Lecture 11

1. Two-phase simplex method

Recall that the two-phase simplex method takes a general LP as input, and, after finitely many steps (e.g., if the lexicographic pivoting rule is used), outputs one of the following:

- LP is infeasible;
- LP is unbounded;
- LP is feasible and bounded, and optimal solution is x_{opt}.

First, we introduce the appropriate (nonnegative) slack and artificial variables:

(1) Let $V = \{j_1, \ldots, j_k\}$ be the list of rows $a_{j1}x_1 + \cdots + a_{jn}x_n \,\square\, b_j$ where $\square$ is $=$ or $\geq$.
(2) Add a slack variable to rows where $\square$ is $\leq$ and *subtract* a slack variable[1] to rows where $\square$ is $\geq$.
(3) If $j \in V$, then add an *artificial variable* to row j.

Second, we begin Phase I, in which we solve an auxiliary LP that tests feasibility of the original LP, by finding the *minimum of the sum of artificial variables.* (If there are no artificial variables, skip to Phase II.) As discussed in the previous lecture, this minimum is 0 if and only if the original LP is feasible, in which case we find an optimal solution x_{aux} for the auxiliary LP. The feasible basis of the auxiliary LP corresponding to the optimal solution x_{aux} may or may not contain artificial variables.

Third, to begin Phase II, if no artificial variables are basic in the last tableau of Phase I, then eliminate the columns of the artificial variables and replace the target row with the original target function to proceed. If some artificial variables are basic (they must have value 0), then eliminate the columns of all nonbasic artificial variables and of any variable from the original LP with a negative entry in the target row. Then, replace the (remaining entries in the) target row with the original target function and proceed.

Exercise 1. Introduce the appropriate slack and artificial variables and solve the feasibility test (Phase I) for the following LP:

$$\begin{aligned} \max \quad 2x_1 + 3x_2 \quad \text{s.t.} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 \leq 4, \\ & x_1 + 3x_2 \geq 36, \\ & x_1 + x_2 = 10, \\ & x \geq 0. \end{aligned}$$

If it is feasible, then proceed to Phase II to either find an optimal solution or show the LP is unbounded.

Exercise 2. Introduce the appropriate slack and artificial variables and solve the feasibility test (Phase I) for the following LP:

$$\begin{aligned} \max \quad x_1 + x_2 \quad \text{s.t.} \quad & 5x_1 - x_2 \geq 5, \\ & 2x_1 + x_2 \geq 3, \\ & \tfrac{1}{2}x_1 - x_2 \leq -4, \\ & x \geq 0. \end{aligned}$$

If it is feasible, then proceed to Phase II to either find an optimal solution or show the LP is unbounded.

[1] These are usually called *excess* or *surplus* variables.

Exercise 3. Introduce the appropriate slack and artificial variables and solve the feasibility test (Phase I) for the following LP:

$$\max \quad 40x_1 + 10x_2 + 7x_5 + 14x_6 \quad \text{s.t.} \quad \begin{aligned} &x_1 - x_2 + 2x_5 = 0, \\ &-2x_1 + x_2 - 2x_5 = 0, \\ &x_1 + x_3 + x_5 - x_6 = 0, \\ &x_2 + x_3 + x_4 + 2x_5 + x_6 = 4, \\ &x \geq 0. \end{aligned}$$

If it is feasible, then proceed to Phase II to either find an optimal solution or show the LP is unbounded.

Solution to Exercise 1. We add a slack variable x_3 to row 1, subtract a slack variable x_4 from row 2, and add artificial variables x_5 and x_6 to rows 2 and 3, so the original LP can be written in equational form as:

$$\max \quad 2x_1 + 3x_2 \quad \text{s.t.} \quad \begin{aligned} &\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + x_3 = 4, \\ &x_1 + 3x_2 - x_4 + x_5 = 36, \\ &x_1 + x_2 + x_6 = 10, \\ &x \geq 0. \end{aligned}$$

The auxiliary LP is to minimize the sum of artificial variables:

$$\min \quad x_5 + x_6 \quad \text{s.t.} \quad \begin{aligned} &\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + x_3 = 4, \\ &x_1 + 3x_2 - x_4 + x_5 = 36, \\ &x_1 + x_2 + x_6 = 10, \\ &x \geq 0. \end{aligned}$$

Setting up a tableau using the feasible basis $B = \{3, 5, 6\}$ we have:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4
x_5	1	3	0	-1	1	0	36
x_6	1	1	0	0	0	1	10
	0	0	0	0	-1	-1	0

Performing row operations to have 0 in the target row on columns of basic variables:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	$\frac{1}{2}$	$\frac{1}{4}$	1	0	0	0	4
x_5	1	3	0	-1	1	0	36
x_6	1	1	0	0	0	1	10
	2	4	0	-1	0	0	46

Choosing x_1 as entering variable, we compute $\theta(x_3) = 8$, $\theta(x_5) = 36$, $\theta(x_6) = 10$, so x_3 is the departing variable, and we obtain the next tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	1	$\frac{1}{2}$	2	0	0	0	8
x_5	0	$\frac{5}{2}$	-2	-1	1	0	28
x_6	0	$\frac{1}{2}$	-2	0	0	1	2
	0	3	-4	-1	0	0	30

There are still positive entries in the target row, so we select x_2 as entering variable, and compute $\theta(x_1) = 16$, $\theta(x_5) = \frac{56}{5}$, $\theta(x_6) = 4$, so x_6 is the departing variable, as we obtain the next tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	
x_3	$\frac{1}{4}$	0	1	0	0	$-\frac{1}{4}$	$\frac{3}{2}$
x_5	-2	0	0	-1	1	-3	6
x_2	1	1	0	0	0	1	10
	-2	0	0	-1	0	-4	6

The above is an optimal solution to the auxiliary LP, so the original LP is not feasible.

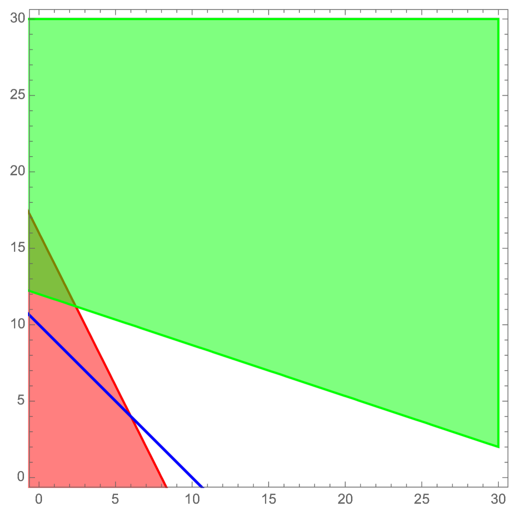

FIGURE 1. Constraints have empty intersection, so LP is infeasible.

Solution to Exercise 2. We first multiply row 3 by -1 so that the right-hand sides of all rows are ≥ 0. Then, subtract slack variables x_3, x_4, and x_5 and add artificial variables x_6, x_7, and x_8, one in each row, so that the LP can be written in equational form as:

$$\begin{aligned} \max \quad x_1 + x_2 \quad \text{s.t.} \quad & 5x_1 - x_2 - x_3 + x_6 = 5, \\ & 2x_1 + x_2 - x_4 + x_7 = 3, \\ & -\tfrac{1}{2}x_1 + x_2 - x_5 + x_8 = 4, \\ & x \geq 0. \end{aligned}$$

The auxiliary LP is to minimize the sum of artificial variables:

$$\begin{aligned} \min \quad x_6 + x_7 + x_8 \quad \text{s.t.} \quad & 5x_1 - x_2 - x_3 + x_6 = 5, \\ & 2x_1 + x_2 - x_4 + x_7 = 3, \\ & -\tfrac{1}{2}x_1 + x_2 - x_5 + x_8 = 4, \\ & x \geq 0. \end{aligned}$$

Setting up a tableau using the feasible basis $B = \{6, 7, 8\}$ we have:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_6	5	-1	-1	0	0	1	0	0	5
x_7	2	1	0	-1	0	0	1	0	3
x_8	$-\frac{1}{2}$	1	0	0	-1	0	0	1	4
	0	0	0	0	0	-1	-1	-1	0

Performing row operations to have 0 in the target row on columns of basic variables:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	
x_6	5	-1	-1	0	0	1	0	0	5
x_7	2	1	0	-1	0	0	1	0	3
x_8	$-\frac{1}{2}$	1	0	0	-1	0	0	1	4
	$\frac{13}{2}$	1	-1	-1	-1	0	0	0	12

Choosing x_1 as entering variable, we compute $\theta(x_6) = 1$ and $\theta(x_7) = \frac{3}{2}$ so x_6 is the departing variable, and we obtain the next tableau. See `lecture11.nb` for the remaining details.

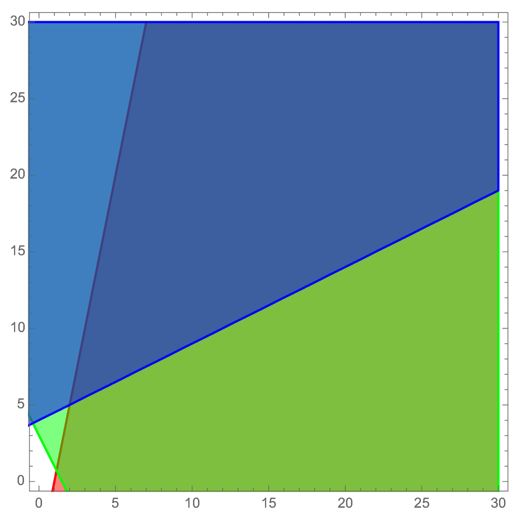

FIGURE 2. Constraints have unbounded intersection, and target function $x_1 + x_2$ takes arbitrarily large values, so LP is unbounded.

Solution to Exercise 3. The variable x_4 only appears in row 4 and with coefficient $+1$, so we may use it as an artificial variable. We add artificial variables x_7, x_8, x_9, to rows 1, 2, 3, respectively.

In equational form, the LP becomes:

$$\max \quad 2x_1 + 3x_2 \quad \text{s.t.} \quad \begin{aligned} &\tfrac{1}{2}x_1 + \tfrac{1}{4}x_2 + x_3 + x_7 = 4, \\ &x_1 + 3x_2 - x_4 + x_5 + x_8 = 36, \\ &x_1 + x_2 + x_6 + x_9 = 10, \\ &x \geq 0. \end{aligned}$$

The auxiliary problem (Phase I) is to minimize the sum $x_7 + x_8 + x_9$ of artificial variables, and results in an optimal tableau where the mininum 0 is achieved, but some basic variables are artificial variables. Thus, to proceed with Phase II, we drop the columns of nonbasic artificial variables and columns with negative entries in the target row. Then, Phase II terminates at the optimal value 9, attained at $x = (0, \frac{2}{3}, \frac{8}{3}, 0, \frac{1}{3}, 0)$. See `lecture11.nb` for details, where all tableaus are presented.