Lecture 15

1. Duality

1.1. Dual LP. Recall from the last lecture that the *dual* LP to

$$\max \quad c^T x \quad \text{s.t.} \quad Ax \leq b, \;\; x \geq 0,$$

is given by

$$\min \quad b^T y \quad \text{s.t.} \quad A^T y \geq c, \;\; y \geq 0.$$

More generally, to find the dual to a general LP, we may use the following table:

primal	dual
max	min
$\leq b_i$	$y_i \geq 0$
$= b_i$	y_i unconstrained
$x_j \geq 0$	$\geq c_j$
x_j unconstrained	$= c_j$
unbounded	infeasible
# constraints	# variables

An interpretation of duality in economic terms is given by the following:

Exercise 1. Suppose you are an industry that produces two types of products, A and B, which are manufactured using three chemical compounds, C_1, C_2, and C_3. The amounts (in kg) of each compound required to manufacture one box of each product, as well as the quantities of these compounds currently available, are given in the table below.

	C_1	C_2	C_3
A	2	3	5
B	3	2	1
Availability	16	19	30

You are able to sell each box of A for \$10, and each box of B for \$12.

a) Find the LP to find how many boxes of A and B should be produced to maximize your profit;
b) A competitor industry is planning to buy from your stock of chemical compounds. How much should would they offer you for each kg of C_1, C_2, and C_3 in order for you to sell them your raw materials? Write this as an LP and recognize this as the dual LP to that in a).

Solution to Exercise 1. a) The LP is

$$\begin{aligned} \max \quad 10x_1 + 12x_2 \quad \text{s.t.} \quad 2x_1 + 3x_2 &\leq 16 \\ 3x_1 + 2x_2 &\leq 19 \\ 5x_1 + x_2 &\leq 30 \\ x_1, x_2 &\geq 0. \end{aligned}$$

b) Let y_1, y_2, y_3 be the prices to be offered for each kg of compounds C_1, C_2, and C_3, respectively. (These are often called *shadow prices.*[1]) The competitor would like to minimize the prices to be paid, with the constraint that the offer will be accepted. In order for the offer to be accepted, the income must be at least as large as if the manufacturer produces and sells A and B. Thus,

$$\begin{aligned} \min \quad 16y_1 + 19y_2 + 30y_3 \quad \text{s.t.} \quad 2y_1 + 3y_2 + 5y_3 &\geq 10 \\ 3y_1 + 2y_2 + y_3 &\geq 12 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

[1] See e.g., `https://en.wikipedia.org/wiki/Shadow_price`

Each constraint above reflects that, with those amounts of C_1, C_2, and C_3, one box of A or B could be produced, so the corresponding price must exceed the price for which A or B are sold. The objective function minimizes the total cost of buying the entire inventory of raw materials.

Note that the shadow price y_j associated to a raw material C_j corresponds to the additional profit that would be created if the amount of that raw material was increased by one unit.

1.2. **Min cut.** The dual to the LP of finding the maximum flow through a network is the LP of finding the minimum cut that interrupts flow on that network.