Lecture 8

1. Simplex method

The *simplex method* is an algorithm to find an optimal solution to LPs, consisting of steps in which we move from one feasible basis $B \subset \{1, \ldots, n\}$ to another $B' = B \cup \{i\} \setminus \{j\}$ by adding an *entering basic variable* x_i and removing a *departing basic variable* x_j, in such a way that we improve the value of the target function $c^T x$. Recall that $B \subset \{1, \ldots, n\}$, $|B| = m$, is a *feasible basis* for an LP with feasible set in equational form

$$S = \{x \in \mathbb{R}^n : Ax = b,\, x \geq 0\},$$

where A is an $m \times n$ matrix, if A_B is invertible and the unique $x_B \in \mathbb{R}^n$ such that $Ax_B = b$ and $x_j = 0$ for all $j \notin B$ satisfies $x_B \geq 0$. Each feasible basis B corresponds to a vertex of S, so the above corresponds to moving from one vertex to another.

As an example, let us consider

$$A = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \quad c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

so that the LP is given by

$$\begin{aligned} \min \quad -x_1 - x_2 \quad \text{s.t.} \quad & x_1 + 2x_2 + x_3 = 4 \\ & 3x_1 + x_2 + x_4 = 6 \\ & x_1 + x_5 = 2 \\ & x \geq 0 \end{aligned}$$

An "obvious" feasible basis is $B = \{3, 4, 5\}$, with $x_B = (0, 0, 4, 6, 2)$. We represent this with the following *tableau*:

(1)

	x_1	x_2	x_3	x_4	x_5	
x_3	1	2	1	0	0	4
x_4	3	1	0	1	0	6
x_5	1	0	0	0	1	2
z	1	1	0	0	0	0

The bottom row indicates that the target function $z = -x_1 - x_2$ takes value 0 at the basic feasible solution $(0, 0, 4, 6, 2)$, which is written as $z + x_1 + x_2 = 0$.

In order to select an *entering variable*, we pick the smallest index $i \notin B$ such that the coefficient in the x_i column of the tableau is ≥ 0; in this case, the entering variable is x_1.

Exercise 1. Show that by increasing x_1 from 0 to a positive value, we will decrease the target function.

To select the *departing variable*, we see how large can we make x_1 while staying in the feasible set. Namely, let us look at the constraints $x_i \geq 0$, $i \in B$, in terms of x_1 and see which would be violated first if we increase x_1 from 0 to a positive quantity (keeping in mind that $x_2 = 0$ since x_2 is not entering as a basic variable):

$$\begin{aligned} 0 &\leq x_3 = 4 - x_1 - 2x_2 \\ 0 &\leq x_4 = 6 - 3x_1 - x_2 \\ 0 &\leq x_5 = 2 - x_1 \end{aligned}$$

Since $x_i \geq 0$, we see from the above inequalities that $x_1 \leq 4$, $x_1 \leq 2$ and $x_1 \leq 2$, respectively. These quantities are sometimes called *θ-ratios*, and are obtained dividing the coefficients on the last column of the tableau by the coefficients in the column of the entering variable:

$$\theta(x_3) = 4, \quad \theta(x_4) = 2, \quad \theta(x_5) = 2.$$

The choice of departing variable is now arbitrary between x_4 and x_5, since their vanishing happens before x_3 vanishes, i.e., they have the smallest θ-ratio $\theta(x_4) = \theta(x_5) = 2$.

Let us proceed selecting x_4, so the new feasible basis is $B' = \{1, 3, 5\}$. The corresponding tableau is obtained by row operations, so that the columns of x_i with $i \in B$ are the columns of an $m \times m$ identity matrix:

(2)

	x_1	x_2	x_3	x_4	x_5	
x_1	1	$\frac{1}{3}$	0	$\frac{1}{3}$	0	2
x_3	0	$\frac{5}{3}$	1	$-\frac{1}{3}$	0	2
x_5	0	$-\frac{1}{3}$	0	$-\frac{1}{3}$	1	0
z	0	$\frac{2}{3}$	0	$-\frac{1}{3}$	0	-2

Note that the value of the target function decreased from $z = 0$ to $z = -2$ at this new basic feasible solution $x = (2, 0, 2, 0, 0)$. However, we have not yet arrived at the minimum, since the coefficient of x_2 is still positive. So we take x_2 to be the new entering variable!

To select a departing variable, let us again check that the constraints $x_i \geq 0$ remain satisfied. From the last tableau, we have:

$$0 \leq x_1 = 2 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_4$$
$$0 \leq x_3 = 2 - \tfrac{5}{3}x_2 + \tfrac{1}{3}x_4$$
$$0 \leq x_5 = \tfrac{1}{3}x_2 + \tfrac{1}{3}x_4$$

Keeping in mind that $x_4 = 0$ since x_4 is not entering as a basic variable, the first inequality gives $x_2 \leq 6$ and the second gives $x_2 \leq \frac{6}{5}$, while the last does not yield any constraint on x_2. In other words, we only consider the θ-ratios that are ≥ 0 and discard those of the form $0/a$ if $a < 0$, such as $0/(-\frac{1}{3})$, since they do not yield any constraints:

$$\theta(x_1) = 6, \quad \theta(x_3) = \tfrac{6}{5}.$$

Thus, we set x_3 as the departing variable, enforcing the strongest constraint on how large x_2 can be when we increase it from 0 to a positive quantity (i.e., it has the smallest θ-ratio).

Performing row operations, we arrive at the tableau for the feasible basis $B'' = \{1, 2, 5\}$,

(3)

	x_1	x_2	x_3	x_4	x_5	
x_1	1	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$
x_2	0	1	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{6}{5}$
x_5	0	0	$\frac{1}{5}$	$-\frac{2}{5}$	1	$\frac{2}{5}$
z	0	0	$-\frac{2}{5}$	$-\frac{1}{5}$	0	$-\frac{14}{5}$

The above tableau corresponds to an optimal solution $x = (\frac{8}{5}, \frac{6}{5}, 0, 0, \frac{2}{5})$ of the LP, where the target function achieves its minimum $z = -\frac{14}{5}$. Indeed, there are no positive entries in the target row. In other words, by the above tableau, the target function is given by:

$$z = -\tfrac{14}{5} + \tfrac{2}{5}x_3 + \tfrac{1}{5}x_4,$$

so increasing either x_3 or x_4 from the current value 0 would increase the value of the target function, of which we are seeking the minimum. We have thus found that minimum!