Assignment 5

1 Statistical thermodynamics of free classical particles

Partition function of classical particles in $3d$ is defined as

$$Z_{\text{class}} = \int d^3p \int d^3r \exp\left[-\beta E(\mathbf{p},\mathbf{r})\right], \tag{1}$$

where $E(\mathbf{p},\mathbf{r})$ is particle's energy. Note that this expression has the unit of (momentum $\times$ distance)^3, unlike the quantum partition function that is dimensionless. Define the density of states of a free classical particle in a box of volume V. By comparing it with the density of states for a quantum particle in a rigid box, find the missing factor in Eq. (1) that would make the classical partition function match the quantum one. This will define a quantum-mechanical "cell" in the phase space of a classical particle. Show that this quantum-mechanical aspect does not contribute into the internal energy and heat capacity of the classical particles.

2 Classical particles with gravity

Using the distribution function

$$f(\mathbf{p},\mathbf{r}) = \frac{1}{Z_{\text{class}}} \exp\left[-\beta E(\mathbf{p},\mathbf{r})\right]$$

for classical particles with gravity, find the dependence of particle's concentration n and pressure P as the function of the height. Set the minimal height (the earth level) to zero. Calculate the heat capacity of this system and compare it with the one for free particles.

3 Classical harmonic oscillators

Consider classical particles with the potential energy

$$V(\mathbf{r}) = \frac{kr^2}{2}$$

in $3d$. Calculate the partition function, internal energy and heat capacity.

4 Phonons in $1d$ and $2d$

Calculate the internal energy and heat capacity of the system of harmonic phonons in one and two dimensions at low temperatures.

5 Two interacting Ising spins

Consider the model of two coupled spins with the Hamiltonian

$$\hat{H} = -g\mu_B B\left(S_{1,z} + S_{2,z}\right) - JS_{1,z}S_{2,z}.$$

Here B is the external magnetic field and J is the so-called exchange interaction, ferromagnetic for $J > 0$ and antiferromagnetic for $J < 0$. The model above in which only z components of the spins are coupled is called Ising model. The energy levels of this system are given by

$$\varepsilon_{m_1 m_2} = -g\mu_B B\left(m_1 + m_2\right) - Jm_1m_2,$$

where the quantum numbers take the values $-S \leq m_1, m_2 \leq S$. Write down the expression for the partition function of the system. Can it be calculated analytically for a general S? If not, perform the calculation for $S = 1/2$ only. Calculate the internal energy, heat capacity, magnetization induced by the magnetic field, and the magnetic susceptibility. Analyze ferro- and antiferromagnetic cases.