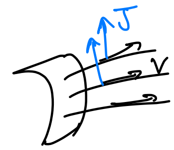
Lecture 20 $\quad$ 4/12/2024

From last time: $J'' + R_V J = 0 \iff \begin{cases} J' = SJ \\ S' + S^2 + R_V = 0 \end{cases} \quad (S = \nabla V)$



Thm. Let $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}^2 E$ be smooth curves with $R_1(t) \geq R_2(t)$, $\forall t$. Let $S_i : [t_0, t_i) \to \mathrm{Sym}^2 E$ be the maximal solutions to $S_i' + S_i^2 + R_i = 0$. If $S_1(t_0) \leq S_2(t_0)$, then $t_1 \leq t_2$ and $S_1(t) \leq S_2(t)$ for all $t \in [t_0, t_1)$.

Next, we apply the above to get a comparison of lengths of Jacobi fields:

Thm. Let $S_1, S_2 : (t_0, t') \to \mathrm{Sym}^2 E$ be smooth curves with $S_1(t) \leq S_2(t)$. Let $J_i : (t_0, t') \to E$ be nonzero sol. to $J_i' = S_i J_i$. Then $t \mapsto \dfrac{\|J_1(t)\|}{\|J_2(t)\|}$ is nonincreasing. Moreover, if $\lim\limits_{t \searrow t_0} \dfrac{\|J_1(t)\|}{\|J_2(t)\|} = 1$, then $\|J_1(t)\| \leq \|J_2(t)\|$ for all $t \in (t_0, t')$. Equality holds for some $t_* \in (t_0, t')$ if and only if $J_i = j \cdot v_i$ on $[t_0, t']$ for some $v_i \in E$ with $S_i v_i = \lambda v_i$, $j' = \lambda j$, and $S_1 \leq \lambda \, \mathrm{Id} \leq S_2$.

<u>Pf:</u> Since $\|J_i(t)\|$ is smooth, we can differentiate:

$$\frac{\|J_i\|'}{\|J_i\|} = \frac{1}{\|J_i\|} \frac{1}{2\sqrt{\langle J_i, J_i \rangle}} 2 \langle J_i', J_i \rangle = \frac{\langle J_i', J_i \rangle}{\|J_i\|^2} = \frac{\langle S_i J_i, J_i \rangle}{\|J_i\|^2}$$

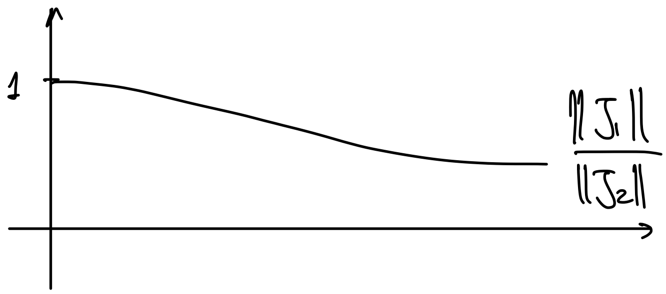
$$\in \left[\lambda_{min}(S_i), \lambda_{max}(S_i) \right]$$

min. and max eigenvalues of $S_i \in \mathrm{Sym}^2 E$.

Thus
$$\left(\log \|J_1\|\right)' = \frac{\|J_1\|'}{\|J_1\|} \leq \lambda_{max}(S_1) \underset{S_1 \leq S_2}{\leq} \lambda_{min}(S_2) \leq \frac{\|J_2\|'}{\|J_2\|} = \left(\log \|J_2\|\right)'$$

i.e. $\left(\log \dfrac{\|J_1\|}{\|J_2\|}\right)' \leq 0$ so $\dfrac{\|J_1\|}{\|J_2\|}$ is non-increasing.

By monotonicity, if $\|J_1\| = \|J_2\|$ at $t = t_0$, and $t = t_*$. then $\|J_1\| = \|J_2\|$, $\forall t \in (t_0, t_*)$ and hence $J_i' = S_i J_i \overset{\text{from } = \text{ in the inequalities.}}{=} \lambda J_i$, from which the stated conclusions follow. $\square$



The following corollaries are originally due to Berger and Rauch:

Thm (Rauch I). Suppose J_i are sol to $J_i'' + R_i J_i = 0$ with $R_1 \geqslant R_2$ and $J_i(0) = 0$, $\|J_1'(0)\| = \|J_2'(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.

Thm (Rauch II). Suppose J_i are sol to $J_i'' + R_i J_i = 0$ with $R_1 \geqslant R_2$ and $J_i'(0) = 0$, $\|J_1(0)\| = \|J_2(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to the first zero of J_1.

Both Rauch I and II follow from comparison theorems above; namely $R_1(t) \geqslant R_2(t)$ and $S_1(0) = S_2(0)$ give $S_1(t) \leqslant S_2(t)$ for all $t \in (0, t_1)$. Then:

Rauch I: use singular initial condition "$S_i(0) = \infty$", i.e., $S_i(t) \sim \frac{1}{t} \mathrm{Id}$ as $t \downarrow 0$

$$J_i' = S_i J_i \Rightarrow t J_i' \sim J_i \text{ as } t \downarrow 0 \rightarrow J_i(0) = 0$$

$$\|J_1'(0)\| = \|J_2'(0)\| \Rightarrow \lim_{t \downarrow 0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = \lim_{t \downarrow 0} \frac{t\|J_1'(t)\|}{t\|J_2'(t)\|} = 1. \text{ Apply Thm.}$$

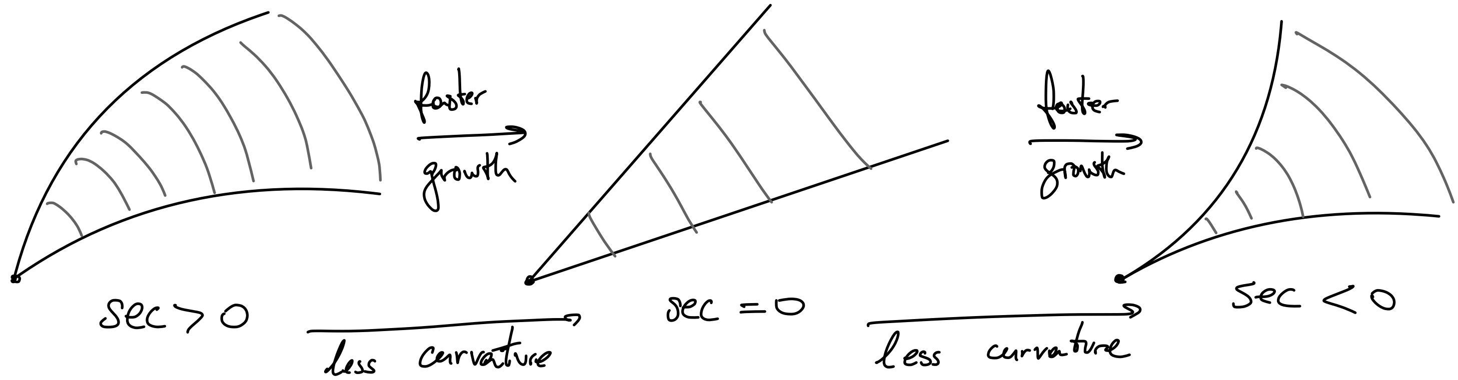
$\square$

Rauch II: use initial condition $S_i(0) = 0$

$$J_i' = S_i \cdot J_i \Rightarrow J_i'(0) = 0.$$

$$\|J_1(0)\| = \|J_2(0)\| \Rightarrow \lim_{t \downarrow 0} \frac{\|J_1(t)\|}{\|J_2(t)\|} = 1. \text{ Apply Thm.} \quad \square$$

Picture to have in mind from Rauch I:

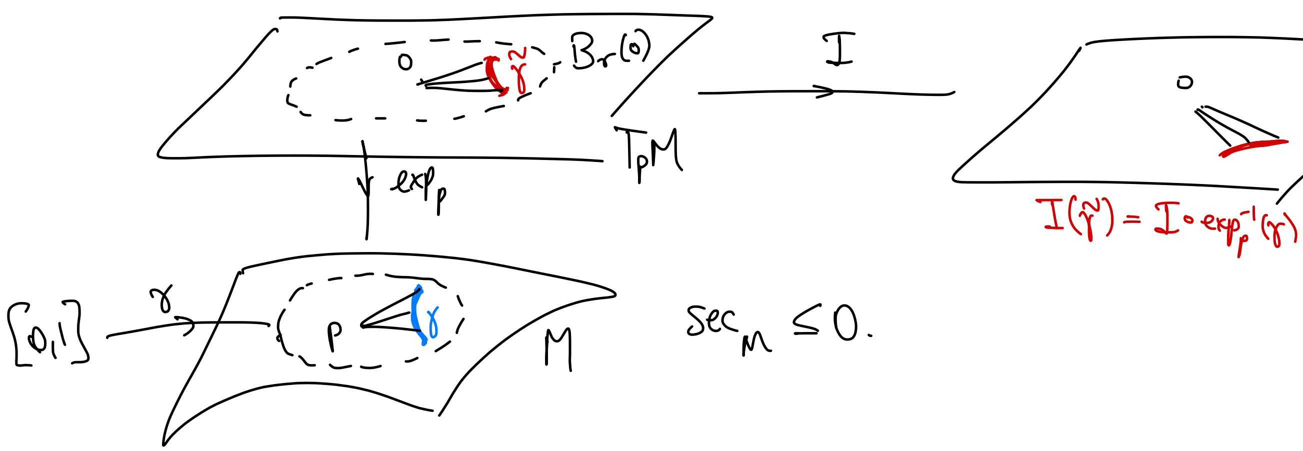


(We knew this for $t \approx 0$ from Taylor Series expansion of $\|J(t)\|^2$ at $t=0$, now this is known for $0 \le t \le t_1$ where t_1 is the first conjugate time.)

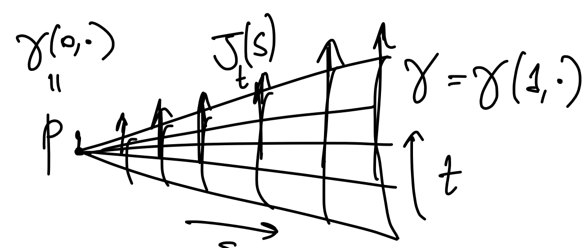
Application of Rauch I:

Cor: Let (M^n, g) be a complete Riem. mfld with $\sec \le 0$, and $r > 0$ s.t. $\exp_p : B_r(0) \longrightarrow M$ is a diffeom. onto its image. Fix a linear isometry $I : T_pM \to \mathbb{R}^n$. Given $\gamma : [0,1] \to \exp_p(B_r(0))$, we have

$$\mathrm{length}_g(\gamma) \geq \mathrm{length}_{\mathbb{R}^n}\left(I \circ \exp_p^{-1}(\gamma)\right).$$



Pf: Let $\tilde{\gamma} = \exp_p^{-1} \gamma$, and consider the "rectangle" $\gamma(s,t) = \exp_p s\tilde{\gamma}(t)$



For fixed t, $s \mapsto \gamma(s,t)$ is a geodesic, and $J_t(s) = \frac{\partial}{\partial t}\gamma(s,t)$ is a Jacobi field along $s \mapsto \gamma(s,t)$, with $J_t(0) = 0$ and $J_t(1) = \dot{\gamma}(t)$.

Since $\sec_M \leq 0$, by Rauch I,

$$\|J_t(s)\| \geq \underbrace{s\,\|J_t'(0)\|}_{(*)\ \text{length of comparison Jacobi field in } \mathbb{R}^n} \quad \text{so} \quad \text{length}_g(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|\,dt = \int_0^1 \|J_t(1)\|\,dt$$

$$\overset{s=1}{\geq} \int_0^1 \|J_t'(0)\|\,dt = \text{length}_{\mathbb{R}^n}(I \circ \exp_p^{-1}\gamma)$$

Indeed, $J_t'(0) = \frac{D}{ds} J_t(s)\Big|_{s=0} = \frac{D}{ds}\frac{\partial}{\partial t} \exp_p s\tilde{\gamma}(t)\Big|_{s=0}$

$$= \frac{D}{dt}\frac{\partial}{\partial s} \exp_p s\tilde{\gamma}(t)\Big|_{s=0} = \frac{D}{dt}\underbrace{d(\exp_p)_0}_{\mathrm{id}}\tilde{\gamma}(t) = \tilde{\gamma}'(t)$$

and so $\text{length}_{\mathbb{R}^n}(I \circ \exp_p^{-1}\gamma) = \int_0^1 \Big\| \frac{\partial}{\partial t} \underbrace{I \circ \exp_p^{-1}(\gamma)}_{\tilde{\gamma}} \Big\|\,dt = \int_0^1 \|J_t'(0)\|\,dt.$ $\square$

(*) In $\mathbb{R}^n$, the Jacobi equation $J'' = 0$ has solutions $J(s) = J(0) + sJ'(0)$, so Jacobi fields with $J(0) = 0$ are given by $J(s) = s\,J'(0)$.

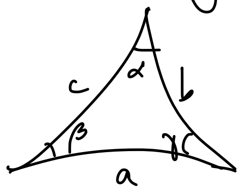
Rmk: Reasoning as above, Rauch I gives a more refined estimate

$$\|J(t)\| \geq t\,\|J'(0)\| > 0$$

for Jacobi fields with $J(0) = 0$ on manifolds with $\sec \leq 0$, compared to our earlier observation (a crucial step in the proof of Cartan–Hadamard Thm) that $J(t) \neq 0$, $\forall t > 0$; cf. Remark in p.2 of Lectures3.pdf.

Def. A geodesic triangle is a triple of minimizing geodesics with endpoints that are match pairwise (as in a triangle).

Cor: A geodesic triangle on a complete manifold with $\sec \le 0$ satisfies

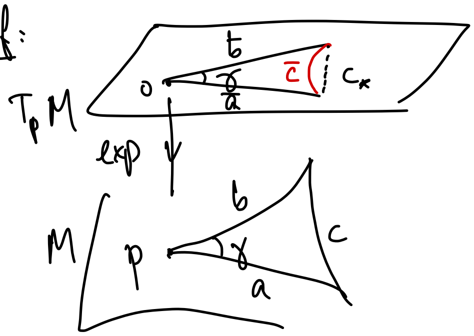


(i) $\ell(c)^2 \geq \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma$ $\quad (\ell = \text{length})$

(ii) $\alpha + \beta + \gamma \le \pi$

If $\sec < 0$, then get strict inequalities.

Pf:



Let $\bar{a}, \bar{b}, \bar{c}$ in T_pM be such that

$a = \exp_p \bar{a}, \quad b = \exp_p \bar{b}, \quad c = \exp_p \bar{c}$

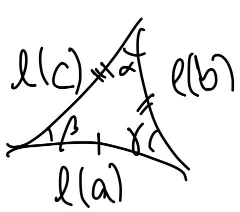
Note that $\bar{a}$ and $\bar{b}$ are straight line segments ($\exp_p$ is radial isometry); with $\ell(\bar{a}) = \ell(a)$ and $\ell(\bar{b}) = \ell(b)$. Let c_* be the straight line segment with same endpoints as $\bar{c}$, so $\ell(\bar{c}) \geq \ell(c_*)$. By the Application of Rauch I, $\ell(c) \geq \ell(\bar{c}) \geq \ell(c_*)$. Thus, altogether:

$$\ell(c)^2 \geq \ell(c_*)^2 \overset{\text{Law of cosines in } T_pM \cong \mathbb{R}^n}{=} \ell(\bar{a})^2 + \ell(\bar{b})^2 - 2\ell(\bar{a})\ell(\bar{b})\cos\gamma$$

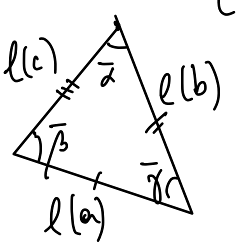
$$\overset{\text{Gauss Lemma}}{=} \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma.$$

To compare angles, since $\ell(a), \ell(b), \ell(c)$ satisfy the triangle inequalities ($\ell(a), \ell(b), \ell(c)$ achieve distances) (b/c every geodesic is minimizing in $\sec \le 0$, i.e., $\ell(a), \ell(b), \ell(c)$ achieve distances) we can build a comparison triangle in $\mathbb{R}^2$, with same side angles, but possibly different angles, say $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. Then, from the above:

$$\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \le \ell(c)^2$$
$$= \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\bar{\gamma}$$



in M

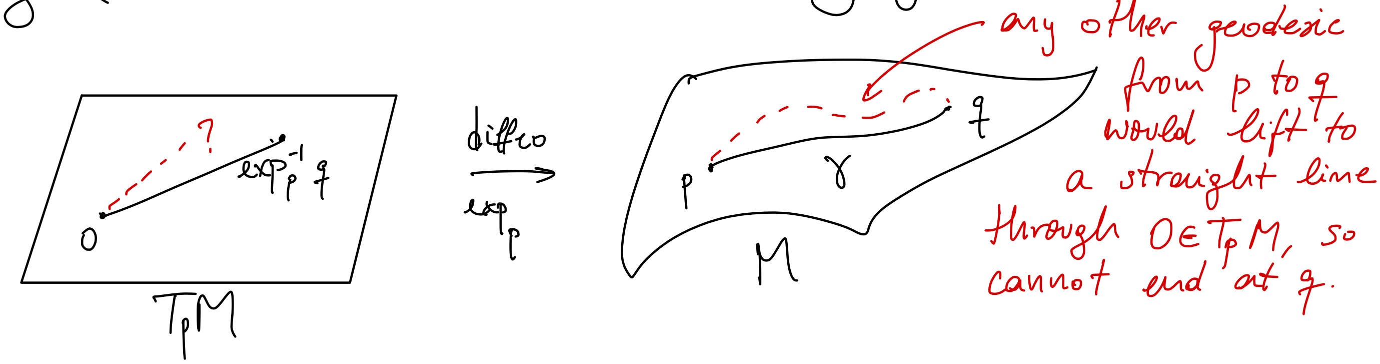


in $\mathbb{R}^2$

$$\Rightarrow \cos\gamma \geq \cos\bar{\gamma} \Rightarrow \gamma \le \bar{\gamma}$$

Same for α, β and get $\alpha + \beta + \gamma \le \bar{\alpha} + \bar{\beta} + \bar{\gamma} = \pi$. $\square$

Rmk: If (M^n, g) is a complete Riem. mfld with $\pi_1 M = \{1\}$ and $\sec \leq 0$, then by Cartan-Hadamard $\exp_p : T_pM \to M$ is a diffeo, so given any $q \in M$ there is a unique geodesic joining p and q, which is hence minimizing (b/c there exists some minimizing geodesic by Hopf–Rinow).

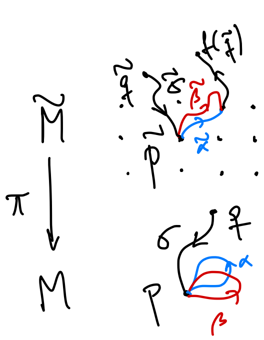


Thus, if M is complete, $\pi_1 M = \{1\}$, and $\sec \leq 0$, then the above facts about geodesic triangles hold for any triangles with geodesic sides (b/c the sides are automatically minimizing.)

Lecture 21 4/17/2024

Def: (M, g) closed Riem. mfld, $(\tilde{M}, \tilde{g})$ universal covering.
A deck transformation $f : \tilde{M} \to \tilde{M}$ is a translation along the geodesic $\tilde{\gamma}$ in $\tilde{M}$ if $f(\tilde{\gamma}) = \tilde{\gamma}$. Note: If $f \neq id$, then $f(\tilde{\gamma}(t)) = \tilde{\gamma}(t+a)$.

From basic topology: $\pi_1(M) \cong Aut(\tilde{M}) = \{ f : \tilde{M} \to \tilde{M} : \text{deck transformation} \}$



$$[\alpha] \in \pi_1(M, p) \longmapsto \left(f_{\alpha, p} : \tilde{M} \longrightarrow \tilde{M} \right) \in Aut(\tilde{M})$$

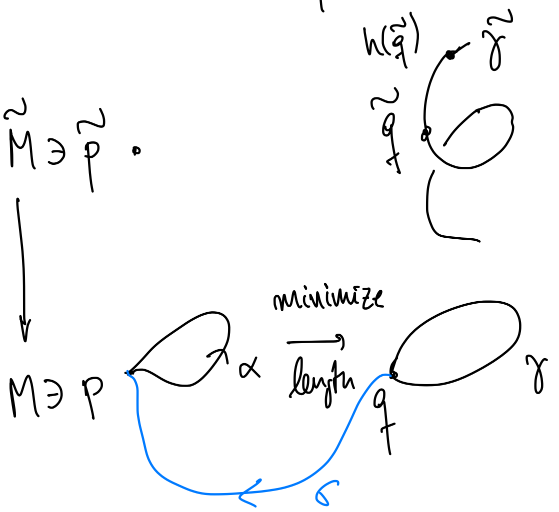
$f_{\alpha,p}(\tilde{q})$ = endpoint of lift of $\sigma^{-1} \alpha \sigma$ to $\tilde{M}$, starting at $\tilde{q}$.

Recall: curves in M are homotopic $\iff$ lifts to $\tilde{M}$ have same endpoint so the above is well-defined.

Prop. Given a deck transformation $f:\tilde{M}\to\tilde{M}$, there exists a geodesic $\tilde{\gamma}$ in $\tilde{M}$ s.t. f is a translation along $\tilde{\gamma}$.

Pf. $f = f_{\alpha,p}$ for some $\alpha \in \pi_1(M,p)$. Let $\gamma \overset{\text{free homotopy}}{\sim} \alpha$ be a closed geodesic. Then

$h = f_{\gamma,\tilde{q}} \in Aut(\tilde{M})$ is s.t. $h(\tilde{\gamma}) = \tilde{\gamma}$; by construction.



closed geodesic
$[\gamma]=[\alpha]$

<u>Claim</u>: $f = h$.

Since h, f are deck transformations, suffices to show $h(\tilde{q}) = f(\tilde{q})$.
As α, γ are freely homotopic, it follows $\sigma^{-1}\alpha\sigma$ is homotopic to γ rel. q. (as elements of $\pi_1(M,q)$) So the endpoints of their lifts are the same, i.e., $h(\tilde{q}) = f(\tilde{q})$ so $h = f$ hence $f(\tilde{\gamma}) = \tilde{\gamma}$. $\square$

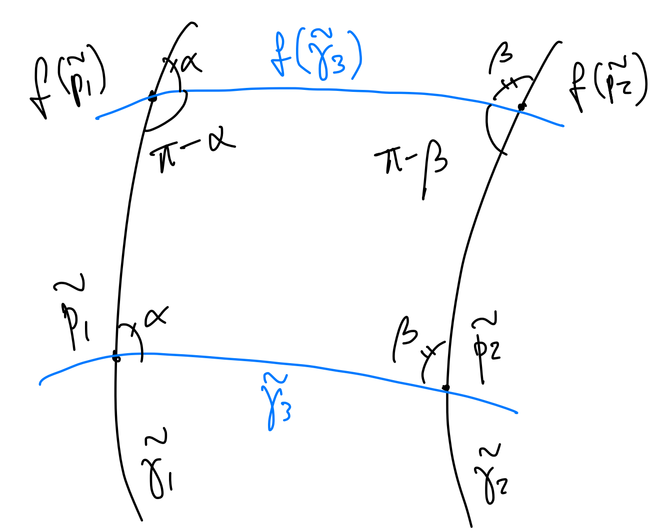
<u>Lemma</u>. If (M,g) is a closed mfld with $\sec < 0$, then a deck transformation $f:\tilde{M}\to\tilde{M}$, $f \neq id$ is a translation along a <u>unique</u> geodesic $\tilde{\gamma}$ in $\tilde{M}$.

Pf. Suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are geodesics in $\tilde{M}$ s.t. f is a translation along $\tilde{\gamma}_i$. Then, if $\tilde{p} \in \tilde{\gamma}_1 \cap \tilde{\gamma}_2$, we have $\tilde{p} \neq f(\tilde{p}) \in \tilde{\gamma}_1 \cap \tilde{\gamma}_2$, but this contradicts injectivity of $\exp_{\tilde{p}}$.

($f \neq id \Rightarrow f$ has no fixed pts.)

by Cartan-Hadamard, $\exp_{\tilde{p}}: T_{\tilde{p}}\tilde{M} \to \tilde{M}$ is a diffeo

Thus $\tilde{\gamma}_1 \cap \tilde{\gamma}_2 = \phi$.



Let $\tilde{p}_i \in \tilde{\gamma}_i$ and $\tilde{\gamma}_3$ be a minimizing geodesic from $\tilde{p}_1$ to $\tilde{p}_2$. As f is an isometry of $\tilde{M}$, the angles α, β in the diagram are the same. Subdividing this quadrangle into two triangles Δ_1, Δ_2 it follows that

$$\sum_{\text{int. angles}} \Delta_1 + \sum_{\text{int. angles}} \Delta_2 \geq 2\pi$$

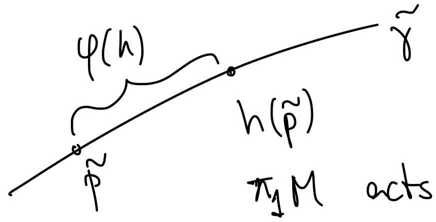
So $\sum_{\text{int angles}} \Delta_i \geq \pi$ for $i=1$ or 2, contradicting Cor. from last Lecture that $\sum_{\text{int. angles}} \Delta < \pi$ if $\sec < 0$. $\square$

Lemma. If (M,g) is a closed mfld with $\sec<0$, then commuting deck transformations are translations along the same geodesic.

Pf. If $f_1, f_2 : \tilde{M} \to \tilde{M}$ are as above, with $f_i(\tilde{\gamma}_i) = \tilde{\gamma}_i$, then $f_2(f_1(\tilde{\gamma}_2)) = f_1(f_2(\tilde{\gamma}_2)) = f_1(\tilde{\gamma}_2)$ so f_2 preserves $f_1(\tilde{\gamma}_2)$ hence $f_1(\tilde{\gamma}_2) = \tilde{\gamma}_2$. By uniqueness proved above, $\tilde{\gamma}_1 = \tilde{\gamma}_2$. $\square$

> **Thm** (Preissmann, 1943). If (M^n, g) is a closed Riem. mfld with $\sec<0$ and $H < \pi_1 M$ is Abelian, $H \neq \{1\}$, then $H \cong \mathbb{Z}$.

Pf. Let $H < \pi_1 M$ be Abelian, and $\tilde{\gamma}$ be the geodesic in $\tilde{M}$ along which every $h \in H$ is a translation. Recall (see Remark at end of last lecture) that given two points in $\tilde{M}$ there is a unique geodesic (hence minimizing) joining them. Fix $\tilde{p} \in \tilde{\gamma}$ and define $\varphi : H \to \mathbb{R}$, $\varphi(h) = \pm \operatorname{dist}(\tilde{p}, h(\tilde{p}))$, according to $h(\tilde{p})$ being before/after $\tilde{p}$ along $\tilde{\gamma}$. Then φ is a group homomorphism and injective, so the subgroup $\varphi(H) < \mathbb{R}$ is either dense or isomorphic to $\mathbb{Z}$. It cannot be dense because $\pi_1 M$ acts properly discontinuously on $\tilde{M}$. $\square$



Cor. If M_1, M_2 are closed manifolds, then $M_1 \times M_2$ does not admit any metric with $\sec < 0$.

Pf. Suppose $(M_1 \times M_2, g)$ has $\sec < 0$; in particular, by Cartan–Hadamard, $\widetilde{M_1 \times M_2} \cong \widetilde{M}_1 \times \widetilde{M}_2 \cong \mathbb{R}^n$ so $\pi_1 M_i \neq \{1\}$ for $i = 1, 2$. Indeed, if, say $\pi_1 M_1 = \{1\}$, then $\widetilde{M}_1 \times \widetilde{M}_2 \cong M_1 \times \widetilde{M}_2 \not\cong \mathbb{R}^n$ because M_1 is closed. Let $h_i \in \pi_1 M_i$ be nontrivial elements and $\langle h_i \rangle$ the corresponding cyclic subgroups. Then $H = \langle h_1 \rangle \oplus \langle h_2 \rangle$ is an Abelian subgroup of $\pi_1 M$ that is not isomorphic to $\mathbb{Z}$. $\square$

E.g., T^n does not have any metric with $\sec < 0$

$\Sigma^2 \times S^1$ does not have any metric with $\sec < 0$

e.g., closed surface of genus ≥ 2

But they have metrics with $\sec \leq 0$!

Rmk. Byers showed that if a closed manifold (M, g) with $\sec < 0$ has $H < \pi_1 M$ a nontrivial <u>solvable</u> subgroup, then $H \cong \mathbb{Z}$. Moreover, $\pi_1 M$ does not admit finite-index cyclic subgroups.

Rmk. It is unknown if any $M_1 \times M_2$ where M_i are closed, $\pi_1 M_i = \{1\}$, can admit metrics with $\sec > 0$. The case $M_1 = M_2 = S^2$ is known as the Hopf Question.

Rmk. The analogous question for $\sec > 0$ was proposed by Chern in 1965: If (M, g) is a closed Riem. mfld w/ $\sec > 0$, and $H < \pi_1 M$ is Abelian, then is H cyclic? By Synge, the answer is affirmative in even dimensions. Ravi Shankar (1998) found infinitely many counter-examples in dimension 7, as there are homogeneous spaces M^7 with $\sec > 0$ and a free action by $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which is isometric. Thus, $M^7 / \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a closed manifold with $\sec > 0$ and fundamental group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Lecture 22 4/19/2024

Comparison theorems for sec: Rauch I, II $\rightsquigarrow$ Preissmann, Cartan–Hadamard, Myers, Synge, ...

also with second variation of length/energy.

Toponogov $\rightsquigarrow$ Gromov's bounds on generators of π_1, total Betti #, ... Alexandrov geometry

maybe later?

Comparison theorem for Ric: Bishop Volume Comparison (today) $\rightsquigarrow$ Milnor (π_1 has polynomial growth)

Rigidity in Myers, (today)

Gromov's compactness theorem, ...

Recall Riccati equation:

$$S' + S^2 + R_V = 0 \quad (\text{in } \mathrm{Sym}^2 E) \quad \overset{\mathrm{tr}}{\rightsquigarrow} \quad \mathrm{tr}\, S' + \underbrace{\mathrm{tr}(S^2)}_{} + \mathrm{Ric}(V) \overset{\circledast}{=} 0 \quad (\text{in } \mathbb{R})$$

this is not a function of tr S ...

like the actual shape operator...

Since $S(V) = \nabla_V V = 0$, can restrict S to $V^\perp$, $S : V^\perp \to V^\perp$, and $S \in \mathrm{Sym}^2 V^\perp$. Let $a = \dfrac{\mathrm{tr}\, S}{n-1}$, and note that

$$S = a\,\mathrm{Id} + S_0, \quad \text{where} \quad \mathrm{tr}\, S_0 = 0.$$

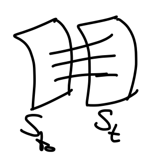
"trace-free part"

So $\langle S_0, I \rangle = 0$. recall $\langle A, B \rangle = \mathrm{tr}\, AB$

Then $\mathrm{tr}(S^2) = \|S\|^2 = a^2 \|\mathrm{Id}\|^2 + \|S_0\|^2 = (n-1)a^2 + \|S_0\|^2$ so $\circledast$ gives

$$a' + a^2 + r = 0, \quad \text{where} \quad r = \frac{1}{n-1}\left(\|S_0\|^2 + \mathrm{Ric}(V)\right) \geq \frac{\mathrm{Ric}(V)}{n-1}$$

Rmk: Geometrically, $a(t) = \dfrac{H}{n-1}$ where $H = \mathrm{tr}\, S$ is the mean curvature of S_t.



Thm. Suppose $S : [t_0, t_1) \to \mathrm{Sym}^2 V^\perp$ is the maximal solution to $S' + S^2 + R = 0$, where $R : \mathbb{R} \to \mathrm{Sym}^2 V^\perp$ is given. Suppose $\exists K \in \mathbb{R}$ s.t.

(i) $\mathrm{tr}\, R \geq (n-1) K$

(ii) $\mathrm{tr}\, S(t_0) \leq (n-1)\bar{a}(t_0)$

where $\bar{a} : [t_0, t_2) \to \mathbb{R}$ is the maximal solution to $\bar{a}' + \bar{a}^2 + K = 0$. Let $a = \dfrac{\mathrm{tr}\, S}{n-1}$

Then $t_1 \leq t_2$ and $a(t) \leq \bar{a}(t)$ for all $t \in [t_0, t_1)$.

Pf: Apply ODE comparison from Lectures 19-20:

> **Thm.** Let $R_1, R_2 : \mathbb{R} \to \mathrm{Sym}^2 E$ be smooth curves with $R_1(t) \geq R_2(t)$, $\forall t$.
> Let $S_i : [t_0, t_i) \to \mathrm{Sym}^2 E$ be the maximal solutions to $S_i' + S_i^2 + R_i = 0$
> If $S_1(t_0) \leq S_2(t_0)$, then $t_1 \leq t_2$ and $S_1(t) \leq S_2(t)$ for all $t \in [t_0, t_1)$.

setting $E = \mathbb{R}$, $R_1 = r$, $R_2 = K$, so (i) $\Rightarrow r \geq K \Rightarrow R_1 \geq R_2$

$$S_1' + S_1^2 + R_1 = 0 \iff a' + a^2 + r = 0$$
$$S_2' + S_2^2 + R_2 = 0 \iff \bar{a}' + \bar{a}^2 + K = 0.$$

$\square$

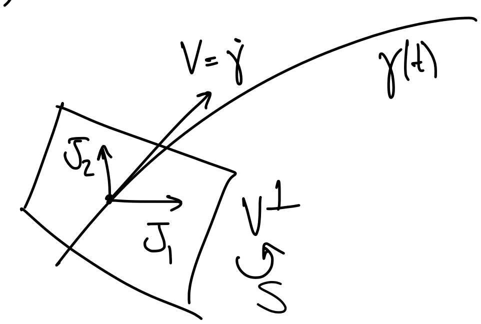
Rmk: Above result remains true if $\bar{a}$ has a pole at t_0; namely

$$S(t) \sim \frac{1}{t - t_0} \mathrm{Id}, \qquad \bar{a} = \frac{sn_K'}{sn_K} \quad \text{where} \begin{cases} sn_K'' + K\, sn_K = 0 \\ sn_K(t_0) = 0 \\ sn_K'(t_0) = 1. \end{cases}$$

Let $J_1, \dots, J_{n-1}$ be Jacobi fields along γ that form a basis of solutions to

$$J' = SJ \qquad (S : V^\perp \to V^\perp)$$

all identified via parallel transport



and set $j = \det(J_1, J_2, \dots, J_{n-1})$.

$$j' = \det(J_1', J_2, \dots, J_{n-1}) + \det(J_1, J_2', J_3, \dots, J_{n-1}) + \dots + \det(J_1, \dots, J_{n-1}')$$
$$= \det(SJ_1, J_2, \dots, J_{n-1}) + \det(J_1, SJ_2, J_3, \dots, J_{n-1}) + \dots + \det(J_1, \dots, SJ_{n-1})$$
$$= \mathrm{tr}\, S \cdot \det(J_1, \dots, J_{n-1}) = \mathrm{tr}\, S \cdot j$$

or: $d(\det)_I X = \mathrm{tr}\, X$; more generally, if A is invertible, $d(\det)_A X = (\det A) \mathrm{tr}(A^{-1} X)$

Let $j(t) = \det A(t)$, where $A(t) = (J_1(t), \dots, J_{n-1}(t))$.

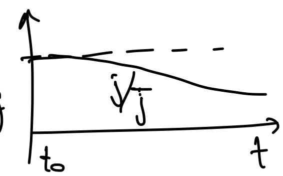
$$j'(t) = d(\det)_{A(t)} A'(t) = (\det A(t))\, \mathrm{tr}\left(A(t)^{-1} A'(t)\right)$$
$$= j(t) \cdot \mathrm{tr}\left(A^{-1}(t) \cdot S(t) \cdot A(t)\right) = (\mathrm{tr}\, S) \cdot j$$

so we have $j' = (n-1)\, a\, j$

Thm. Let $S : [t_0, t_1) \to \mathrm{Sym}^2 V^\perp$ and $a = \frac{1}{n-1} \operatorname{tr} S$ be s.t. $a \le \bar{a}$, and $j' = (n-1) a j$. Choose $\bar{j}$ s.t. $\bar{j}' = (n-1)\bar{a}\bar{j}$. Then $j/\bar{j}$ is nonincreasing.

Pf: Once again, apply ODE comparison from before!

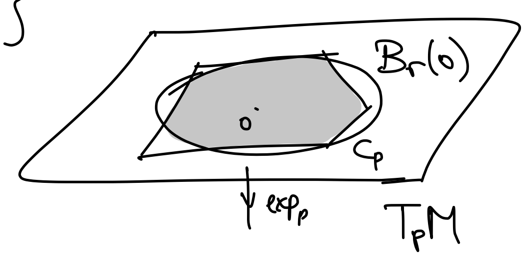
$$(n-1)a \le (n-1)\bar{a} \implies \left(\log \frac{j}{\bar{j}}\right)' \le 0 \implies j/\bar{j} \text{ nonincreasing}$$

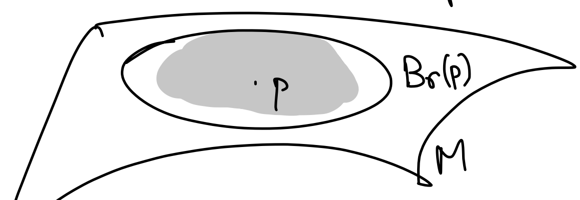


Thm (Bishop Volume Comparison). Let (M^n, g) be a Riem. mfld with $\mathrm{Ric} \geq (n-1)K$ and $\overline{M}$ be the simply-connected Riem. mfld with $\sec_{\overline{M}} \equiv K$. Then $\forall p \in M$, $\mathrm{Vol}(B_r(p)) \le \mathrm{Vol}(\overline{B_r})$, where $B_r(p) \subset M$ and $\overline{B_r} \subset \overline{M}$ are balls of radius r. Moreover, equality holds if and only if $B_r(p) \underset{\text{isom}}{\cong} \overline{B_r}$.

Pf: We will show that $r \mapsto \dfrac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\overline{B_r})}$ is nonincreasing; the conclusion follows since $\displaystyle\lim_{r \downarrow 0} \frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\overline{B_r})} = 1$ because both approach Euclidean balls as $r \downarrow 0$.

Let $\mathrm{cut}(v) = \max\{t_* > 0 : \exp_p tv \text{ is min. geod. on } [0, t_*]\}$ and $C_p = \{tv : t \le \mathrm{cut}(v), \|v\| = 1\} \subset T_pM$. Then $\exp_p : C_p \to M$ is a diffeom. onto its image, so:





$\exp_p C_p = \mathrm{Cut}(p)$ is the cut locus of p

$$\mathrm{Vol}(B_r(p)) = \int_{B_r(p)} 1 \, d\mathrm{vol} = \int_{\exp_p(B_r(0) \cap C_p)} 1 \, d\mathrm{vol}$$

Change of variables formula

$$\overset{\text{}}{=} \int_{B_r(0) \cap C_p} \det\left(d(\exp_p)_u\right) du$$

Polar coord.

$$= \int_{S^{n-1}(1)} \int_0^{r(v)} \det\left(d(\exp_p)_{tv}\right) t^{n-1} \, dt \, dv$$

Recall: $B_r(p) = \exp_p(B_r(0)) = \exp_p(B_r(0) \cap C_p)$

where $r(v) = \min\{r, \mathrm{cut}(v)\}$ for $v \in T_pM$, $\|v\| = 1$, i.e. $v \in S^{n-1}(1) \subset T_pM$.

Since $d(\exp_p)_{tv}\, e_i = \frac{1}{t}\left(d(\exp_p)_{tv}\, te_i\right) = \frac{1}{t} J_i(t)$ is the Jacobi field along $t \mapsto \exp_p tv$ with $J_i(0)=0$ and $J_i'(0)=e_i$, it follows that

$$\det\left(d(\exp_p)_{tv}\right) = \frac{1}{t^{n-1}} \det\left(J_1(t), \ldots, J_{n-1}(t)\right)$$ and hence:

$$\mathrm{Vol}(B_r(p)) = \int_{S^{n-1}(1)} \int_0^{r(v)} \underbrace{\det(J_1(t), \ldots, J_{n-1}(t))}_{j_v(t)}\, dt\, dv$$

if needed, extend $j_v(t)$ as $j_v(t)=0$ for $t > cut(v)$.

By previous result, $j_v(t)/\bar{j}(t)$ is non increasing on $[0,r]$, where $\bar{j}(t) = \det(\bar{J}_1, \ldots, \bar{J}_{n-1})$, for corresponding Jacobi fields $\bar{J}_i$ on $\bar{M}$.

Set $$q(t) = \frac{1}{\mathrm{Vol}(S^{n-1}(1))} \int_{S^{n-1}(1)} \frac{j_v(t)}{\bar{j}(t)}\, dv,$$ which is also non-increasing (because it is an average of nonincreasing quantities). As before,

$$\mathrm{Vol}(\bar{B}_r) = \int_{S^{n-1}(1)} \int_0^r \bar{j}(t)\, dt\, dv \overset{\substack{\text{(space w/ sec} \equiv K \\ \text{is isotropic)}}}{=} \mathrm{Vol}(S^{n-1}) \int_0^r \bar{j}(t)\, dt$$

Thus,

$$\frac{\mathrm{Vol}(B_r(p))}{\mathrm{Vol}(\bar{B}_r)} = \frac{\int_{S^{n-1}(1)} \int_0^r j_v(t)\, dt\, dv}{\mathrm{Vol}(S^{n-1}(1)) \cdot \int_0^r \bar{j}(t)\, dt} \overset{\text{Fubini}}{=} \frac{\int_0^r q(t) \cdot \bar{j}(t)\, dt}{\int_0^r \bar{j}(t)\, dt}$$

is non increasing, because RHS is the ($\bar{j}$ – weighted) average(*) of the non increasing function $q(t)$ over growing intervals.

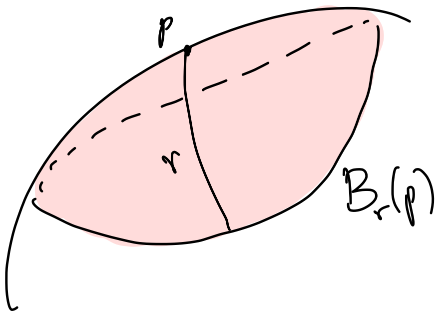
(*) More explicitly: if $\phi, \psi > 0$, and $t \mapsto \dfrac{\phi(t)}{\psi(t)}$ is non increasing, then

$$r \mapsto \frac{\int_0^r \phi(t)\,dt}{\int_0^r \psi(t)\,dt} = \frac{\int_0^{\bar r} \frac{\phi(s)}{\psi(s)}\,ds}{\int_0^{\bar r} ds}$$ is non increasing, where $\begin{cases} ds = \psi(t)\,dt \\ \bar r = s(r). \end{cases}$

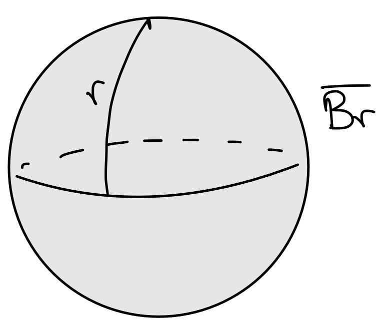
Rigidity statement follows from rigidity statements in ODE comparison: if $\forall v \in S^{n-1}(1)$, $\forall\, 0 \le t \le r$, $j_v(t) = \bar{J}(t)$, then $a(t) = \bar a(t)$, for all $0 \le t \le r$; so $R(t) = \bar R(t) = K\,\mathrm{Id}$. Thus $B_r(p)$ has constant curvature $\sec \equiv K$ and is hence isometric to $\bar{B_r}$. $\square$

<u>Remark:</u> Similarly, one can prove $r \mapsto \dfrac{\mathrm{Vol}(\partial B_r(p))}{\mathrm{Vol}(\partial \bar{B_r})}$ is non increasing.

Geometrically:



$\mathrm{Ric} \geq (n-1)K$



$\sec \equiv K$
(so $\mathrm{Ric} = (n-1)K$)

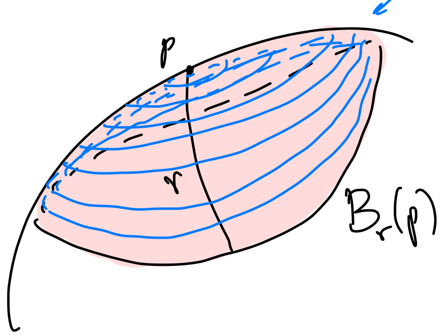
$\mathrm{Vol}(B_r(p)) \leq \mathrm{Vol}(\bar{B_r})$

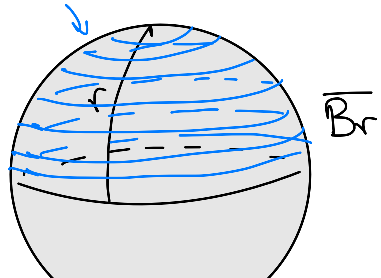
$\Updownarrow$

$B_r(p) \underset{\text{isom}}{\cong} \bar{B_r}$

With <u>stronger</u> control on curvature $\sec \geq K$ we know that:

longer / shorter





so "integrating" get the above.

BUT

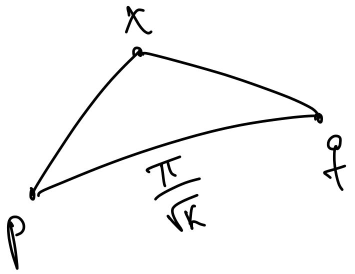
$\mathrm{Ric} \geq K(n-1)$ is enough for this "integral" control.

Rigidity in Myers Theorem

(Originally by Shiu-Yuen Cheng, with different proof)
↳ student of S.S. Chern

$$\boxed{\textbf{Thm.}\ \text{Let } (M^n,g) \text{ be a complete Riem. mfld with } Ric \geq K\cdot(n-1) > 0 \text{ and } \operatorname{diam}(M^n,g) = \operatorname{diam}\left(S^n\left(\tfrac{1}{\sqrt{K}}\right)\right) = \tfrac{\pi}{\sqrt{K}}.\ \text{Then } (M^n,g) \underset{\text{isom.}}{\cong} S^n\left(\tfrac{1}{\sqrt{K}}\right).}$$

Pf: Let $p,q\in M$ be points at maximal distance, i.e. $\operatorname{dist}(p,q)=\frac{\pi}{\sqrt{K}}$. Then, for all $r>0$, the balls $B_r(p)$ and $B_{\frac{\pi}{\sqrt{K}}-r}(q)$ are disjoint: if $d(p,x)<r$ and $d(x,q)<\frac{\pi}{\sqrt{K}}-r$, then



$$\frac{\pi}{\sqrt{K}} = d(p,q) \leq d(p,x)+d(x,q) < \frac{\pi}{\sqrt{K}}$$

so no such x can exist. Thus,

$$M \supseteq B_r(p)\ \dot{\cup}\ B_{\frac{\pi}{\sqrt{K}}-r}(q) \quad \text{(disjoint union)}$$

hence $\operatorname{Vol}(M) \overset{\circledast}{\geq} \operatorname{Vol}(B_r(p)) + \operatorname{Vol}(B_{\frac{\pi}{\sqrt{K}}-r}(q))$. From Bishop Vol. Comp.,

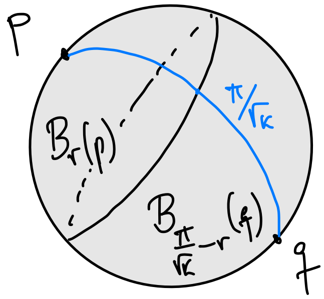
$$r \longmapsto \frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(\overline{B_r})} \text{ is non increasing; in particular,}$$

$$\frac{\operatorname{Vol}(B_r(x))}{\operatorname{Vol}(\overline{B_r})} \geq \frac{\operatorname{Vol}(B_{\frac{\pi}{\sqrt{K}}}(x))}{\operatorname{Vol}(\overline{B_{\frac{\pi}{\sqrt{K}}}})} = \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^n(1/\sqrt{K}))} \quad \text{b/c} \begin{cases} \overline{B_{\frac{\pi}{\sqrt{K}}}} = S^n(1/\sqrt{K}) \\ B_{\frac{\pi}{\sqrt{K}}}(x) = M \end{cases}$$

i.e. $\operatorname{Vol}(B_r(x)) \geq \dfrac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^n(1/\sqrt{K}))}\operatorname{Vol}(\overline{B_r})$. Thus, applying this in $\circledast$:

$$\operatorname{Vol}(M) \geq \frac{\operatorname{Vol}(M)}{\operatorname{Vol}(S^n(1/\sqrt{K}))}\underbrace{\left(\operatorname{Vol}(\overline{B_r}) + \operatorname{Vol}(\overline{B_{\frac{\pi}{\sqrt{K}}-r}})\right)}_{\operatorname{Vol}(S^n(1/\sqrt{K}))} = \operatorname{Vol}(M), \text{ so all}$$

the inequalities using Bishop Vol. Comp. above are equalities. Thus, from rigidity in the equality case of Bishop Vol. Comp., we have $B_r(p) \underset{\text{isom}}{\cong} \overline{B_r}$ and $B_{\frac{\pi}{\sqrt{K}}-r}(q) \underset{\text{isom}}{\cong} \overline{B_{\frac{\pi}{\sqrt{K}}-r}}$, thus $M \underset{\text{isom}}{\cong} S^n(1/\sqrt{K})$.



Indeed, there is no room for any $M \setminus \overline{(B_r(p) \cup B_{\frac{\pi}{\sqrt{K}}-r}(q))}$ because that would increase the diameter. $\square$

Open problem: If (M^n, g) has $\mathrm{Ric} \geq (n-1)K > 0$ and $\mathrm{Vol}(M,g) > \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{K}))$, then $M \underset{\substack{\text{homeo?}\\ \text{diffeo?}}}{\cong} S^n$.

Exercise: a) Find counter-example with $\mathrm{Vol}(M,g) = \frac{1}{2}\mathrm{Vol}(S^n(1/\sqrt{K}))$.

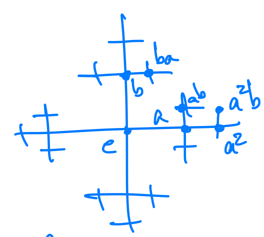
HW5

b) Prove that (M^n, g) as above is simply-connected.

Hint: if M is not simply connected, take its universal covering.

Lecture 23 5/1/2024

A quick taste of Geometric Group Theory:



Cayley graph of $F = \langle a, b \rangle$

- If Γ is finitely generated, fix a finite generating set G, with $e \in G$ and $G^{-1} = G$. Then define growth function for Γ:

$$N_K^G = \#\{g \in \Gamma : g = g_1 \cdots g_K, \text{ with } g_i \in G\}$$

In terms of the Cayley graph with the word metric, this is the cardinality of the closed ball of radius K around $e \in \Gamma$

\# of group elements that can be written as product of K generators in the fixed generating set G.

- If G' is another choice of generating set for Γ as above, then $N_K^{G'} \geq N_{CK}^{G}$ and $N_K^{G} \geq N_{DK}^{G'}$ for some constants $C, D > 0$, so can ignore choice of gen. set G for questions below.

• Q: How does N_k grow with k? Polynomially? Exponentially?

> **Thm** (Milnor '68). If (M^n, g) is complete and has $\mathrm{Ric} \geq 0$, then any finitely generated subgroup $\Gamma < \pi_1 M$ has $\underline{N_k \leq C \cdot k^n}$.

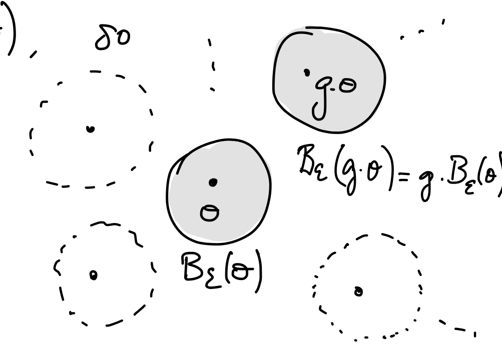
i.e., "polynomial growth"

Pf: Choose $o \in \widetilde{M}^n$, and let $V(r) = \mathrm{Vol}(B_r(o))$. By Bishop Volume Comp.,

$$V(r) \leq \mathrm{Vol}\left(B_r^{\mathbb{R}^n}(0)\right) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n.$$

Let $G = \{g_1, \ldots, g_p\}$ be the fixed generating set for $\Gamma < \pi_1 M$ and $\mu = \max_{1 \leq i \leq p} \mathrm{dist}(o, g_i \cdot o)$.

Then $B_{\mu \cdot k}(o)$ has at least N_k^G distinct points of the form $g \cdot o$, with $g \in \Gamma$. Choose $\varepsilon > 0$ s.t. $g \cdot B_\varepsilon(o) \cap B_\varepsilon(o) = \emptyset$ if $g \neq e$. Then $B_{\mu \cdot k + \varepsilon}(o)$ has at least N_k^G disjoint subsets of the form $g \cdot B_\varepsilon(o)$, so



$$N_k^G \cdot V(\varepsilon) = \mathrm{Vol}\Big(\coprod_{\substack{g = g_{i_1} \cdots g_{i_k} \\ g_i \in G}} g \cdot B_\varepsilon(o)\Big) \leq V(\mu k + \varepsilon)$$

Thus
$$N_k^G \leq \frac{V(\mu k + \varepsilon)}{V(\varepsilon)} \overset{\text{Bishop}}{\leq} \frac{\widetilde{C}\,(\mu k + \varepsilon)^n}{V(\varepsilon)} \leq C \cdot k^n. \qquad \square$$

(recall: M compact $\Rightarrow$ $\pi_1 M$ is finitely generated)

> **Thm** (Milnor '68). If (M, g) is a closed Riem. mfld with $\sec < 0$, and $\pi_1 M = \langle G \rangle$, $|G| < \infty$, then $\underline{N_k^G \geq a^k}$ for some $a > 1$.

i.e., "exponential growth"

Ex: Fundamental group of hyperbolic manifold Σ^n has exponential growth; thus, cannot be π_1 of mfld w/ $\mathrm{Ric} \geq 0$.
So, cannot "improve" the above Thm to $\mathrm{scal} > 0$, as $\Sigma^2 \times S^{n-2}(\varepsilon)$ has $\mathrm{scal} > 0$ for $n \geq 4$ and $\varepsilon > 0$ suff. small, if Σ^2 is a hyperbolic surface.

Conjecture (Milnor, 1968). If (M^n, g) is complete and has $\mathrm{Ric} \geq 0$, then $\pi_1 M$ is finitely generated.

- For $n=3$, it was proven by [Liu, 2013] and indep. [Pan, 2017].

  Inventiones paper, uses minimal surfaces

  Crelle paper, uses Cheeger–Colding theory and $\mathrm{Ric} \geq 0$ limit spaces

- In November 2023, a counter-example (M^7, g) with $\mathrm{Ric} \geq 0$ and $\pi_1 M^7 = \mathbb{Q}/\mathbb{Z}$ was announced by Bruè–Naber–Semola, using a sophisticated gluing method to produce a "smooth fractal structure".

One of the founding achievements of Geometric Group Theory is:

Thm (Gromov '81) A finitely generated group Γ has polynomial growth if and only if Γ is virtually nilpotent ($\exists N \triangleleft \Gamma$ nilpotent with $[\Gamma : N] < \infty$.)

So, if $\Gamma < \pi_1 M$ is fin. gen. and M has $\mathrm{Ric} \geq 0$, then Γ is virtually nilpotent. Conversely, if Γ is fin. gen. and virtually nilpotent, then $\Gamma = \pi_1 M$ for some M with $\mathrm{Ric} \geq 0$ (Wilking '2000)

This paper also shows that if a counter-example M to Milnor's conjecture exists, then it has a covering space $\hat{M} \to M$ with $\pi_1 \hat{M}$ abelian and not fin. gen (e.g., $\pi_1 \hat{M} = \mathbb{Q}/\mathbb{Z}$.)

Stronger results about $\pi_1 M$ can be proven with stronger curvature assumptions:

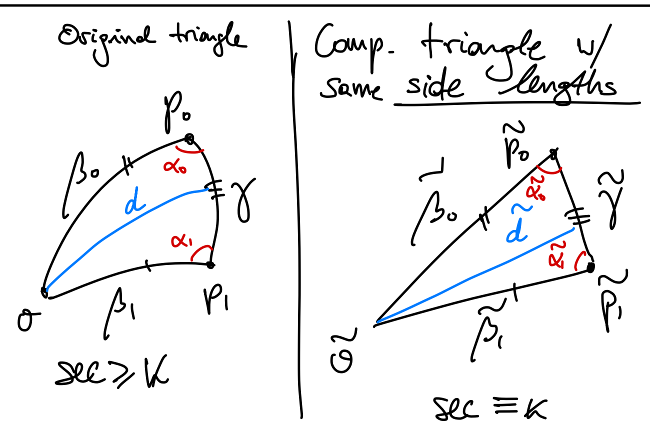
Toponogov Triangle Comparison

See, e.g., Eschenburg's Comparison Geometry Notes for a proof.

Here and throughout: if $K>0$, then assume all lengths are $< \frac{\pi}{\sqrt{K}}$.

Triangle Version

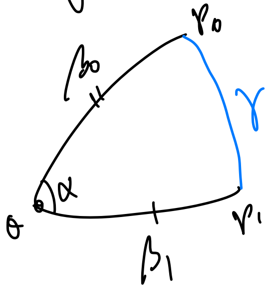
If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, $\gamma : [0, L] \to M$ geod from p_0 to p_1, β_i min. geod from o to p_i, then $d = \mathrm{dist}_g(o, \gamma(t)) \geq \tilde{d} = \mathrm{dist}_{\tilde{g}}(\tilde{o}, \tilde{\gamma}(t))$ for all $t \in [0, L]$ and $\alpha_i \geq \tilde{\alpha}_i$.



Hinge Version

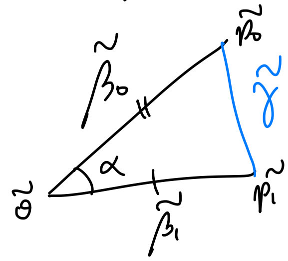
If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, β_i min. geod. from o to p_i, Then $\ell(\gamma) \leq \ell(\tilde{\gamma})$; where $\gamma, \tilde{\gamma}$ are the min. geod. that close the hinge: $\ell(\gamma) = \mathrm{dist}_g(p_0, p_1)$, $\ell(\tilde{\gamma}) = \mathrm{dist}_{\tilde{g}}(\tilde{p}_0, \tilde{p}_1)$.

Original triangle



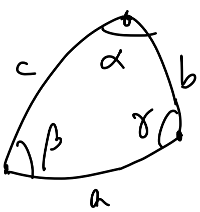
$\sec \geq K$

Comp. triangle w/ same hinge: $\ell(\beta_i) = \ell(\tilde{\beta}_i)$, $\alpha = \tilde{\alpha}$



$\sec \equiv K$

Corollary: A geodesic triangle on a manifold with $\sec \geq 0$ satisfies



(i) $\ell(c)^2 \leq \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma$ $\quad$ ℓ = length

(ii) $\alpha + \beta + \gamma \geq \pi$ $\quad$ If $\sec > 0$, then get strict inequalities.

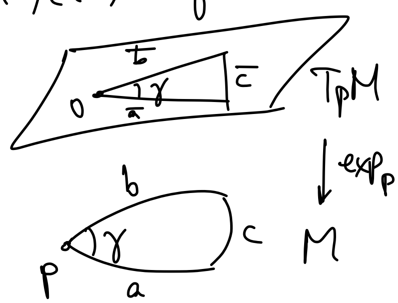
Pf: (i) is immediate:

$$\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \overset{\text{Gauss Lemma}}{=} \ell(\bar{a})^2 + \ell(\bar{b})^2 - 2\ell(\bar{a})\ell(\bar{b})\cos\gamma$$

$$\overset{\text{Law of Cosines in } \mathbb{R}^2}{=} \ell(\bar{c})^2$$

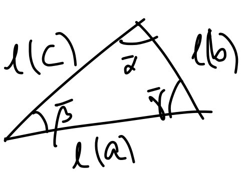
$$\overset{\text{Toponogov (Hinge)}}{\geq} \ell(c)^2.$$

where $a = \exp_p \bar{a}$, $b = \exp_p \bar{b}$, $c = \exp_p \bar{c}$.



(ii) Follows from (i) as in the $\sec \leq 0$ case: build comparison triangle in $\mathbb{R}^2$ with side lengths $\ell(a), \ell(b), \ell(c)$, and angles $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$. Then $\ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\gamma \geq \ell(c)^2$

$$= \ell(a)^2 + \ell(b)^2 - 2\ell(a)\ell(b)\cos\bar{\gamma}$$



So $\cos\gamma \leq \cos\bar{\gamma}$ hence $\gamma \geq \bar{\gamma}$. Similarly for α, β and get

$$\alpha + \beta + \gamma \geq \bar{\alpha} + \bar{\beta} + \bar{\gamma} = \pi. \qquad \square$$

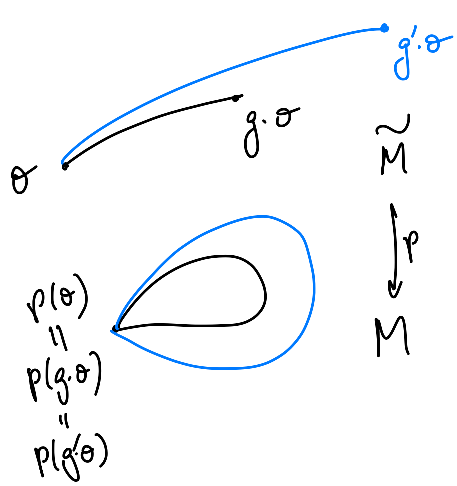
Combining above w/ earlier work on $\sec \leq 0$

As before...

Cor: (M^n, g) has $\sec \geq 0$ (≤ 0) iff $\forall p \in M$, $\exp_p : C_p \subset T_pM \xrightarrow{\cong} M$ is distance non-increasing (non-decreasing).

Thm (Gromov 1978). If (M^n, g) has $\sec \geq 0$, then $\pi_1 M$ can be generated by $\leq \sqrt{2n\pi} \cdot 2^{n-2}$ elements. If (M^n,g) has $\sec \geq -K^2$ and $\operatorname{diam}(M) \leq D$, then $\pi_1 M$ can be generated by $\leq \frac{1}{2}\sqrt{2n\pi}\,\big(2 + 2\cosh(2KD)\big)^{\frac{n-1}{2}}$. Note: If $K \to 0$, then this becomes $\sqrt{2n\pi} \cdot 2^{n-2}$.

Pf: (Case $K=0$). Fix $o \in \widetilde{M}$ and consider the isometric action of $\Gamma = \pi_1 M$, by deck transformations. Define displacement of $g \in \Gamma$: $|g| = \operatorname{dist}(o, g\cdot o)$.



Clearly, a min. geod. from o to $g \cdot o$ in $\widetilde{M}$ projects to geodesic loop based at $p(o) \in M$, which has minimal length in its homotopy class. For any given $R > 0$, there are only finitely many $g \in \Gamma$ with $|g| \leq R$, because otherwise an infinite seq. $g_i \in \Gamma$ with $|g_i| \leq R$ would produce an infinite seq. $g_i \cdot o$ of points in $B_R(o)$, which has a limit and contradicts the covering property.

Thus, we can define $g_1 \in \Gamma$ s.t. $|g_1| = \min_{g \in \Gamma} |g|$, and $g_2 \in \Gamma$ with $|g_2| = \min_{g \in \Gamma \setminus \langle g_1 \rangle} |g|$; inductively, define a sequence $g_1, g_2, \ldots \in \Gamma$ of generators ("short basis") with $|g_1| \leq |g_2| \leq \ldots$ and $|g_{i+1}| = \min_{g \in \Gamma \setminus \langle g_1, \ldots, g_i \rangle} |g|$. (Keep adding elements g_i until a set of generators is achieved!)

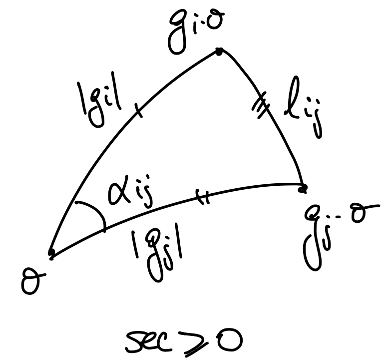
Set $l_{ij} = \operatorname{dist}(g_i \cdot o, g_j \cdot o)$ for all $i < j$. Then $l_{ij} \geq |g_j|$, for otherwise $\bar{g} = g_i^{-1} g_j$ would have

$$|\bar{g}| = l_{ij} < |g_j| \quad \text{and} \quad \langle g_1, \ldots g_i, \ldots, g_j \rangle = \langle g_1, \ldots, g_i, \ldots, \bar{g} \rangle$$

hence contradict the min. choice of g_j above.

Note that all sides of the triangles $o, g_i \cdot o, g_j \cdot o$ are min geodesics.

By Toponogov, applied to the triangle $g_i\cdot o, g_j\cdot o, o$, we have that $\alpha_{ij} \geq \tilde{\alpha}_{ij}$.



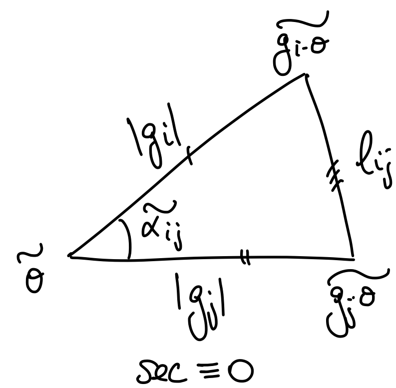
Law of cosines in $\mathbb{R}^2$:

$$l_{ij}^2 = |g_i|^2 + |g_j|^2 - 2|g_i||g_j| \cos\tilde{\alpha}_{ij}$$

$$\Rightarrow \quad \cos(\tilde{\alpha}_{ij}) = \frac{|g_i|^2 + |g_j|^2 - l_{ij}^2}{2|g_i|\cdot|g_j|}$$

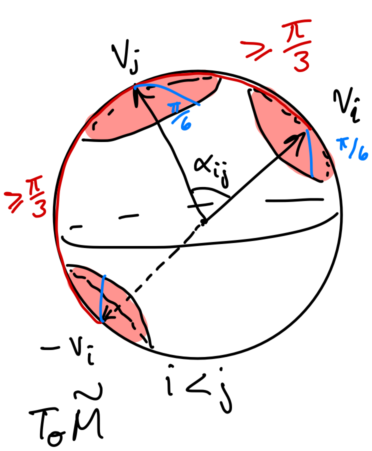
$|g_i| \leq |g_j| \leq l_{ij}$ if $i<j$

$$\leq \frac{|g_i|^2 + \cancel{(|g_j|^2 - |g_j|^2)}}{2\cdot |g_i|^2} = \frac{1}{2}$$



$$\Rightarrow \quad \alpha_{ij} \geq \tilde{\alpha}_{ij} \geq \frac{\pi}{3}.$$

Let $v_i \in T_o\tilde{M}$ be the unit vector tangent to the min. geod. from o to $g_i\cdot o$. By the above, the distance (on the unit sphere in $T_o\tilde{M}$) between v_i and v_j is $\alpha_{ij} \geq \frac{\pi}{3}$, so the balls of radius $\frac{\pi}{6}$ centered at v_i and v_j must be disjoint. (This already proves there can be only finitely many v_i's, hence finitely many g_i's so $\Gamma = \pi_1 M$ is finitely generated.) Moreover, as $|g_i^{-1}| = |g_i|$, we must also have that distance from $-v_i$ to v_j is $\geq \frac{\pi}{3}$ if $i<j$, therefore the number of v_i's is:



$$\#\{g_i\} = \#\{v_i\} \leq \frac{\mathrm{Vol}(\mathbb{RP}^{n-1}(1))}{\mathrm{Vol}(B^{S^{n-1}}_{\pi/6}(v))}$$

$\leftarrow$ Volume of the set of $\pm v \in S^{n-1} \subset T_o\tilde{M}$.

$\leftarrow$ Volume of each disjoint ball around $\pm v_i \in S^{n-1}$.

Standard computations give:

Volume of spherical ball of radius r is $\geqslant$ volume of Euclidean ball of radius $\sin r$. $(0<r<\pi/2)$

- $\mathrm{Vol}\left(B^{S^{n-1}}_{\pi/6}(v)\right) \geqslant \mathrm{Vol}\left(B^{\mathbb{R}^{n-1}}_{\underbrace{\sin\pi/6}_{=1/2}}(0)\right) = \dfrac{\pi^{\frac{n-1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)2^{n-1}}$

(Γ = Gamma function)

- $\mathrm{Vol}\left(\mathbb{RP}^{n-1}(1)\right) = \frac{1}{2}\mathrm{Vol}\left(S^{n-1}(1)\right) = \dfrac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}$

log-concavity of Γ:
$$\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \leq \sqrt{\frac{n}{2}}$$

So $\#\{g_i\} = \#\{v_i\} \leq \dfrac{\pi^{n/2}\,\Gamma\left(\frac{n+1}{2}\right)2^{n-1}}{\Gamma\left(\frac{n}{2}\right)\cdot \pi^{\frac{n-1}{2}}} = \sqrt{\pi}\,\dfrac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\,2^{n-1} \leq \sqrt{2n\pi}\cdot 2^{n-2}.$

For case $\sec \geqslant -K^2$, see Eschenburg's notes. $\square$

Using <u>Bishop Volume Comparison</u>, <u>Toponogov Triangle Comparison</u>, Critical point theory for distance functions and topological constructions, Gromov proved the following:

> <u>Thm</u> (Gromov '1981).
> i) If (M^n, g) is a complete mfld with $\sec \geqslant 0$, then $\sum_{k=0}^{n} b_k(M) \leq C(n)$.
> ii) If (M^n, g) is a closed mfld with $\sec \geqslant -K^2$ and $\mathrm{diam} \leq D$, then $\sum_{k=0}^{n} b_k(M) \leq C(n)^{1+KD}$.

Cannot replace the hypothesis $\sec \geqslant 0$ to $\mathrm{Ric} > 0$ because:

"Docking station"

> <u>Thm</u> (Sha-Yang '90s). $\forall \ell \in \mathbb{N}$, $\#^\ell S^2 \times S^2$ and $\#^k \mathbb{CP}^2 \#^\ell \overline{\mathbb{CP}}^2$ have $\mathrm{Ric} > 0$.

also $\#^\ell S^n \times S^m$ for any $n, m \geqslant 2$, $\ell \geqslant 1$.

> <u>Thm.</u> (Perelman '97). $\forall \ell \in \mathbb{N}$, $\#^\ell \mathbb{CP}^2$ has a metric with $\mathrm{Ric} > 0$, $\mathrm{diam} = 1$ and $\mathrm{Vol} \geqslant V > 0$.

Thus, since $b_2(\#^\ell S^2 \times S^2) = 2\ell$ and $b_2(\#^k \mathbb{CP}^2 \#^\ell \overline{\mathbb{CP}}^2) = k+\ell$, only <u>finitely many</u> of these manifolds can have $\sec \geqslant 0$. Currently,

<u>Note</u>: $\mathrm{scal} > 0$ is preserved by $\#$; indeed by surgeries of codimension $\geqslant 3$.

- only S^4 and $\mathbb{CP}^2$ are known to have $\sec > 0$ and
- only $S^2 \times S^2$ and $\mathbb{CP}^2 \# \pm \mathbb{CP}^2$ are known to have $\sec \geqslant 0$.

⚠ Related <u>open</u> question: is there a simply-connected closed mfld that admits $\mathrm{scal} > 0$ but does not admit $\mathrm{Ric} > 0$?

Conjecturally, the above is the complete list of simply-connected 4-mflds with $\sec > 0$ and $\sec \geqslant 0$.

<u>Note</u>: As $\ell \nearrow +\infty$, Perelman's $\#^\ell \mathbb{CP}^2$ converges to $B^4 \cup B^4$ flat (double disk: )