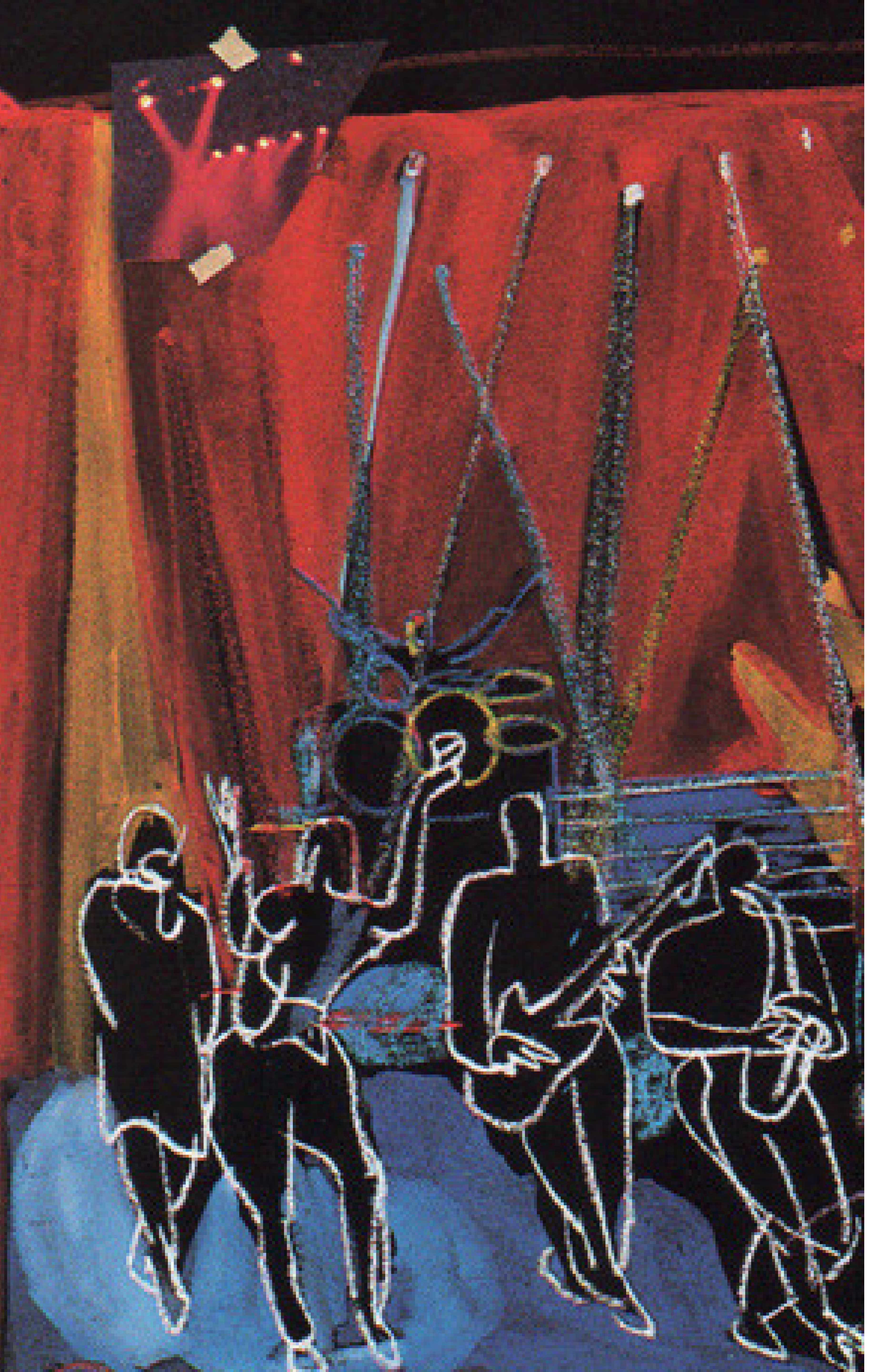


PHY-141

Logarithm Rules

Luis Anchordoqui

Outline

- Acoustics utilizes logarithmic scales (decibels (dB) for intensity and octaves for frequency) to match the human ear's logarithmic perception of sound

- Decibels measure amplitude, where a $10\ \mathrm{dB}$ increase feels twice as loud, while octaves group frequencies by doubling, splitting sound into manageable, musically relevant, and often logarithmic octave bands

- In today's class we briefly detour into the mathematical rules governing logarithms

Setting the Stage: Exponents

- While this lecture covers the logarithm rules, we will start with a review of exponents

- Because logarithms and exponents are closely linked, understanding the latter is necessary for mastering the former

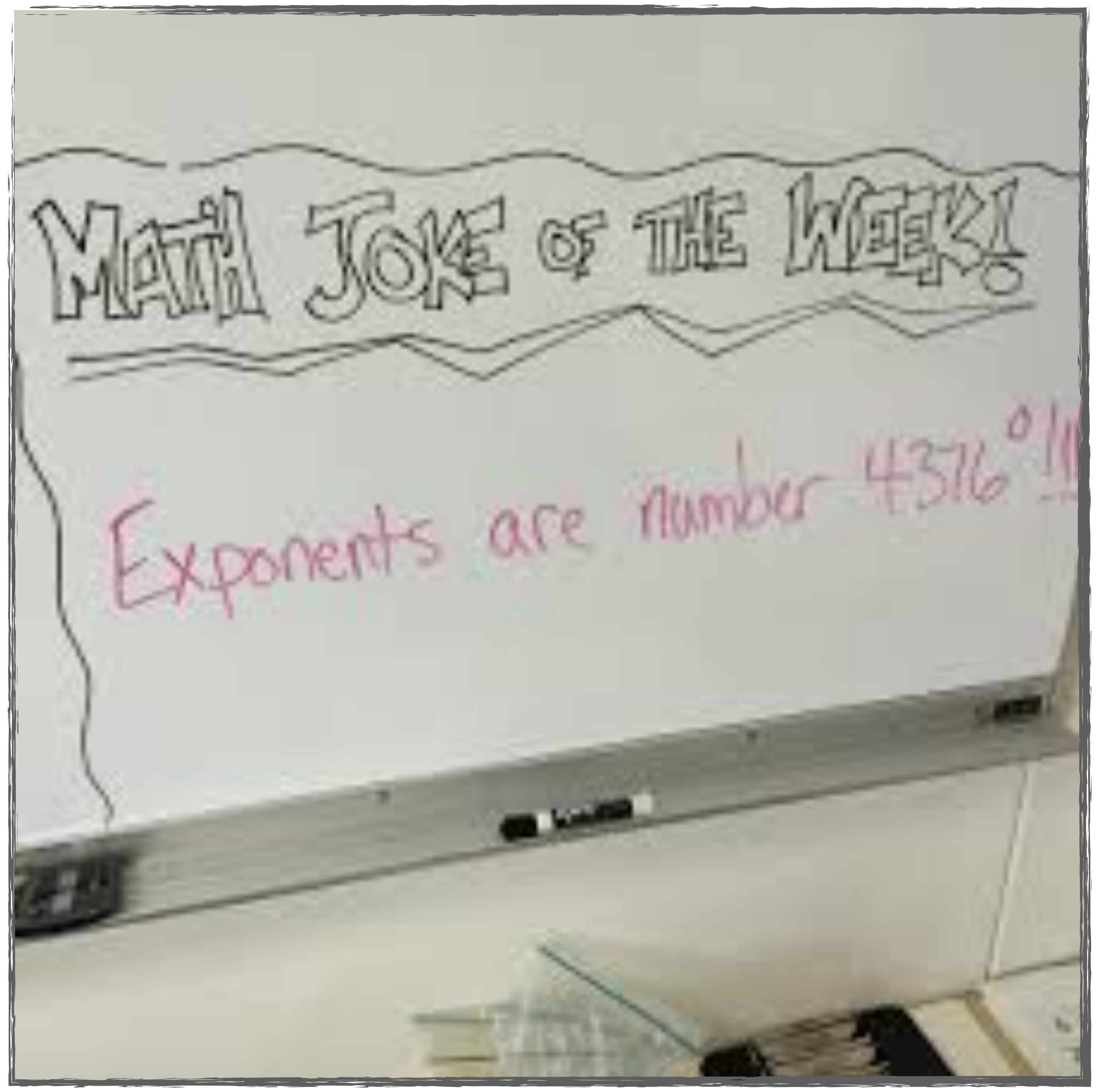
- In an expression like 2^4, the number 4 is the exponent (or power) whereas the number 2 is the base

- For example, in $81 = 9^2$, the number 2 is the exponent and 9 is the base

"Yeah, they've got their hidden agendas, but we've got our hidden logarithms."

Exponents (cont'd)

➢ To motivate the study of logarithms, consider the relationship between exponents and multiplication



➢ We know that $16 = 2^4$ and $8 = 2^3$

➢ If we wish to multiply 16 and 8, the standard method is direct long-multiplication, which yields 128

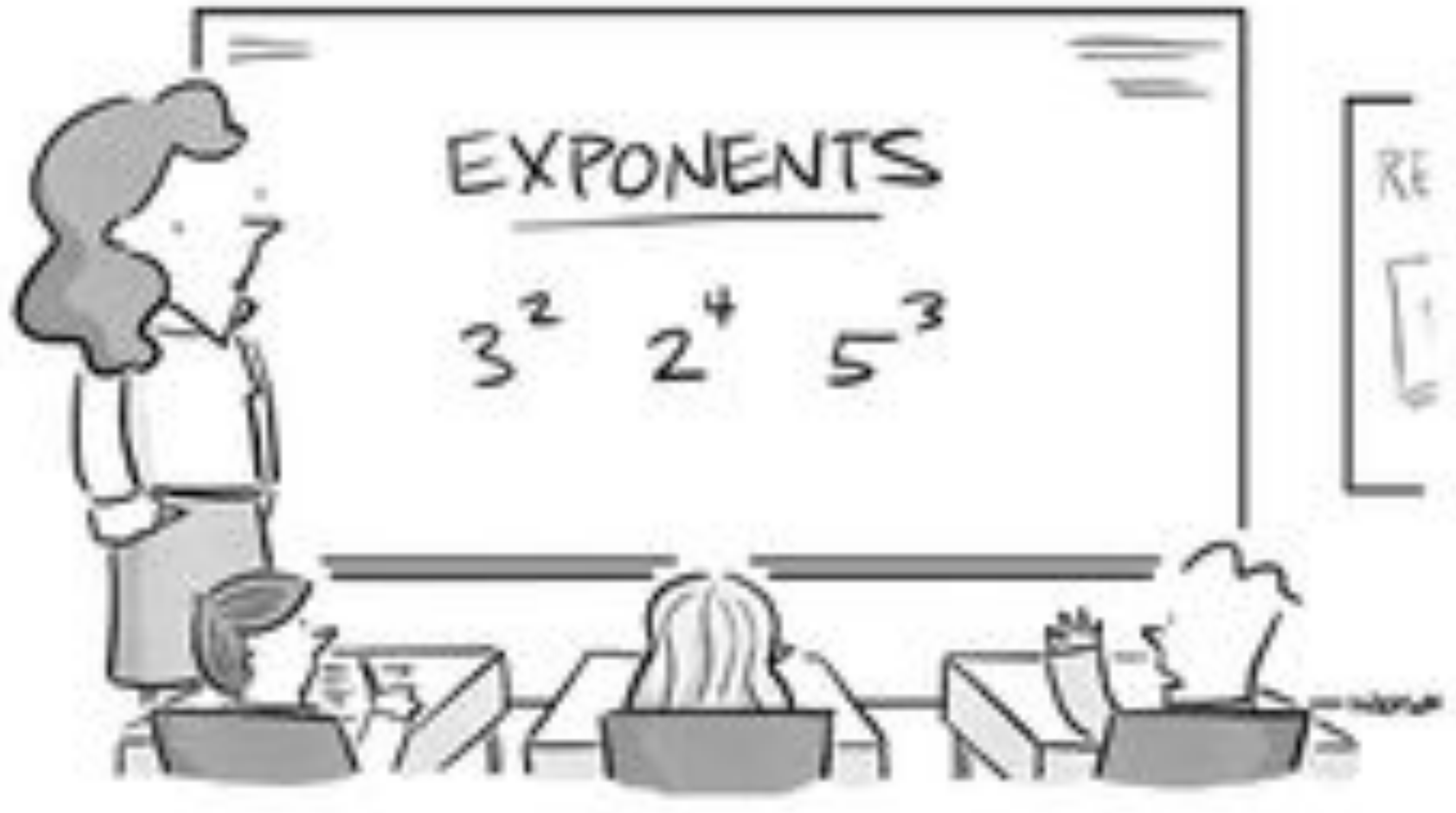
➢ However, this process becomes increasingly tedious as numbers grow larger

Exponents (cont'd)

- Alternatively, let's work out this calculation more efficiently by utilizing the properties of powers
- First, express each number as a power of a common base
- In this case, both 16 and 8 are powers of 2 ☛ $16 = 2^4$ and $8 = 2^3$
- Instead of multiplying the decimal values, multiply the exponential expressions
- According to the product rule for exponents, $a^m \cdot a^n = a^{m+n}$
- By adding the exponents, we simplify the operation ☛ $16 \cdot 8 = 2^4 \cdot 2^3 = 2^{4+3} = 2^7$
- Finally, convert the resulting power back into a standard integer
- Since $2^7 = 128$, we arrive at the same product without performing traditional long-multiplication

Exponents (cont'd)

- In a similar manner, dividing 16 by 8 can be expressed as $2^4/2^3$
- Following the laws of indices, we subtract the exponents $(4-3)$ to find the resulting power of 1
- This gives 2^1, or simply 2
- By utilizing a lookup table of the powers of 2, we could easily determine that $2^7 = 128$, which allows us to find the product of $16 \cdot 8$ through simple addition of their exponents $(4+3=7)$
- This fundamental insight inspired Napier and Briggs to invent logarithms, providing a method to simplify complex calculations by substituting multiplication with addition and division with subtraction



"Why did they quit being ponents?"

Understanding Logarithms Intuitively

- The exponential equation $16 = 2^4$ can be rewritten in an equivalent logarithmic form as $\log_2 16 = 4$ which is read as: log base 2 of 16 equals 4

- In this conversion, the base of the exponent (2) becomes the base of the logarithm, while the exponent (4) equals the result of the logarithm

- Because these forms are equivalent, they imply one another

☛ $16 = 2^4 \Leftrightarrow \log_2 16 = 4$

Logarithms Explained: A Simple Definition

- The two sets of statements, one using exponents and the other logarithms, are equivalent

- Generally, we *define the logarithm as*

  ☛ If $x = a^n$ then equivalently $\log_a x = n$

- Note that for any base a we have

  $a = a^1$ and so $\log_a a = 1$

  🪵😄(😄) = 1

THERE ARE

4 ($\log_{36} 6$)

TYPES OF PEOPLE IN THIS WORLD:

THOSE WHO UNDERSTAND LOGARITHMS AND THOSE WHO DON' T

First Law of Logarithms

- ➢ The *first law of logarithms* states that the logarithm of a product is equivalent to the sum of the logarithms of the individual factors
- ➢ To understand this law assume ☛ $x = a^n$ and $y = a^m$ that rearranges to the following logarithmic forms ☛ $\log_a x = n \quad \text{and} \quad \log_a y = m$
- ➢ Using the rule of exponents ☛ $xy = a^n \cdot a^m = a^{n+m}$
- ➢ Converting the expression $xy = a^{n+m}$ into its logarithmic equivalent yields $\log_a(xy) = n + m$
- ➢ By substituting the original definition s$n = \log_a x$ and $m = \log_a y$, we arrive at identity

☛ $\log_a(xy) = \log_a x + \log_a y$ 🪵(😘) = 🪵(😙) + 🪵(❤️)

- ➢ Multiplying two numbers and taking the log equals adding the logs of those numbers

Second Law of Logarithms

- The *second law of logarithms* states that the logarithm of a number raised to an exponent is evaluated by multiplying the exponent by the logarithm of the base number

- Assume $x = a^n$, or equivalently $\log_a x = n$

- Now, raise both sides of $x = a^n$ to the power m, $x^m = (a^n)^m$

- Using the rules of exponents we can write $x^m = (a^n)^m$ as $x^m = a^{nm}$

- Taking the quantity x^m as a single term, the logarithmic form is ☛ $\log_a x^m = nm = m \log_a x$

🪵(😅) = 💧🪵(😄)

Third Law of Logarithms

➢ The *third law of logarithms* states that the logarithm of a quotient is equal to the difference between the logarithms of the numerator and the denominator

➢ Assuming again that the relations $x = a^n$ and $y = a^m$ that rearranges to the following logarithmic forms $\log_a x = n$ and $\log_a y = m$ hold, we can write using the rules of exponents

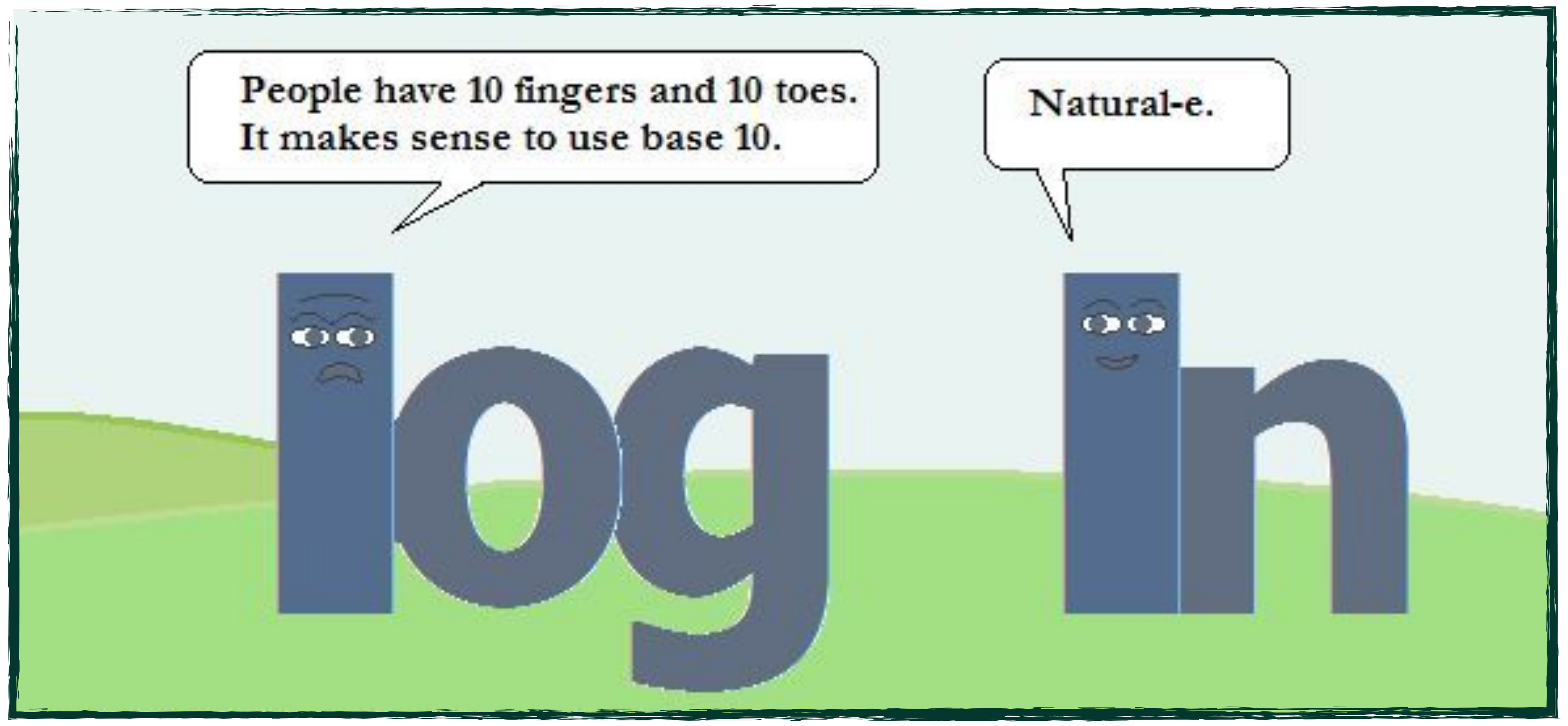
$$\frac{x}{y} = \frac{a^n}{a^m} = a^{n-m}$$

or equivalently, in logarithmic form ☛ $\log_a \frac{x}{y} = n - m$

which can be rewritten as ☛ $\log_a \frac{x}{y} = \log_a x - \log_a y$ 🪵(⛈️) = 🪵(☁️) - 🪵(⚡)

Log of One and Natural Log

- Recall that any non-zero base raised to the power of zero equals, i.e. $a^0 = 1$
- In logarithmic form, this identity is expressed as ☛ $\log_a 1 = 0$
- The logarithm of 1 in any base is 0
- The natural logarithm $(\ln x)$ is a logarithm with the base e, where is an irrational number approximately equal to 2.71828



Logarithm Rules: Summary

🪵(😘) = 🪵(😗) + 🪵(❤️)

🪵(🌩️) = 🪵(☁️) - 🪵(⚡)

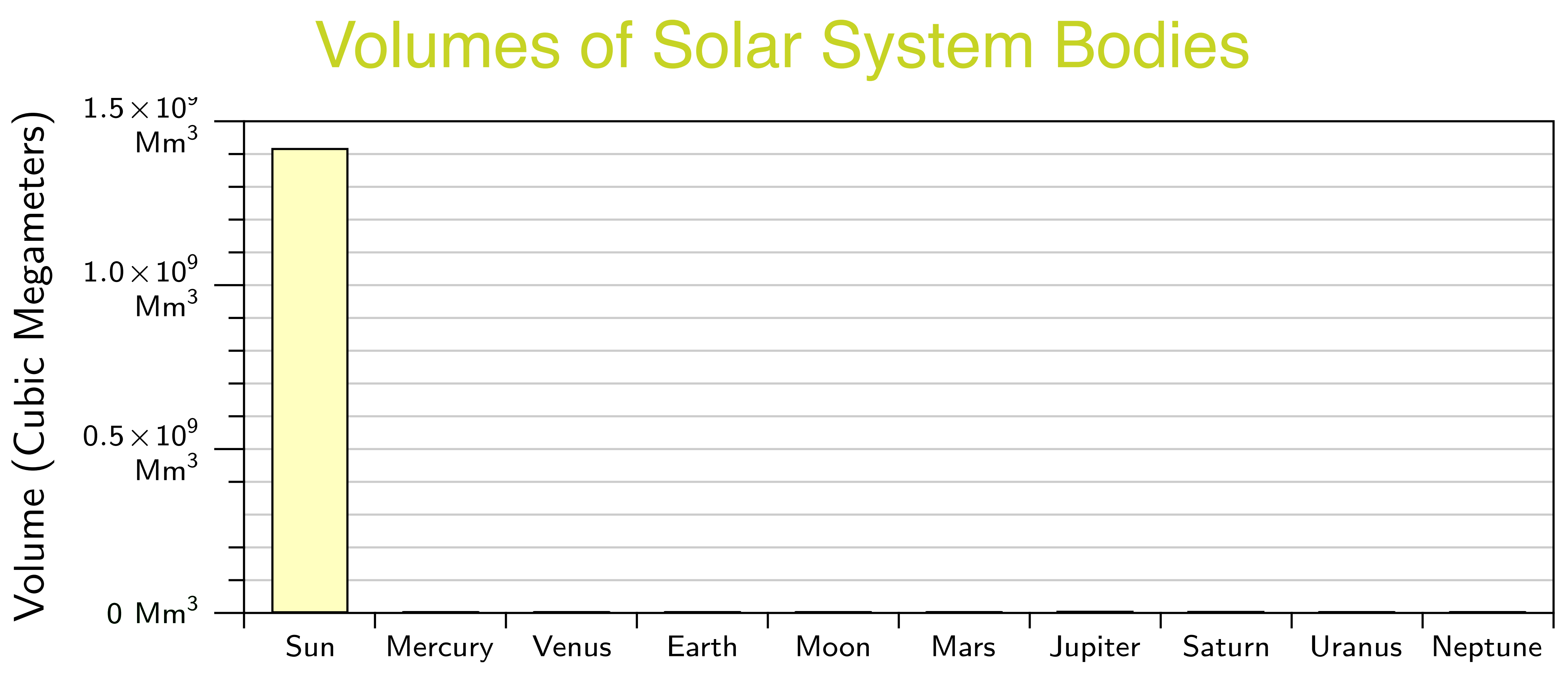
🪵(😅) = 💧🪵(😁)

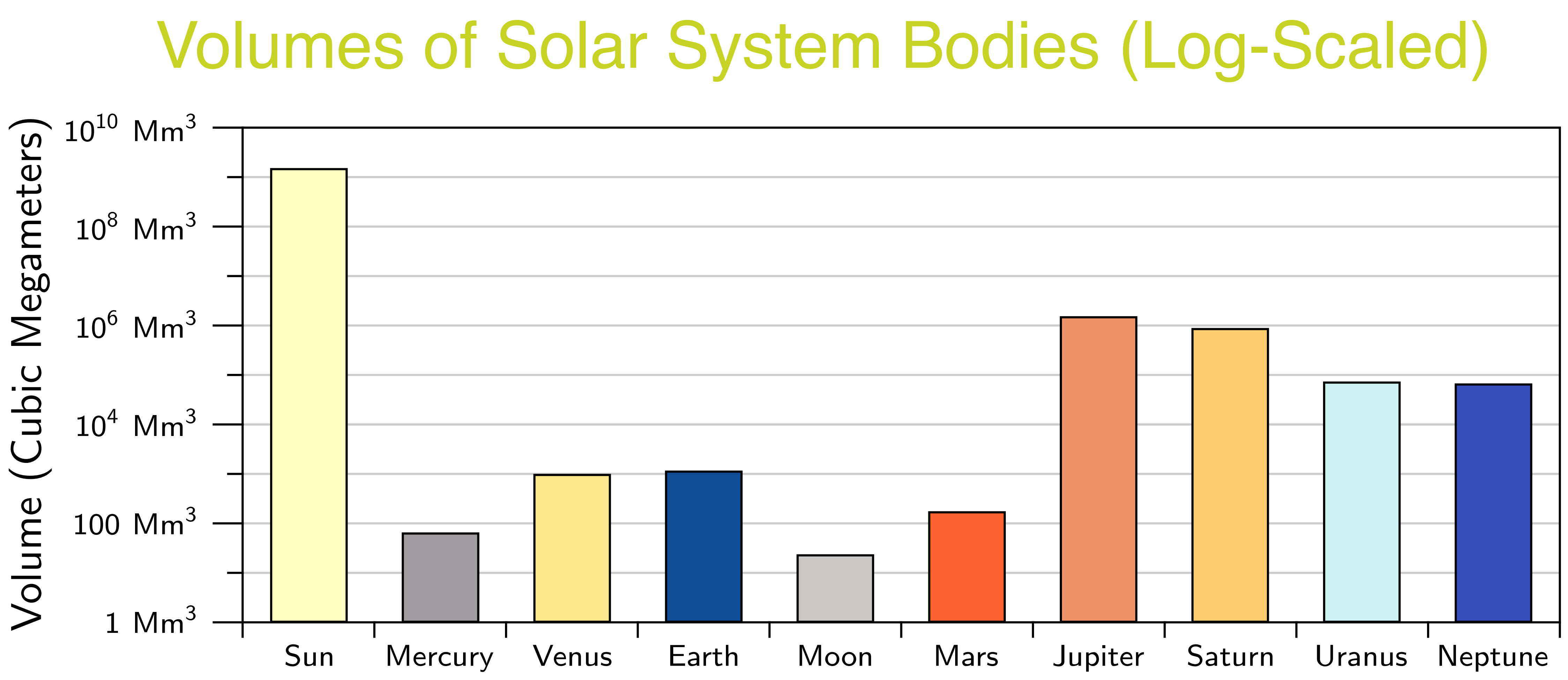
$🪵_{😁}$(😁) = 1

🪵(1) = 0

The Power of Logarithms: Simplifying Complexity and Scaling Data

- Logarithms are incredibly useful for comparing, at a glance, items that vary greatly in size
- For example, if we wanted to graph the volumes of major solar system bodies using cubic megameters (Mm^3), a useful unit roughly equivalent to the distance from Madrid to Paris, we would face massive scaling issues
- Logarithms allow us to visualize these vast differences clearly

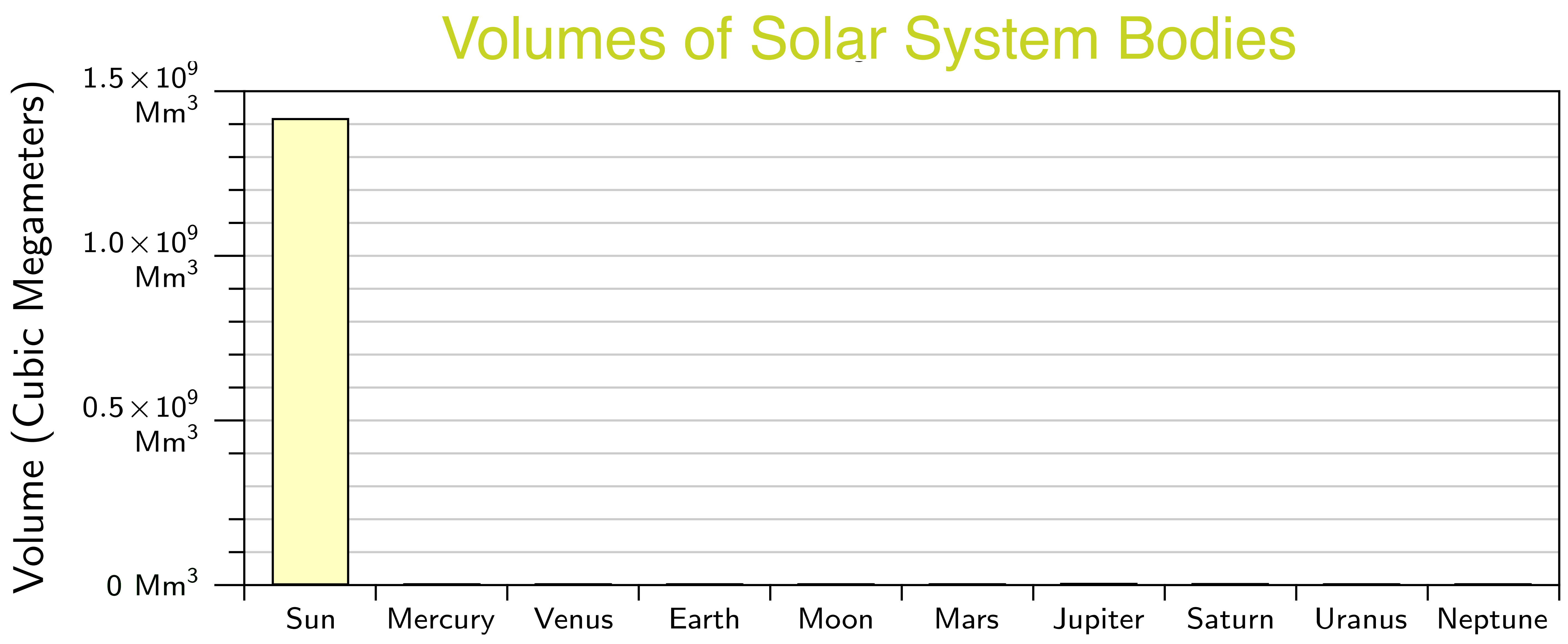




Tales from Interplanetary Space

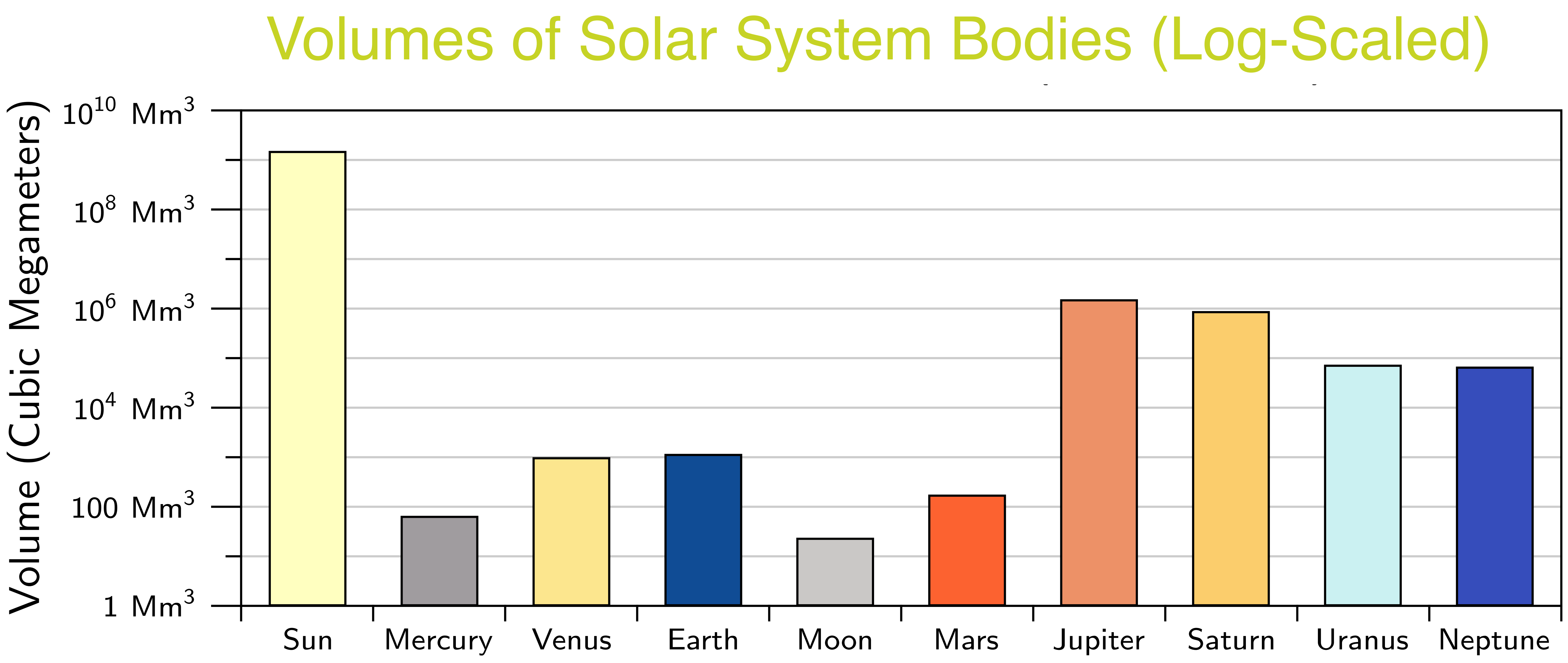
- The traditional bar graph in the figure below illustrates a common data visualization issue: when one value (the Sun) is vastly larger than the others, it becomes impossible to distinguish the sizes of the remaining objects

- We cannot easily compare e.g., Saturn to Neptune; the graph only highlights that the Sun is huge, which is not particularly informative



Tales from Interplanetary Space

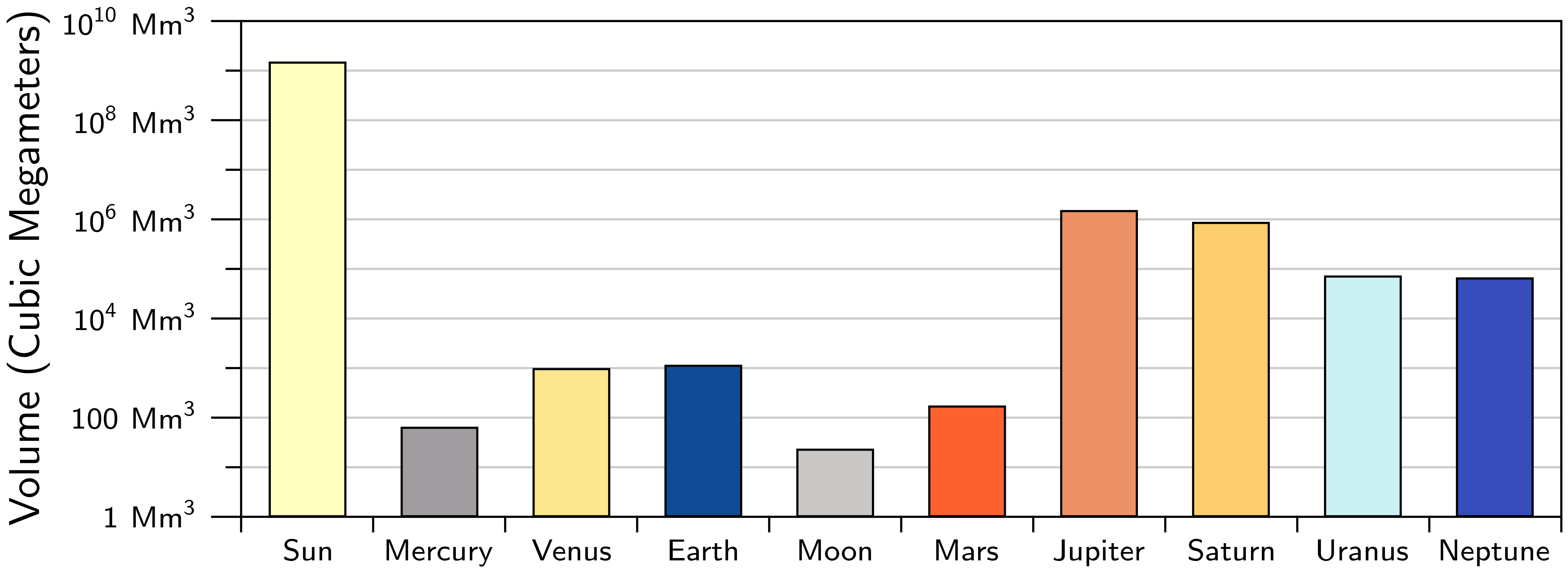
- By using a logarithmic scale, where each vertical notch represents a 10-fold increase rather than a linear step, we gain much better insight
- This method allows us to visualize data spanning several orders of magnitude
- In the figure below we show the sizes of the same objects but the vertical axis is now in $\log_{10}$ scale



Tales from Interplanetary Space

- Although the Sun remains the largest, we can now compare the planets effectively
- Jupiter and Saturn are roughly a thousand times smaller than the Sun
- Uranus and Neptune are ten times smaller than Jupiter/Saturn
- Earth is approximately a million times smaller than the Sun, a distinction hidden in the figure of the previous slide

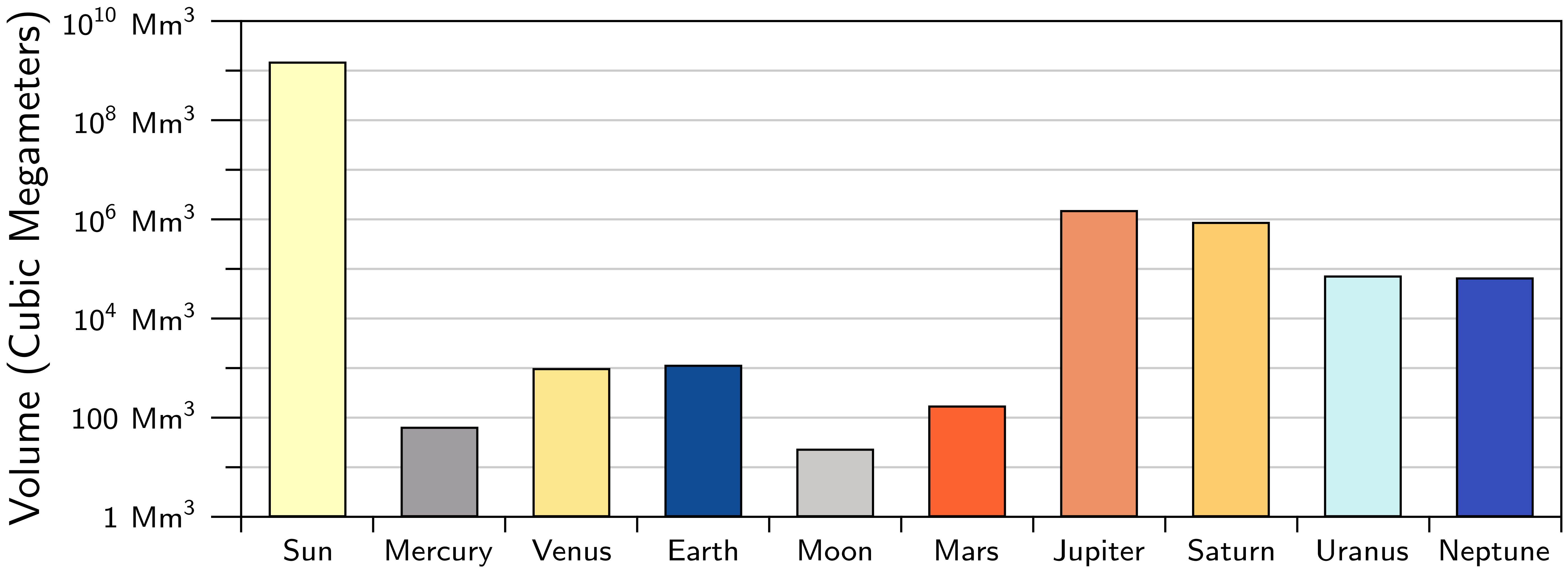
Volumes of Solar System Bodies (Log-Scaled)



A Word of Caution

- Log-scaled graphs are powerful tools, but they can be deceptive
- Jupiter A casual glance might suggest Earth is one-sixth the size of the Sun because its bar appears to be one-sixth the height, but that is incorrect; it is actually six whole orders of magnitude smaller
- Logarithmic scales should be used when the data spans massive ranges, providing clarity that linear scales cannot

Volumes of Solar System Bodies (Log-Scaled)



If $\log_2 p = 4$ and $\log_2 g = 2$, what is $\log_2(p/g)$?

A. 2

B. 4

C. 8

D. 16

If $\log_2 p = 4$ **and** $\log_2 g = 2$**, what is** $\log_2(p/g)$**?**

A. 2

B. 4

C. 8

D. 16

Evaluate: $\log_{10} 1000 + \log_{10} 0.1$

A. 0

B. 1

C. 2

D. 3

Evaluate: $\log_{10} 1000 + \log_{10} 0.1$

A. 0

B. 1

C. 2

D. 3

What is the logarithm of 1 to any base?

A. 0

B. 1

C. depends on the base

D. infinite

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A. 0

B. 1

C. depends on the base

D. infinite

If $\log_2 m = 5$ and $\log_2 n = 2$, what is $\log_2(m/n)$?

A. 3

B. 4

C. 5

D. 6

If $\log_2 m = 5$ and $\log_2 n = 2$, what is $\log_2(m/n)$?

A. 3

B. 4

C. 5

D. 6

If $\log_2 a = 3$ **and** $\log_2 b = 4$, **what is** $\log_2(ab)$ **?**

A. 12

B. 34

C. 64

D. 7

If $\log_2 a = 3$ and $\log_2 b = 4$, what is $\log_2(ab)$?

A. 12

B. 34

C. 64

D. 7

How does the logarithm of a product relate to the sum of individual logarithms?

A. $\log_b(xy) = \log_b x + \log_b y$

B. $\log_b(xy) = \log_b x - \log_b y$

C. $\log_b(xy) = \log_b x \cdot \log_b y$

D. $\log_b(xy) = \log_b x / \log_b y$

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D. $\log_b(xy) = \log_b x / \log_b y$

What is the logarithmic identity for the logarithm o the reciprocal of a number?

A. $\log_b(1/a) = \log_b a$

B. $\log_b(1/a) = 0$

C. $\log_b(1/a) = -\log_b a$

D. $\log_b(1/a) = 1$

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D. $\log_b(1/a) = 1$

Evaluate: $\log_5 25$

A. 1

B. 2

C. 3

D. 5

Evaluate: $\log_5 25$

A. 1

B. 2

C. 3

D. 5