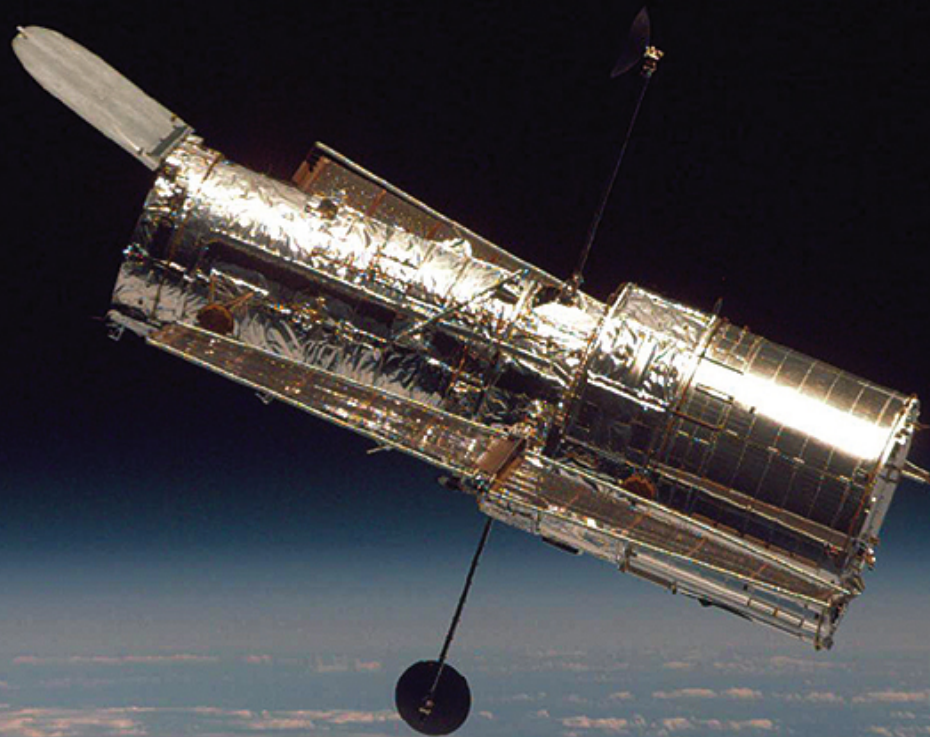


Conservation Theorems: Angular Momentum

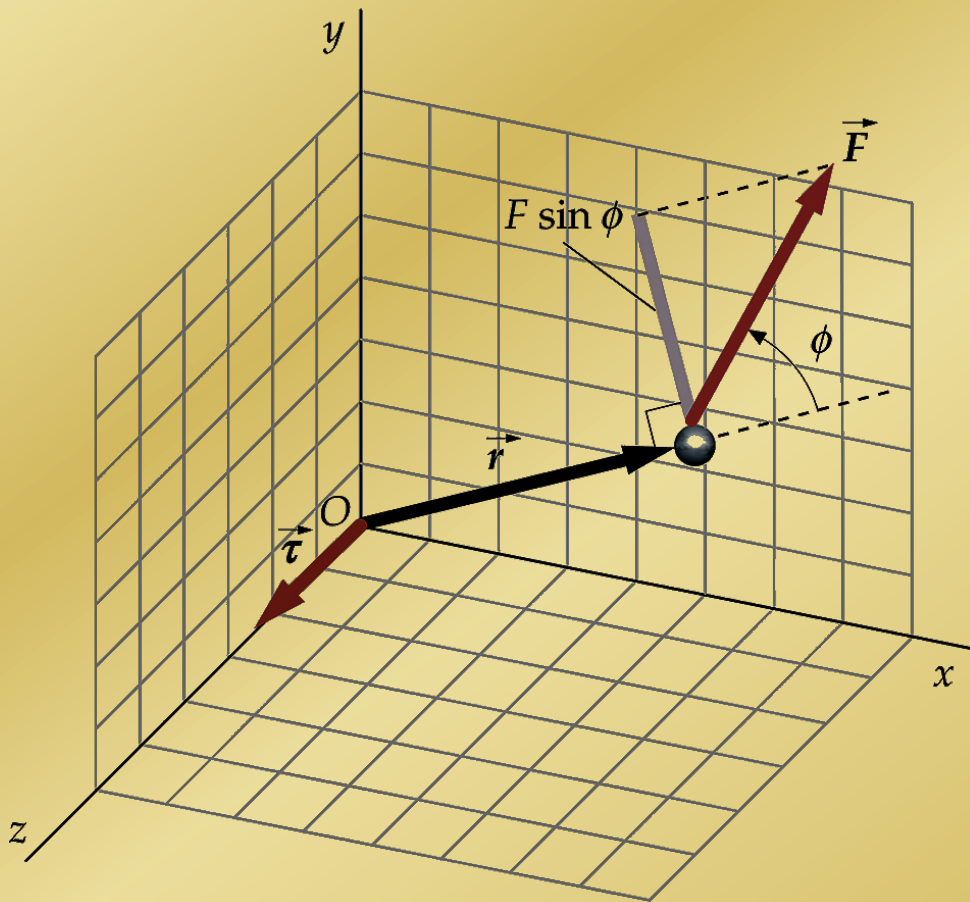


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Vector Nature of Rotation

The torque is expressed mathematically as a vector product of $\vec{r}$ and $\vec{F}$

$$\vec{\tau} = \vec{r} \times \vec{F}$$



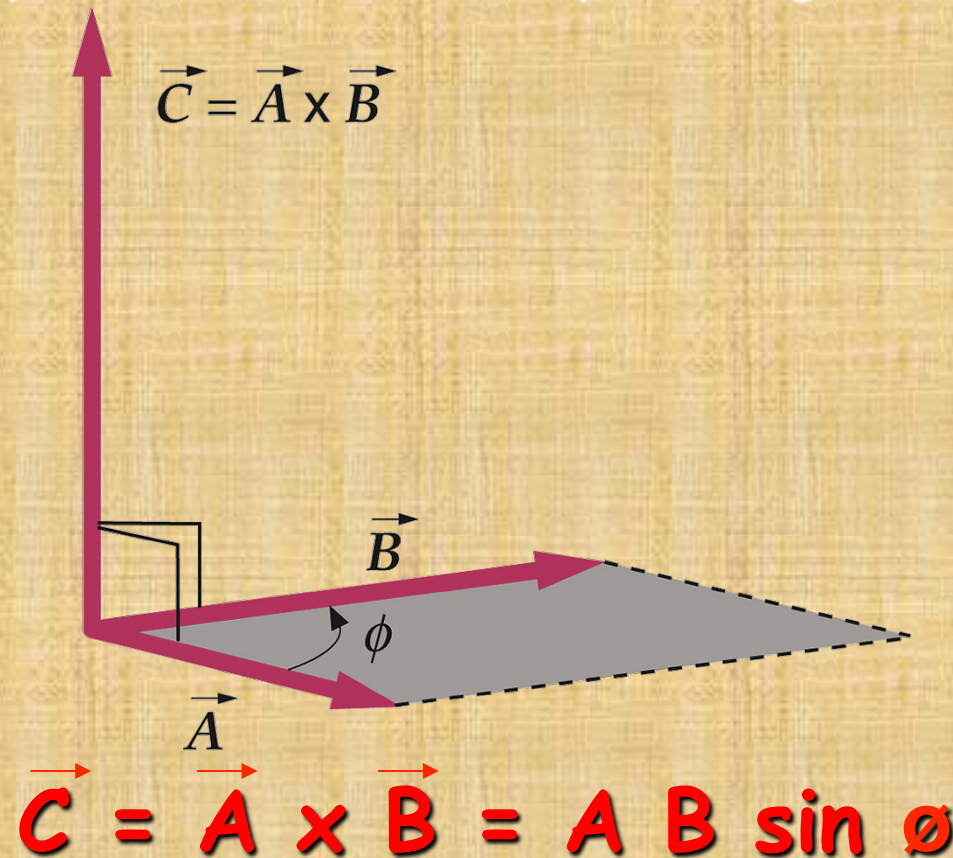
If $\vec{F}$ and $\vec{r}$ are both perpendicular to the z axis

$\vec{\tau}$ is parallel to the z axis

Vector product

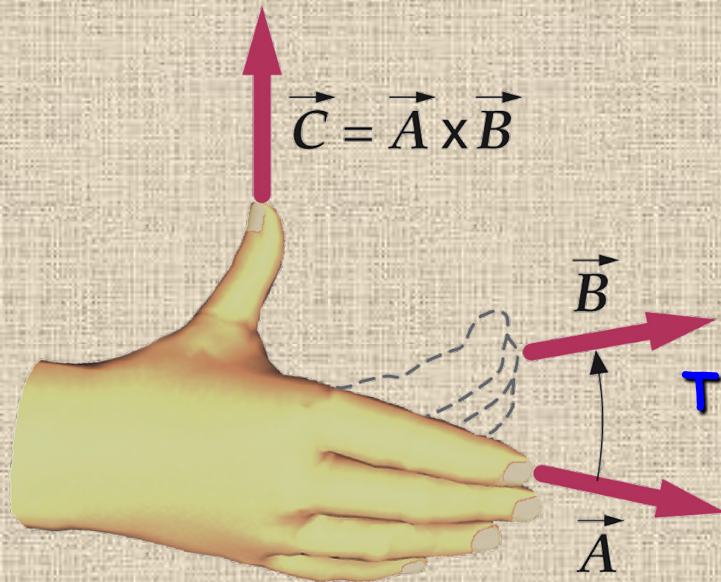
The vector product of two vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ that is perpendicular to both A and B and has a magnitude $|\vec{C}| = |\vec{A}||\vec{B}| \sin \phi$

$|\vec{C}|$ equals the area of the parallelogram shown

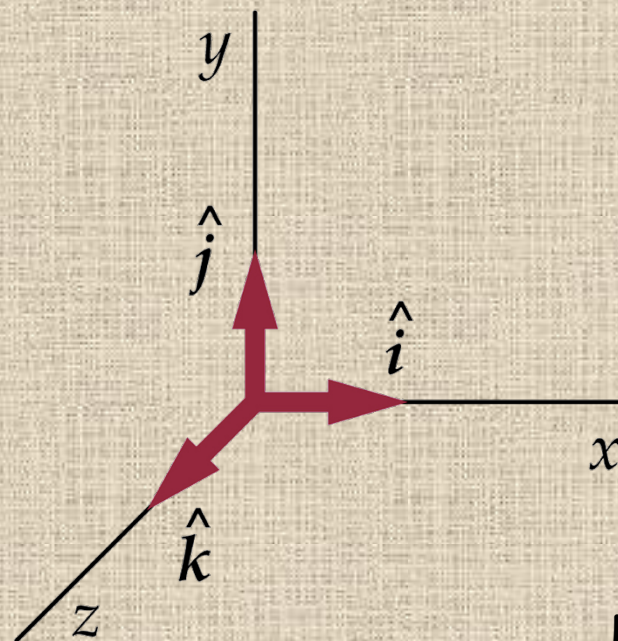


Vector product (cont'd)

The direction of $\vec{A} \times \vec{B}$ is given by the right-hand rule when the fingers are rotated from the direction of $\vec{A}$ toward $\vec{B}$ through an angle θ



This defines a right-handed cartesian system

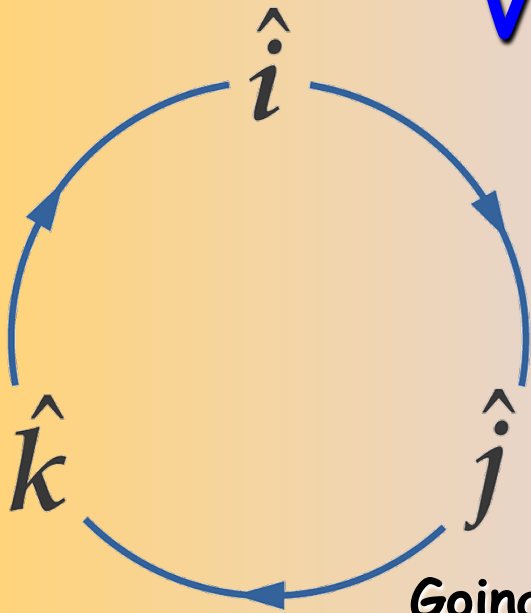


Vector product (cont'd)

If we take the vector product by going around the figure in the direction of the arrows (clockwise)

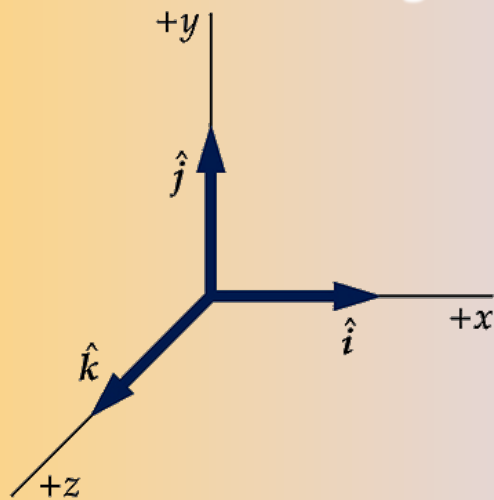
the sign is positive

$$\hat{i} \times \hat{j} = \hat{k}$$

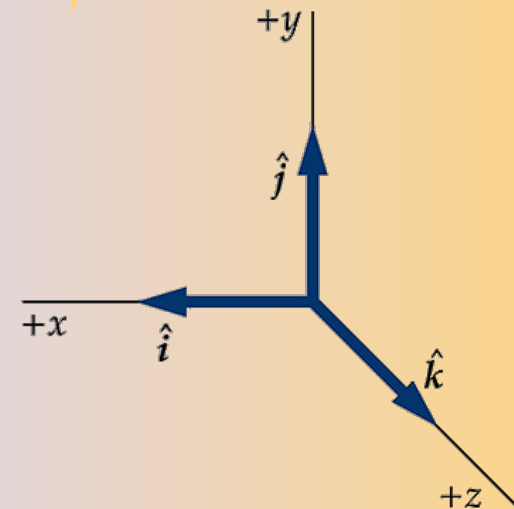


Going around against the arrows $\Rightarrow$ the sign is negative

$$\hat{i} \times \hat{k} = -\hat{j}$$



Right-handed system ($\hat{i} \times \hat{j} = \hat{k}$)



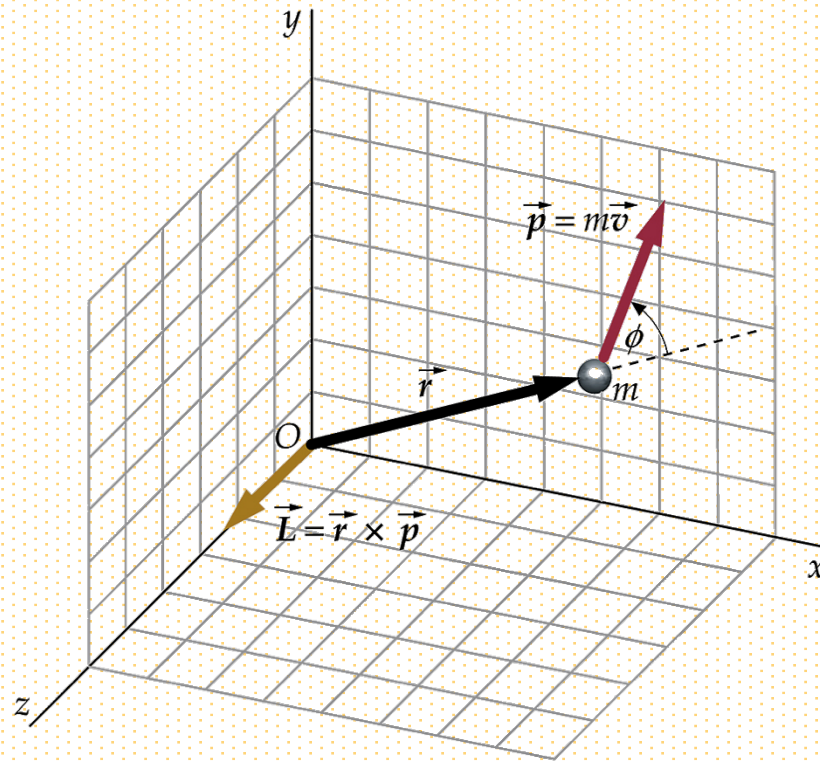
Left-handed system ($\hat{i} \times \hat{j} \neq \hat{k}$)

Throughout this course we adopt right handed coordinate systems

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Angular momentum

The angular momentum $\vec{L}$ of the particle relative to the origin O is defined to be the vector product of $\vec{r}$ and $\vec{p}$

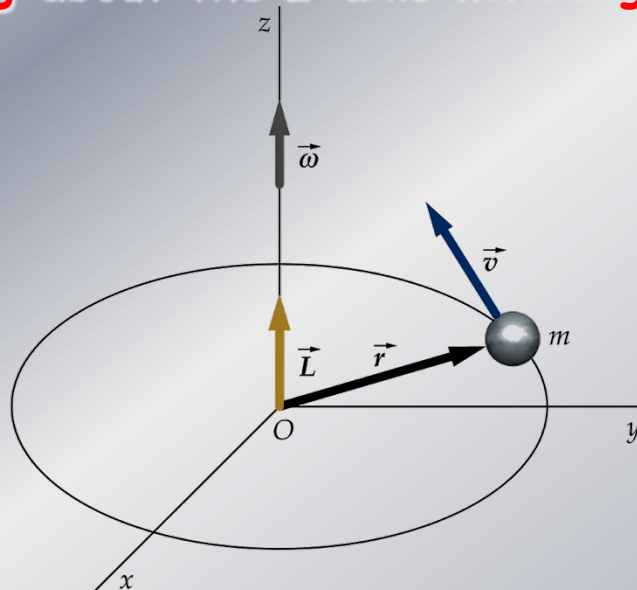


$$\vec{L} = \vec{r} \times \vec{p}$$

Angular momentum (cont' d)

The figure shows a particle of mass m attached to a circular disk of negligible mass moving in a circle in the xy plane with its center at the origin

The disk is spinning about the z -axis with angular speed ω



$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v} = r m v \hat{k} = m r^2 \omega \hat{k} = m r^2 \vec{\omega}$$

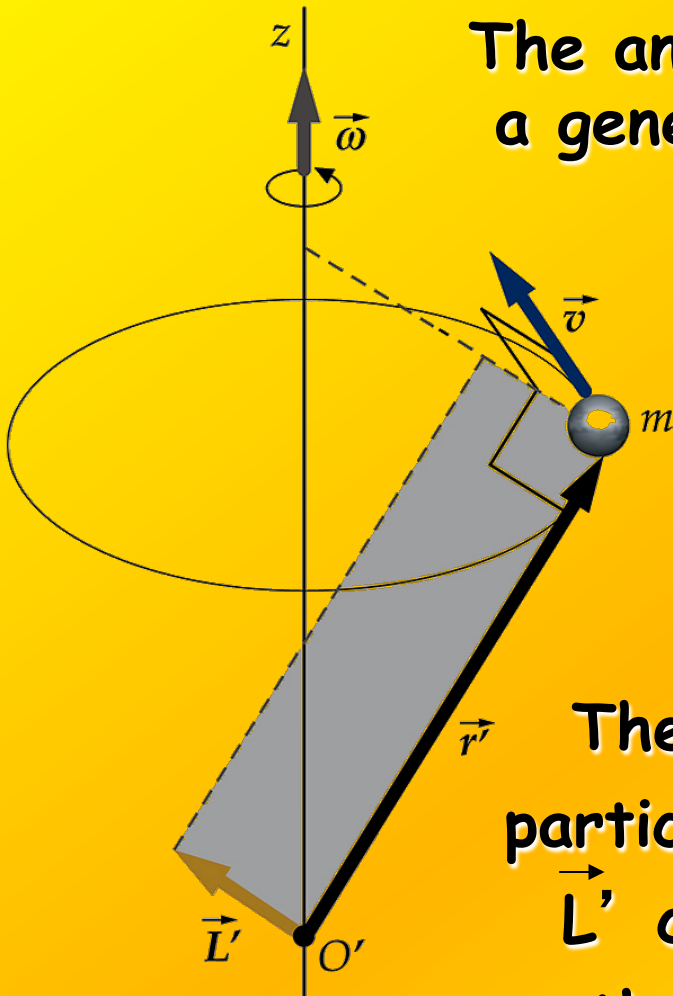
The angular momentum is in the same direction as the angular velocity vector.

Because mr^2 is the moment of inertia for a single particle we have

$$\vec{L} = I\vec{\omega}$$

Angular momentum (cont' d)

The angular momentum of this particle about a general point on the z axis is not parallel to the angular velocity vector

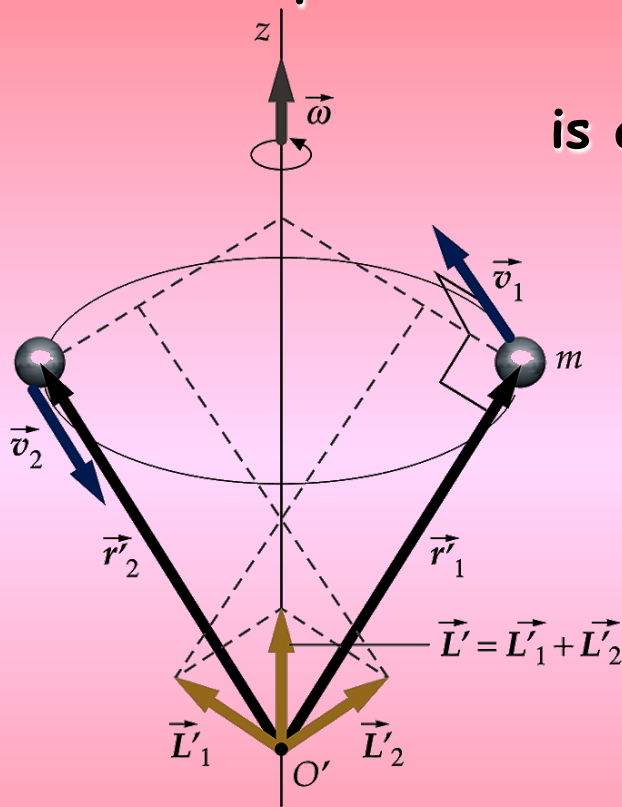


The angular momentum $\vec{L}'$ for the same particle attached to the same disk but with $\vec{L}'$ computed about a point on the z axis that is not at the center of the circle

Angular momentum (cont' d)

We now attach a second particle of equal mass to the spinning disk at a point diametrically opposite to the first particle

The total angular momentum $\vec{L}' = \vec{L}'_1 + \vec{L}'_2$ is again parallel to the angular velocity vector $\vec{\omega}$



In this case the axis of rotation passes through the center of mass of the two-particle system and the mass distribution is symmetric about this axis

Such an axis is called a symmetry axis

For any system of particles that rotates about a symmetry axis the total angular momentum (which is the sum of the angular momenta of the individual particles) is parallel to the angular velocity

$$\vec{L} = I\vec{\omega}$$

Conservation of angular momentum

☛ The angular momentum of a particle
(with respect to an origin from which the position vector $\vec{r}$ is measured) is

$$\vec{L} = \vec{r} \times \vec{p}$$

☛ The torque (or moment of force) with respect to the same origin is

$$\vec{\tau} = \vec{r} \times \vec{F}$$



Position vector from the origin to the point where the force is applied

$$\vec{\tau} = \vec{r} \times \dot{\vec{p}}$$



$$\dot{\vec{L}} = \frac{d}{dt} (\vec{r} \times \vec{p}) = (\dot{\vec{r}} \times \vec{p}) + (\vec{r} \times \dot{\vec{p}})$$

$$\text{But of course } \Rightarrow \dot{\vec{r}} \times \vec{p} = \dot{\vec{r}} \times m\vec{v} = m (\dot{\vec{r}} \times \dot{\vec{r}}) = 0$$

$$\dot{\vec{L}} = \vec{r} \times \dot{\vec{p}} = \vec{\tau}$$

If $\vec{\tau} = 0 \Rightarrow \dot{\vec{L}} = 0 \Rightarrow \vec{L}$ is a vector constant in time

If the net external torque acting on a system about some point is zero, the total angular momentum of the system about that point remains constant.

A figure skater can increase his spin rotation rate from an initial rate of 1 rev every 2 s to a final rate of 3 rev/s.

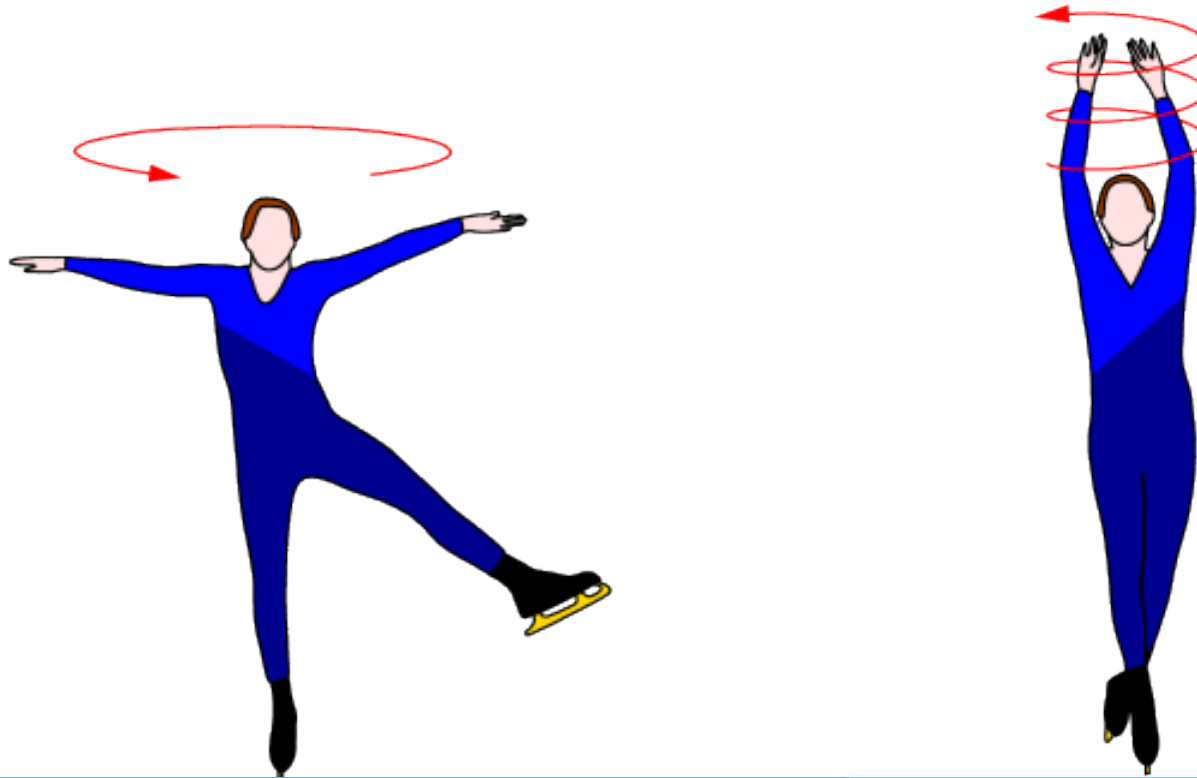
If his initial moment of inertia was 4.6 kg m^2 what is his final moment of inertia? How does he physically accomplish this change?



$$I_f = 0.77 \text{ kg m}^2$$

A figure skater can increase his spin rotation rate from an initial rate of 1 rev every 2 s to a final rate of 3 rev/s.

If his initial moment of inertia was 4.6 kg m^2 what is his final moment of inertia? How does he physically accomplish this change?



$$I_f = 0.77 \text{ kg m}^2$$



(a) What is the angular momentum of a figure skater spinning at 3.5 rev/s with arms in close to her body, assuming her to be a uniform cylinder with height of 1.5 m, radius of 15 cm, and mass of 55 kg?

$$L = 14 \text{ kg m}^2/\text{s}$$

(b) How much torque is required to slow her to a stop in 5 s, assuming she does not move her arms.

$$\tau = -2.7 \text{ mN}$$

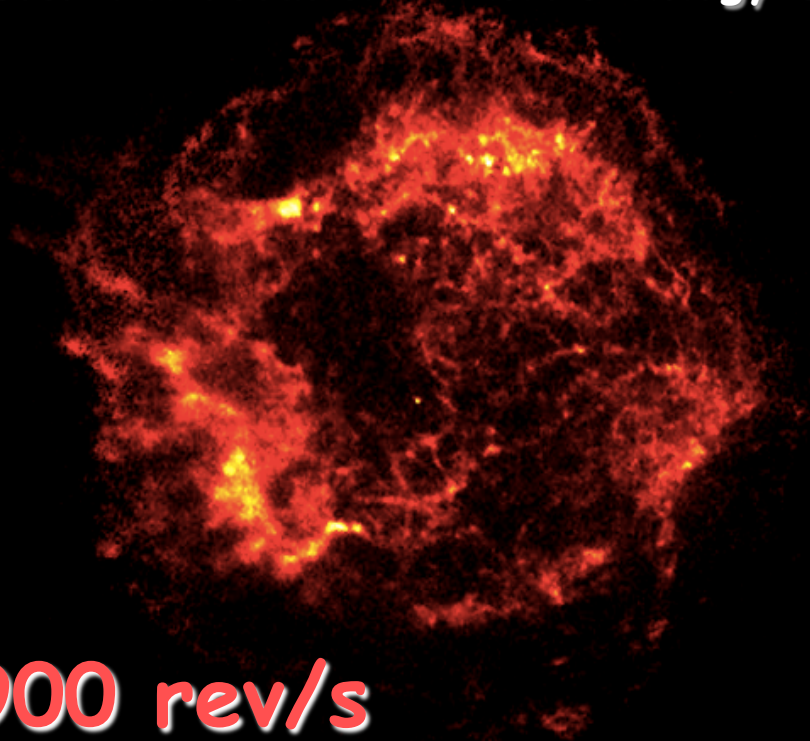
Jerry running away from Tom jumps on the outside edge of a freely turning ceiling fan of moment of inertia I and radius R .

If m is the mass of Jerry, by what ratio does the angular velocity change?



(a) Use conservation of angular momentum to estimate the angular velocity of a neutron star which has collapsed to a diameter of 20 km, from a star whose radius was equal to that of the Sun (7×10^8 m), of mass $1.5 M_{\odot}$ and which rotated like our Sun once a month.

(b) By what factor the rotational kinetic energy change after the collapse?



$$\frac{k_f}{k_i} = 5 \times 10^9$$

$$\omega_f = 1900 \text{ rev/s}$$

The bright dot in the middle is believed to be the hot young neutron star the result of a supernova explosion from about 300 years ago.

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Angular Momentum of a System of Particles

Newton's second law for angular motion

The net external torque about a fixed point acting on a system equals the rate of change of the angular momentum of the system about the same point

$$\vec{\tau}_{\text{net, ext}} = \frac{d\vec{L}_{\text{sys}}}{dt}$$

Angular impulse

$$\Delta\vec{L}_{\text{sys}} = \int_{t_i}^{t_f} \vec{\tau}_{\text{net, ext}} dt$$

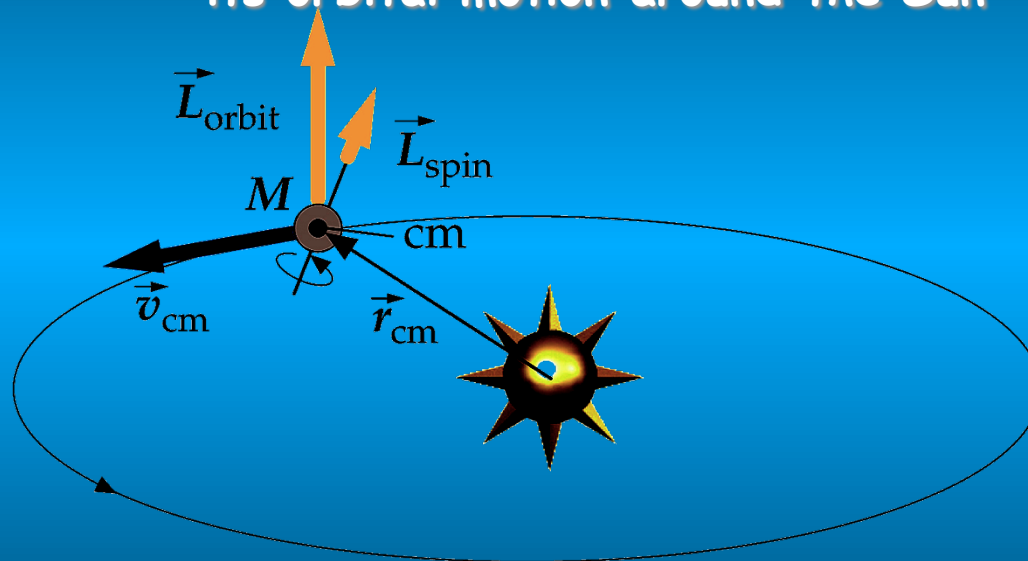
It is often useful to split the total angular momentum of a system about an arbitrary point O into orbital angular momentum and spin angular momentum

$$\vec{L}_{\text{sys}} = \vec{L}_{\text{orbit}} + \vec{L}_{\text{spin}}$$

Angular Momentum of a System of Particles

Newton's second law for angular motion (cont'd)

Earth has spin angular momentum due to its spinning motion about its rotational axis and it has orbital angular momentum about the center of the Sun due to its orbital motion around the Sun



$$\vec{L}_{\text{orbit}} = \vec{r}_{\text{cm}} \times M_{\oplus} \vec{v}_{\text{cm}} = M_{\oplus} r_{\text{cm}}^2 \omega_{\text{yearly}} = 2.7 \times 10^{40} \text{ kg m}^2/\text{s}$$

Assuming the Earth is a uniform sphere

$$\vec{L}_{\text{spin}} = \vec{I}_{\oplus} \vec{\omega}_{\text{daily}} = \frac{2}{5} M_{\oplus} R_{\oplus}^2 \omega_{\text{daily}} = 7.1 \times 10^{33} \text{ kg m}^2/\text{s}$$

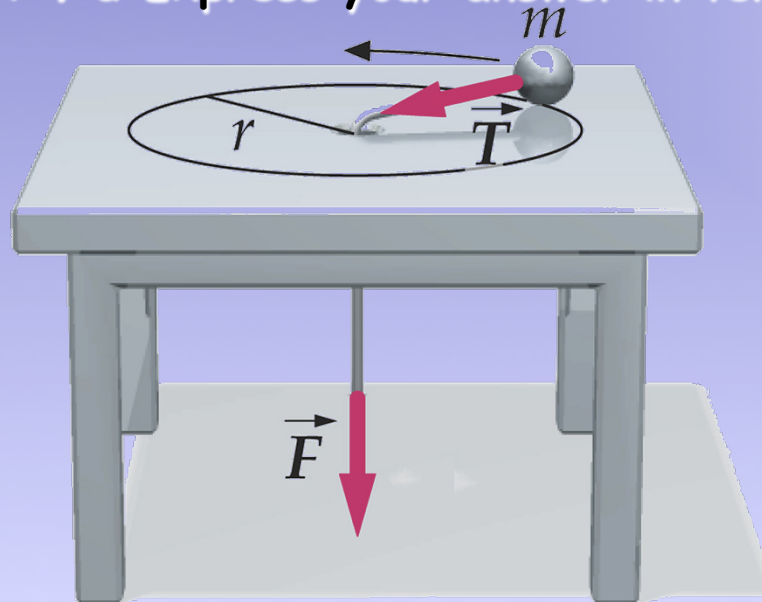
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Pulling Through a Hole

A particle of mass m moves with speed v_0 in a circle with radius r_0 on a frictionless table top. The particle is attached to a string that passes through a hole in the table as shown in the figure

The string is slowly pulled downward until the particle is a distance r from the hole, after which the particle moves in a circle of radius r

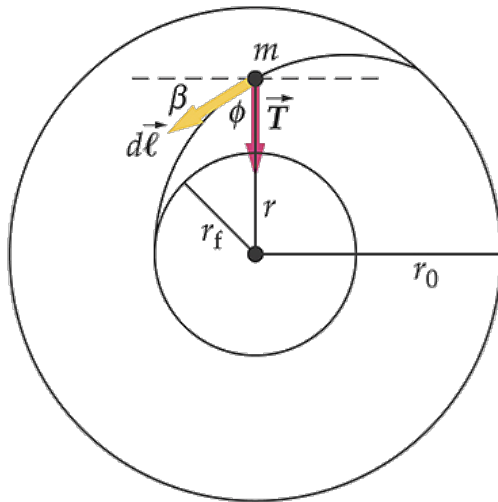
- Find the final velocity in terms of r_0 , v_0 , and r
- Find the tension when the particle is moving in a circle of radius r in terms of m , r , and the angular momentum $\vec{L}$
- Calculate the work done on the particle by the tension force $\vec{T}$ by integrating $\vec{T} \cdot d\vec{r}$. d Express your answer in terms of r and L_0



Pulling Through a Hole (cont' d)

$$L_f = L_0 \Rightarrow v_f = \frac{r_0 v_0}{r_f}$$

Because the particle is being pulled in slowly the acceleration is virtually the same as if the particle were moving in a circle



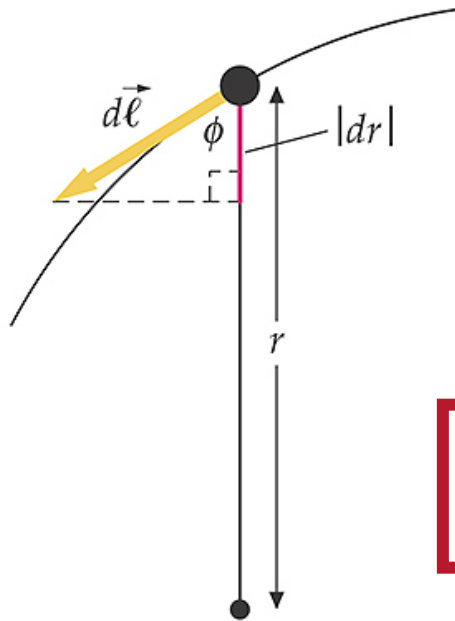
$$T \approx m \frac{v^2}{r}$$

$$\vec{L} = \vec{r} \times \vec{p} = r m v \cos \beta \approx r m v$$

$$\beta \ll 1 \rightarrow \cos \beta \approx 1$$

$$T = m \frac{v^2}{r} = \frac{m}{r} \left(\frac{L}{m r} \right)^2 = \frac{L^2}{m r^3}$$

Pulling Through a Hole (cont' d)



$$dr = -|dr|$$

$$dW = \vec{T} \cdot d\vec{\ell} = T d\ell \cos \phi$$

$$|dr| = d\ell \cos \phi \Rightarrow dW = T |dr| = -T dr$$

$$W = - \int_{r_0}^{r_f} T dr = - \frac{L^2}{m} \int_{r_0}^{r_f} r^{-3} dr = \frac{L^2}{2m} \left(\frac{1}{r_f^2} - \frac{1}{r_0^2} \right)$$