Lecture 23

1. Nonnegative v. sums of squares

We say that a polynomial $p \in \mathbb{R}[x]_{2d}$ in n variables $x_1, \dots, x_n$ of degree $\leq 2d$ is

i) *nonnegative* if $p(x_1, \dots, x_n) \geq 0$ for all $x \in \mathbb{R}^n$;

ii) a *sum of squares (sos)* if there exist polynomials $q_j \in \mathbb{R}[x]_d$ such that $p(x) = \sum_{j=1}^{k} q_j(x)^2$.

Clearly, every sos polynomial is nonnegative. Positive multiples and convex combinations of nonnegative polynomials (respectively, sos polynomials) are again nonnegative (respectively, sos), thus

$$P_{n,2d} := \{p \in \mathbb{R}[x]_{2d} : p \text{ is nonnegative}\}$$
$$\Sigma_{n,2d} := \{p \in \mathbb{R}[x]_{2d} : p \text{ is sos}\}$$

are convex cones in the vector space $\mathbb{R}[x]_{2d} \cong \mathbb{R}^N$, where $N = \binom{n+d}{d}$.

Exercise 1. Prove that $P_{n,2d}$ is a closed semialgebraic set. Find an explicit description of $P_{1,2}$ as a semialgebraic set.

Solution to Exercise 1. This is a consequence of Quantifier Elimination, by eliminating the quantifier $\forall x$ in $\forall x, p(x) \geq 0$. For example, for univariate quadratic polynomials, we have that

$$\begin{aligned} P_{1,2} &= \{p(x) = ax^2 + bx + c : \forall x, p(x) \geq 0\} \\ &= \{p(x) = ax^2 + bx + c : \min_{x \in \mathbb{R}} p(x) \geq 0\} \\ &= \{p(x) = ax^2 + bx + c : a > 0 \text{ and } p(-b/2a), \text{ or } a = b = 0 \text{ and } p(0) \geq 0\} \\ &= \{ax^2 + bx + c : a > 0 \text{ and } b^2 - 4ac \leq 0, \text{ or } a = b = 0 \text{ and } c \geq 0\} \\ &= \{ax^2 + bx + c : a \geq 0,\ c \geq 0,\ 4ac - b^2 \geq 0\}. \end{aligned}$$

Note that $P_{1,2}$ is actually *basic* semialgebraic, and it is also a spectrahedron:

$$\begin{aligned} P_{1,2} &= \left\{ ax^2 + bx + c : \forall x, \begin{pmatrix} 1 \\ x \end{pmatrix}^T \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} \geq 0 \right\} \\ &= \left\{ ax^2 + bx + c : \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix} \succeq 0 \right\} \end{aligned}$$

As it turns out, $P_{n,2d}$ is semialgebraic but it is not *basic* semialgebraic if $2d \geq 4$.[1] In particular, $P_{n,2d}$ is not a spectrahedron if $2d \geq 4$.

Since $\Sigma_{n,2d} \subset P_{n,2d}$, a natural question is whether the converse inclusion $\Sigma_{n,2d} \supset P_{n,2d}$ holds, i.e., if nonnegative polynomials are sos. Remarkably, this is almost never the case:

Theorem 1 (Hilbert, 1888)**.** *The only cases in which $\Sigma_{n,2d} = P_{n,2d}$ are:*

a) *Univariate polynomials, i.e., $n = 1$;*
b) *Quadratic polynomials, i.e., $2d = 2$;*
c) *Bivariate quartics, i.e., $n = 2$, $2d = 4$.*

An example of $p \in P_{2,6} \setminus \Sigma_{2,6}$ is the Motzkin polynomial $p(x, y) = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$.

Exercise 2. Use the arithmetic-geometric inequality applied to $\{x^4y^2, x^2y^4, 1\}$ to show that the Motzkin polynomial is nonnegative. Think how you could try to show it is not sos.

[1] For a more details, see p. 52 in [BPT13].

Solution to Exercise 2. By the arithmetic-geometric inequality, we have that

$$\frac{x^4y^2 + x^2y^4 + 1}{3} \geq \sqrt[3]{(x^4y^2)(x^2y^4)} = x^2y^2,$$

that is, $p(x, y) \geq 0$. If $p(x, y)$ was sos, then one can show it would be a sum of terms of the form $(ax^2y + bxy^2 + cxy + d)^2$ but no such term has a negative coefficient for x^2y^2.

Exercise 3. Use the computations in Exercise 1 to show that $\Sigma_{1,2} = P_{1,2}$. Think about how this could be generalized prove the equality in Theorem 1 b).

Solution to Exercise 3. Let us use the spectrahedral description

$$P_{1,2} = \left\{ ax^2 + bx + c : M(a, b, c) := \begin{pmatrix} c & b/2 \\ b/2 & a \end{pmatrix} \succeq 0 \right\}.$$

Since $M(a, b, c) \succeq 0$, there exists a matrix P such that $M(a, b, c) = P^T P$. Using, e.g., the Cholesky decomposition, we find that, with $a \geq 0,\ c \geq 0,\ 4ac - b^2 \geq 0$, assuming for simplicity $c > 0$,[2]

$$P = \begin{pmatrix} \sqrt{c} & \frac{b}{2\sqrt{c}} \\ 0 & \frac{\sqrt{4ac-b^2}}{2\sqrt{c}} \end{pmatrix}.$$

Thus, setting $v = \begin{pmatrix} 1 \\ x \end{pmatrix}$, we have

$$ax^2 + bx + c = v^T M(a, b, c)\, v = v^T (P^T P) v = (Pv)^T (Pv).$$

Since $Pv = \left(\frac{bx}{2\sqrt{c}} + \sqrt{c}, \frac{x\sqrt{4ac-b^2}}{2\sqrt{c}} \right)$, the above yields the following sos decomposition:

$$ax^2 + bx + c = \left(\frac{bx}{2\sqrt{c}} + \sqrt{c} \right)^2 + \left(\frac{x\sqrt{4ac - b^2}}{2\sqrt{c}} \right)^2.$$

Therefore, $P_{1,2} \subset \Sigma_{1,2}$ and hence $P_{1,2} = \Sigma_{1,2}$.

One can prove the equality in Theorem 1 b) using a similar reasoning, identifying quadratic polynomials in $P_{n,2}$ with $(n+1)\times(n+1)$ positive-semidefinite matrices operating on $v = (1, x_1, x_2, \ldots, x_n)$. In particular, $\Sigma_{n,2} = P_{n,2}$ is a spectrahedron.

More generally, for general d, it can be shown that $\Sigma_{n,2d}$ is a spectrahedral shadow[3] of dimension $\binom{n+2d}{2d}$, see [BPT13, Cor 3.40]. In particular, SDP can be used to test if $p \in \Sigma_{n,2d}$ and find polynomials $q_j \in \mathbb{R}[x]_d$ such that $p = \sum_j q_j^2$. Let us illustrate this with an example of $p \in \Sigma_{1,4}$.

Exercise 4. ([BPT13, Ex. 3.35]) Use the following steps to find an sos decomposition for the nonnegative polynomial $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 5$.

i) Let $v = (1, x, x^2)$ be the vector consisting of monomials of degree ≤ 2 and $M \in \mathrm{Sym}^2(\mathbb{R}^3)$ be a symmetric matrix. Determine affine-linear constraints on M equivalent to $p(x) = v^T M v$, by matching coefficients.
ii) Use SDP to find $M \succeq 0$ satisfying the above constraints.
iii) Find P such that $M = P^T P$ and show that Pv yields the desired sos decomposition.

Solution to Exercise 4. Using basic calculus, we see that $p(x) \geq 4$ for all $x \in \mathbb{R}$, thus we have that $p \in P_{1,4} = \Sigma_{1,4}$, hence an sos decomposition exists.

[2] How would you handle the case $c = 0$? What are the possible values of a, b?
[3] But $\Sigma_{n,2d}$ is not a spectrahedron, unless $2d = 2$.

i) Writing $M = \begin{pmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{pmatrix}$, we have that

$$v^T M v = a_{00} + 2a_{01}x + x^2(2a_{02} + a_{11}) + 2a_{12}x^3 + a_{22}x^4.$$

So, matching coefficients with $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 5$ we find the linear constraints

$$a_{00} = 5,\ a_{01} = 2,\ a_{02} = 3 - a_{11}/2,\ a_{12} = 2,\ a_{22} = 1,$$

and a_{11} is free.

ii) Substituting these affine-linear constraints in M, we have

$$M = \begin{pmatrix} 5 & 2 & 3 - \frac{a_{11}}{2} \\ 2 & a_{11} & 2 \\ 3 - \frac{a_{11}}{2} & 2 & 1 \end{pmatrix}.$$

By Sylvester's criterion, it is easy to see that the above is positive-semidefinite if and only if $4 \leq a_{11} \leq 8$.

iii) Take, e.g., $a_{11} = 6$, so that $M = \begin{pmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 1 \end{pmatrix} \succ 0$, and use the Cholesky decomposition to find P such that $M = P^T P$. Namely, we have

$$P = \begin{pmatrix} \sqrt{5} & \frac{2}{\sqrt{5}} & 0 \\ 0 & \sqrt{\frac{26}{5}} & \sqrt{\frac{10}{13}} \\ 0 & 0 & \sqrt{\frac{3}{13}} \end{pmatrix},$$

hence $p(x) = v^T M v = v^T (P^T P) v = (Pv)^T P v$ yields the sos decomposition

$$p(x) = (Pv)^T P v = \left(\frac{2x}{\sqrt{5}} + \sqrt{5} \right)^2 + \left(\sqrt{\frac{10}{13}} x^2 + \sqrt{\frac{26}{5}} x \right)^2 + \left(\sqrt{\frac{3}{13}} x^2 \right)^2.$$

As usual, this decomposition is not unique. For instance, the matrix

$$Q = \begin{pmatrix} 0 & 2 & 1 \\ \sqrt{2} & \sqrt{2} & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix}$$

also satisfies $M = Q^T Q$ and hence yields the sos decomposition

$$p(x) = (Qv)^T Q v = (x^2 + 2x)^2 + (\sqrt{2} + \sqrt{2}x)^2 + (\sqrt{3})^2.$$

Can you find an sos decomposition with only 2 squares?[4]

References

[BPT13] G. Blekherman, P. A. Parrilo, and R. R. Thomas. *Semidefinite optimization and convex algebraic geometry*, vol. 13 of MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2013.

[4]For nonnegative univariate polynomials, it is always possible to find an sos decomposition with only 2 squares!