Lecture 3

1. Linear programs in 2 variables

A *linear program (LP)* in 2 variables $x = (x_1, x_2)$ is an optimization problem of the form

$$\begin{aligned} \min \quad c_1x_1 + c_2x_2 \quad \text{s.t.} \quad & a_{11}\,x_1 + a_{12}\,x_2 \le b_1, \\ & a_{21}\,x_1 + a_{22}\,x_2 \le b_2, \\ & \dots \\ & a_{m1}\,x_1 + a_{m2}\,x_2 \le b_m, \\ & x_1 \ge 0,\ x_2 \ge 0. \end{aligned}$$

Note that the above LP can be rewritten in matrix notation as

$$\begin{aligned} \min \quad c^T\,x \quad \text{s.t.} \quad & A\,x \le b, \\ & x \ge 0, \end{aligned} \tag{1}$$

where $x = (x_1, x_2)$, inequalities such as $v \le w$ between vectors are defined to mean the coordinate-wise inequalities between the corresponding entries $v_i \le w_i$, and $A = (a_{ij})$ is an $m \times 2$ matrix, $b = (b_i) \in \mathbb{R}^m$, where the indices have ranges $1 \le i \le m$ and $1 \le j \le 2$.

Recall that $(\cdot)^T$ denotes the *transpose* of a vector, or of a matrix. In particular, $x^T y$ is nothing but the dot product of the vectors x and y, also often written $x \cdot y$ or $\langle x, y\rangle$.

Exercise 1. Find the matrix A and vectors b, c so that the problem from Lecture 1:

$$\begin{aligned} \min \quad 1.00\,x_1 + 1.20\,x_2 \quad \text{s.t.} \quad & 8x_1 + 6x_2 \ge 11, \\ & 4x_1 + 12x_2 \ge 16, \\ & x_1 \ge 0,\ x_2 \ge 0, \end{aligned}$$

can be written in the above form (1). (Remember that $u \le v$ if and only if $-u \ge -v$.)

Exercise 2. How can you relate the *maximization* problem

$$\begin{aligned} \max \quad c^T\,x \quad \text{s.t.} \quad & A\,x \le b, \\ & x \ge 0, \end{aligned}$$

to (1)? We shall also refer to an optimization problem as above as a linear program (LP).

The set of points $x \in \mathbb{R}^2$ that satisfy the *constraints* $A\,x \le b$ and $x \ge 0$ is called the *feasible region*, and points in the feasible region are called *feasible solutions*. A feasible solution is called an *optimal solution* if it achieves the min/max of the *target function* $c^T\,x$. The feasible region of an LP in 2 variables is an intersection of finitely many half spaces $a_{i1}x_1 + a_{i2}x_2 \le b_i$ hence it is a convex polygon in $\mathbb{R}^2$. Note that it may be empty (problem is *infeasible*) or noncompact.

Exercise 3. Give examples of LPs with the following feasible regions: (a) the empty set, (b) the first quadrant, (c) the horizontal strip $[0, +\infty) \times [0, 1]$, (d) the square $[0, 1] \times [0, 1]$, (e) the triangle with vertices $(0, 0)$, $(2, 0)$ and $(0, 3)$.

2. Brute force solution and geometric solution

In order to solve an LP in 2 variables, one may follow a very direct geometric method:

(i) Identify the feasible region $S \subset \mathbb{R}^2$, which is a polygon: more precisely, find the coordinates of all vertices (extremal points) of S;
(ii) Compute the target function at all vertices;
(iii) Order the results; the *smallest* value is the min.

As we shall see later, this is a rather crude and brue force approach, which would be extremely slow in larger problems. However, it is a first step in our journey to solving LPs.

Exercise 4. Implement the above strategy in the following LP:

$$\begin{aligned} \max \quad 4\,x_1 - 2\,x_2 \quad \text{s.t.} \quad & 2x_1 + 4x_2 \leq 12 \\ & x_1 + x_2 \leq 5 \\ & x_2 \leq 5/2 \\ & x_1 - x_2 \leq 4 \\ & x_1 \geq 0,\ x_2 \geq 0. \end{aligned}$$

What is the optimal solution? Repeat replacing max with min.

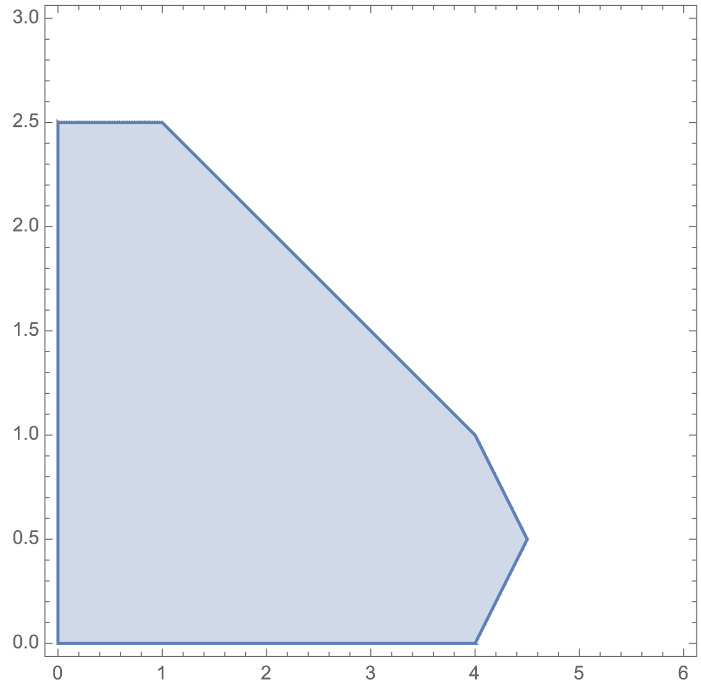
Exercise 5. Explain the procedure you used to find the vertices. How well would it scale if the number of sides of the polygon grows?

An improvement on the above is to find the optimal solution by considering levelsets

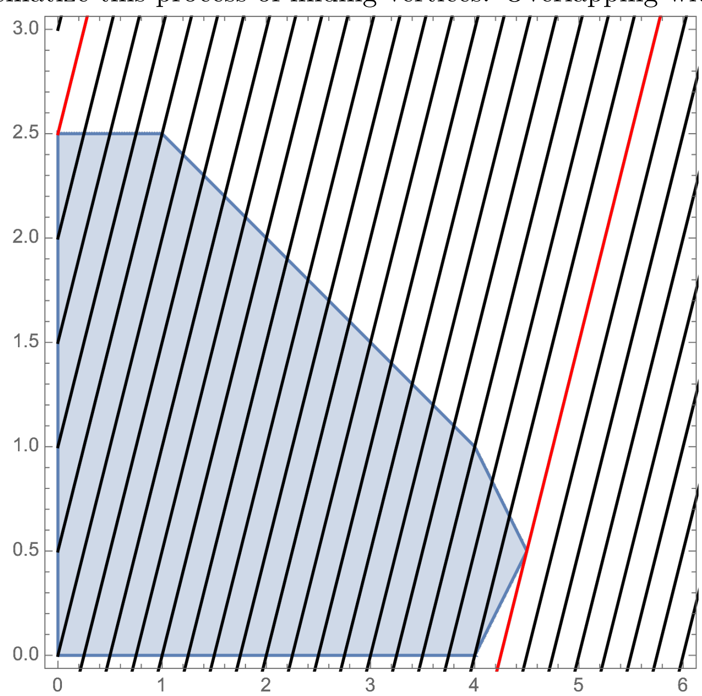
$$L_t = \{x \in \mathbb{R}^2 : c^T\, x = t\}$$

of the target function for varying $t \in \mathbb{R}$. The gradient of $x \mapsto c^T\, x$ is clearly the vector c^T, which is therefore orthogonal to the levelsets L_t. "Bringing in" levelset lines L_t from infinity until they touch the feasible region, we can geometrically identify which vertex is the optimal solution, and then find its coordiantes by solving the equations that correspond to the lines intersecting at that vertex.

Solution to Exercise 4. The feasible region is as follows:



Finding all pairwise intersections of the lines defining the boundary of the feasible region, whose equations are the constraints with $\leq$ replaced with $=$, and discarding those solutions that are not feasible, we conclude that the vertices are: $(4,1)$, $(1,5/2)$, $(9/2,1/2)$, $(0,5/2)$, $(4,0)$, $(0,0)$. We will see later how to systematize this process of finding vertices. Overlapping with levelsets, we find:



Thus, $\min c^T x = -5$ is achieved at $(0,5/2)$, and $\max c^T x = 17$ is achieved at $(9/2,1/2)$.