MATH71000, Spring 2024 **Renato Ghini Bettiol**

Midterm Exam

Due: Mar 29, 2024

1. On a Riemannian manifold (M, g), let $x\colon U \subset M \to x(U) \subset \mathbb{R}^n$ be a local chart which determines metric components $\mathrm{g}_{ij} = \mathrm{g}\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big)$ and Christoffel symbols Γ_{ij}^k, i.e., such that the Levi–Civita connection satisfies $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$. Prove that $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ satisfies $R(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})\frac{\partial}{\partial x_k} = \sum_\ell R_{ijk}{}^\ell \frac{\partial}{\partial x_\ell}$ where
$$R_{ijk}{}^\ell = \frac{\partial \Gamma_{jk}^\ell}{\partial x_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x_j} + \sum_m \Gamma_{jk}^m \Gamma_{im}^\ell - \Gamma_{ik}^m \Gamma_{jm}^\ell.$$

We have that $\big[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big] = 0$ since these are coordinate vector fields, so

$$\begin{aligned}
R(\tfrac{\partial}{\partial x_i}, \tfrac{\partial}{\partial x_j})\tfrac{\partial}{\partial x_k} &= \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \tfrac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_k} \\
&= \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_m \Gamma_{jk}^m \tfrac{\partial}{\partial x_m} \right) - \nabla_{\frac{\partial}{\partial x_j}} \left(\sum_m \Gamma_{ik}^m \tfrac{\partial}{\partial x_m} \right) \\
&= \sum_m \left(\tfrac{\partial \Gamma_{jk}^m}{\partial x_i} \tfrac{\partial}{\partial x_m} + \Gamma_{jk}^m \nabla_{\frac{\partial}{\partial x_i}} \tfrac{\partial}{\partial x_m} \right) - \left(\tfrac{\partial \Gamma_{ik}^m}{\partial x_j} \tfrac{\partial}{\partial x_m} + \Gamma_{ik}^m \nabla_{\frac{\partial}{\partial x_j}} \tfrac{\partial}{\partial x_m} \right) \\
&= \sum_m \left(\tfrac{\partial \Gamma_{jk}^m}{\partial x_i} \tfrac{\partial}{\partial x_m} + \Gamma_{jk}^m \sum_\ell \Gamma_{im}^\ell \tfrac{\partial}{\partial x_\ell} \right) - \left(\tfrac{\partial \Gamma_{ik}^m}{\partial x_j} \tfrac{\partial}{\partial x_m} + \Gamma_{ik}^m \sum_\ell \Gamma_{jm}^\ell \tfrac{\partial}{\partial x_\ell} \right) \\
&= \sum_\ell \left(\tfrac{\partial \Gamma_{jk}^\ell}{\partial x_i} + \sum_m \Gamma_{jk}^m \Gamma_{im}^\ell \right) \tfrac{\partial}{\partial x_\ell} - \sum_\ell \left(\tfrac{\partial \Gamma_{ik}^\ell}{\partial x_j} + \sum_m \Gamma_{ik}^m \Gamma_{jm}^\ell \right) \tfrac{\partial}{\partial x_\ell} \\
&= \sum_\ell \underbrace{\left(\tfrac{\partial \Gamma_{jk}^\ell}{\partial x_i} + \sum_m \Gamma_{jk}^m \Gamma_{im}^\ell - \tfrac{\partial \Gamma_{ik}^\ell}{\partial x_j} - \sum_m \Gamma_{ik}^m \Gamma_{jm}^\ell \right)}_{R_{ijk}{}^\ell} \tfrac{\partial}{\partial x_\ell}
\end{aligned}$$

2. Let $(N, \mathrm{d}y^2)$ be a 1-dimensional Riemannian manifold. Endow $M = (a,b) \times N$ with the metric $\mathrm{g} = \mathrm{d}r^2 + f(r)^2 \mathrm{d}y^2$, where $f\colon (a,b) \to \mathbb{R}$ is positive and smooth. Consider the vector fields $X = \frac{\partial}{\partial r}$ tangent to meridians in M and $Y = \frac{\partial}{\partial y}$ tangent to parallels in M.

   a) Compute $\mathrm{g}(R(X,Y)Y, X)$. Here, you may use the formula for R in Problem 1 and the answer to Problem 6a) in HW2.

   b) Show that $\sec_{\mathrm{g}}(X \wedge Y) = -\dfrac{f''(r)}{f(r)}$.

   c) Compute the volume form $\mathrm{vol}_{\mathrm{g}}$ of g.

For the remaining items, assume $(N, \mathrm{d}y^2) = (\mathbb{S}^1, \mathrm{d}\theta^2)$ is the unit circle (of length 2π).

d) If g extends smoothly to $r = a$ but $f(a) = 0$, what prevents $\sec_{\mathrm{g}}(X \wedge Y)$ from blowing up? Compute the sectional curvature at $r = a$ in terms of f.

e) Compute $\int_M \sec_{\mathrm{g}}(X \wedge Y)\,\mathrm{vol}_{\mathrm{g}}$. What happens to this quantity when the metric extends smoothly to both $r = a$ and $r = b$ with $f(a) = f(b) = 0$? Explain.

a) We use a chart $x\colon (a,b) \times N \to (a,b) \times \mathbb{R}$ with coordinate fields $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial r} = X$ and $\frac{\partial}{\partial x_2} = \frac{\partial}{\partial y} = Y$. From HW2 Problem 6a), we have that the only nonzero Christoffel symbols are

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{f'(r)}{f(r)}, \quad \Gamma_{22}^1 = -f(r)f'(r).$$

Thus, by the formula for R in Problem 1, since $\mathrm{g}(X,X) = 1$ and $\mathrm{g}(X,Y) = 0$,

$$\begin{aligned}
\mathrm{g}(R(X,Y)Y,X) &= \mathrm{g}({R_{122}}^1 X + {R_{122}}^2 Y, X) \\
&= {R_{122}}^1 \\
&= \frac{\partial \Gamma_{22}^1}{\partial x_1} - \frac{\partial \Gamma_{12}^1}{\partial x_2} + \sum_m \Gamma_{22}^m \Gamma_{1m}^1 - \Gamma_{12}^m \Gamma_{2m}^1 \\
&= \frac{\partial \Gamma_{22}^1}{\partial r} + (\Gamma_{22}^1 \Gamma_{11}^1 - \Gamma_{12}^1 \Gamma_{21}^1) + (\Gamma_{22}^2 \Gamma_{12}^1 - \Gamma_{12}^2 \Gamma_{22}^1) \\
&= -f'(r)^2 - f(r)f''(r) - \frac{f'(r)}{f(r)}(-f(r)f'(r)) \\
&= -f(r)f''(r).
\end{aligned}$$

b) From the previous item, since $\mathrm{g}(X,X) = 1$, $\mathrm{g}(X,Y) = 0$, and $\mathrm{g}(Y,Y) = f(r)^2$, it follows that

$$\sec_{\mathrm{g}}(X \wedge Y) = \frac{-f(r)f''(r)}{f(r)^2} = -\frac{f''(r)}{f(r)}.$$

Note that $\sec_{\mathrm{g}}$ depends only on r and not on $y \in N$, in accordance with $Y = \frac{\partial}{\partial y}$ being a Killing vector field.

c) $\mathrm{vol}_{\mathrm{g}} = \sqrt{\det \mathrm{g}}\,\mathrm{d}r\,\mathrm{d}y = f\,\mathrm{d}r\,\mathrm{d}y$

d) If $(N, \mathrm{d}y^2) = (\mathbb{S}^1, \mathrm{d}\theta^2)$ is the unit circle (of length 2π) and the metric extends smoothly to $r = a$ and $f(a) = 0$, then $f'(a) = 1$ and $f^{(2k)}(0) = 0$ for all $k \in \mathbb{N}$, i.e., all derivatives of even order vanish at $r = a$. In particular, $f''(a) = 0$, which prevents $\sec_{\mathrm{g}}$ from blowing up at $r = a$. Moreover, by the L'Hospital rule, $\sec_{\mathrm{g}}(X \wedge Y) = -\frac{f'''(a)}{f'(a)} = -f'''(a)$.

e) By the items above, and the Fundamental Theorem of Calculus,

$$\int_M \sec_{\mathrm{g}}(X \wedge Y)\,\mathrm{vol}_{\mathrm{g}} = \int_0^{2\pi} \int_a^b -\frac{f''(r)}{f(r)}\, f(r)\,\mathrm{d}r\mathrm{d}\theta = 2\pi(f'(a) - f'(b)).$$

If the metric g extends smoothly to both $r = a$ and $r = b$ with $f(a) = f(b) = 0$, then the manifold (M, g) is isometric to the open and dense subset of a Riemannian sphere $(\mathbb{S}^2, \overline{\mathrm{g}})$ obtained by puncturing it twice (at $r = a$ and $r = b$), where the Riemannian metric $\overline{\mathrm{g}}$ is smooth on $\mathbb{S}^2$ and restricts to g on M. A consequence of smoothness of g at $r = a$ and $r = b$ is that $f'(a) = 1$ and $f'(b) = -1$. This recovers the Gauss–Bonnet formula for the integral of the Gauss curvature $K_{\overline{\mathrm{g}}}$ on $(\mathbb{S}^2, \overline{\mathrm{g}})$,

$$\int_{\mathbb{S}^2} K_{\overline{\mathrm{g}}} \,\mathrm{vol}_{\overline{\mathrm{g}}} = \int_M \sec_{\mathrm{g}}(X \wedge Y)\,\mathrm{vol}_{\mathrm{g}} = 4\pi = 2\pi\chi(\mathbb{S}^2).$$

3. Consider the metric $\mathrm{g} = \frac{1}{x^2}\,\mathrm{d}x^2 + x^2\,\mathrm{d}y^2$ on $M = (0, \infty) \times \mathbb{R}$. Replace x by an arclength parameter $r = r(x)$ to recognize that g is isometric to a warped product $\mathrm{d}r^2 + f(r)^2\,\mathrm{d}y^2$. Compute its curvature and use the outcome to show that given $p, q \in M$, and given an orthonormal basis $\{e_1, e_2\}$ of T_pM and an orthonormal basis $\{\overline{e_1}, \overline{e_2}\}$ of T_qM, there exists an isometry φ of (M, g) such that $\varphi(p) = q$ and $\mathrm{d}\varphi(p)e_i = \overline{e_i}$ for $i = 1, 2$.

The arclength parameter in the x direction is $r = r(x)$ such that $\mathrm{d}r = r'(x)\,\mathrm{d}x = \frac{1}{x}\,\mathrm{d}x$, hence $r = \log x$. It follows that $x = e^r$, hence $\psi^*\mathrm{g} = \mathrm{d}r^2 + e^{2r}\mathrm{d}y^2$, where $\psi\colon \mathbb{R}^2 \to M$, $\psi(r, y) = (e^r, y)$. Thus, g is isometric to the warped product metric $\mathrm{d}r^2 + f(r)^2\mathrm{d}y^2$ on $(-\infty, \infty) \times \mathbb{R}$ with $f(r) = e^r$. From the Problem 2), its sectional curvature is

$$\sec = -\frac{f''(r)}{f(r)} = -\frac{e^r}{e^r} = -1,$$

so it is locally isometric to the hyperbolic plane $\mathbb{H}^2$, by Cartan's theorem. In particular, given $p, q \in M$, and the linear isometry $I\colon T_pM \to T_qM$ defined by the prescribed orthonormal bases, that is, $Ie_i = \overline{e_i}$, $i = 1, 2$, the hypothesis of Cartan's theorem are satisfied since the curvature tensor R of (M, g) is $R(X, Y)Z = \mathrm{g}(X, Z)Y - \mathrm{g}(Y, Z)X$ at all points. As (M, g) is simply-connected and complete, by the Cartan–Ambrose–Hicks theorem, there exists a global isometry $\varphi\colon M \to M$ such that $\varphi(p) = q$ and $\mathrm{d}\varphi(p) = I$.

4. For $n \geq 2$, does there exist a Riemannian metric g on $\mathbb{R}^n$ whose distance function is $\mathrm{dist}_{\mathrm{g}}(p, q) = \max\limits_{1 \leq i \leq n} |p_i - q_i|$, for all $p = (p_1, \dots, p_n), q = (q_1, \dots, q_n) \in \mathbb{R}^n$? Explain.

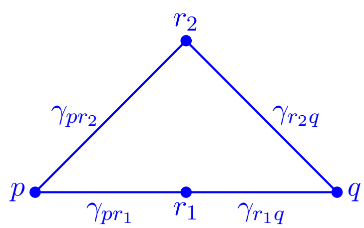
No. Suppose there exists such a Riemannian metric g on $\mathbb{R}^n$, $n \geq 2$, and consider the points $p = (0, 0, \dots, 0)$, $q = (\varepsilon, 0, \dots, 0)$, $r_1 = (\varepsilon/2, 0, \dots, 0)$, $r_2 = (\varepsilon/2, \varepsilon/2, \dots, 0)$, so

$$\mathrm{dist}_{\mathrm{g}}(p, q) = \varepsilon, \quad \mathrm{dist}_{\mathrm{g}}(p, r_i) = \varepsilon/2, \quad \mathrm{dist}_{\mathrm{g}}(r_i, q) = \varepsilon/2,$$

for $i = 1, 2$. Let $\varepsilon > 0$ be sufficiently small so that there exists a unique minimizing geodesic γ_{xy} between each pair of points x and y among $\{p, q, r_1, r_2\}$. (It follows from the Gauss Lemma that within any sufficiently small ball on (M, g) there is a unique minimizing geodesic between any pair of points.) As these geodesics are minimizing,

$$L_{\mathrm{g}}(\gamma_{pr_i}) = \varepsilon/2 \quad \text{and} \quad L_{\mathrm{g}}(\gamma_{r_iq}) = \varepsilon/2.$$

Concatenating γ_{pr_1} and γ_{r_1q} we obtain a curve γ_1 of length ε that joins p to q; similarly, concatenating γ_{pr_2} and γ_{r_2q} we obtain a curve γ_2 of length ε that joins p to q. Both γ_1 and γ_2 are minimizing geodesics from p to q, but they are not the same curve since $r_1 \neq r_2$. This contradicts the uniqueness of the minimizing geodesic γ_{pq} from p to q.



5. Can a complete manifold (M^n, g) with $\sec_{\mathrm{g}} \leq -1$ admit a complete metric with $\sec \geq 1$? Explain.

   No. If (M^n, g) is complete and $\sec_{\mathrm{g}} \leq -1$, then (by the Cartan–Hadamard Theorem) its universal cover is diffeomorphic to $\mathbb{R}^n$. If M^n also supported a complete Riemannian metric with $\sec \geq 1$, in particular $\mathrm{Ric} \geq (n-1)$, then its universal cover would be compact (by Myers' Theorem), so no such metric can exist.

6. Let (M, g) be a complete manifold and fix $p \in M$.

   a) Prove that M is noncompact if and only if there exists a unit speed geodesic $\gamma\colon \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\mathrm{dist}_{\mathrm{g}}(\gamma(t), p) = t$ for all $t \geq 0$; in particular, γ is such that $\mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) = |t - s|$ if $t, s \geq 0$.

   b) Can one arrange for γ to be such that $\mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in \mathbb{R}$?

   a) If M is compact, then $f(t) = \mathrm{dist}(p, \gamma(t))$ is continuous and hence bounded for any curve $\gamma(t)$. Thus, existence of a unit speed geodesic $\gamma\colon \mathbb{R} \to M$ such that $\gamma(0) = p$ and $\mathrm{dist}_{\mathrm{g}}(\gamma(t), p) = t$ for all $t \geq 0$ implies that M is noncompact.

      Conversely, if (M, g) is complete and noncompact, there exists a sequence $q_n \in M$ such that $\mathrm{dist}_{\mathrm{g}}(p, q_n) = L_n \nearrow +\infty$ as $n \nearrow +\infty$ by the Hopf–Rinow Theorem. Also by the Hopf–Rinow Theorem, there exist minimizing unit speed geodesics $\gamma_n\colon [0, L_n] \to M$ such that $\gamma_n(0) = p$ and $\gamma_n(L_n) = q_n$. The corresponding initial velocities $v_n := \dot\gamma_n(0)$ form a sequence on the unit sphere in T_pM which hence admits a convergent subsequence. Up to reindexing, let us assume that $v_n \to v$ itself converges to a unit vector $v \in T_pM$. Let $\gamma\colon \mathbb{R} \to M$, $\gamma(t) = \exp_p tv$.

      We claim that $\mathrm{dist}_{\mathrm{g}}(p, \gamma(t)) = t$ for all $t \geq 0$. If not, there exists $t_* > 0$ with $\mathrm{dist}_{\mathrm{g}}(p, \gamma(t_*)) < t_*$. Note that $t_* < L_n$ for n sufficiently large, as $L_n \nearrow +\infty$. By construction, we have that $\gamma_n(t_*) = \exp_p(t_* v_n) \to \gamma(t_*)$ as $n \nearrow +\infty$. Thus,

$\mathrm{dist}_{\mathrm{g}}(\gamma_n(t_*), \gamma(t_*)) < t_* - \mathrm{dist}_{\mathrm{g}}(p, \gamma(t_*))$ for all n sufficiently large. Since the geodesic $\gamma_n : [0, L_n] \to M$ is minimizing, by the triangle inequality, for n large,

$$\begin{aligned} t_* = \mathrm{dist}_{\mathrm{g}}(p, \gamma_n(t_*)) &\leq \mathrm{dist}_{\mathrm{g}}(p, \gamma(t_*)) + \mathrm{dist}_{\mathrm{g}}(\gamma(t_*), \gamma_n(t_*)) \\ &< \mathrm{dist}_{\mathrm{g}}(p, \gamma(t_*)) + \big(t_* - \mathrm{dist}_{\mathrm{g}}(p, \gamma(t_*))\big) = t_*, \end{aligned}$$

a contradiction, which proves the claim. Moreover, for any $t, s \geq 0$, we have

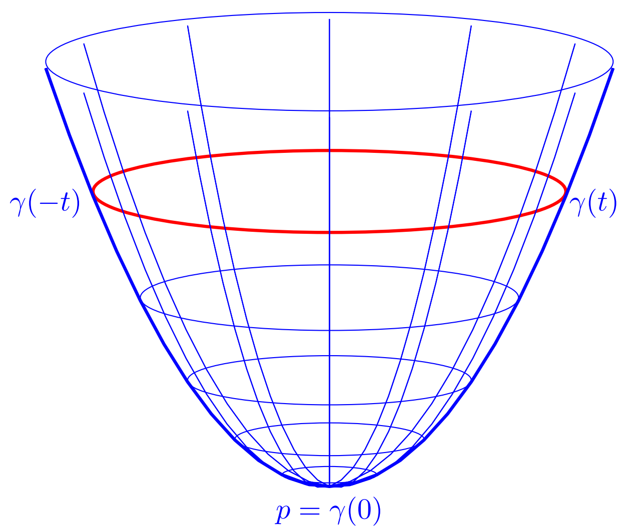
$$\begin{aligned} s = \mathrm{dist}_{\mathrm{g}}(p, \gamma(s)) \leq \mathrm{dist}_{\mathrm{g}}(p, \gamma(t)) + \mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) = t + \mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) \\ t = \mathrm{dist}_{\mathrm{g}}(p, \gamma(t)) \leq \mathrm{dist}_{\mathrm{g}}(p, \gamma(s)) + \mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) = s + \mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) \end{aligned}$$

hence

$$|t - s| = \max\{t - s, s - t\} \leq \mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s))$$

and $\mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) \leq |t - s|$ as γ is a curve of length $|t - s|$ joining $\gamma(t)$ and $\gamma(s)$.

b) Even though the above geodesic γ is a *ray*, i.e., satisfies $\mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \geq 0$, it is not always possible to arrange for it to be a *line*, i.e., satisfy $\mathrm{dist}_{\mathrm{g}}(\gamma(t), \gamma(s)) = |t - s|$ for all $t, s \in \mathbb{R}$. For example, if (M, g) is the paraboloid $(\mathbb{R}^2, \mathrm{g})$ and p is the origin, then unit speed geodesics $\gamma : \mathbb{R} \to M$ with $\gamma(0) = p$ are meridians. All of them are rays, none of them are lines. Indeed, if $t > 0$ is large enough, then $\mathrm{dist}_{\mathrm{g}}(\gamma(-t), \gamma(t)) < 2t$ since one one can use a parallel as shortcut.



It can be shown that if M has at least two *ends*, i.e., M is *disconnected at infinity*, e.g., M diffeomorphic to $\mathbb{S}^{n-1} \times \mathbb{R}$, then it has lines.