1 Moments of inertia

(5 points) Calculate tensors of inertia with respect to the principal axes of the following bodies:

a) Hollow sphere of mass M and radius R.

b) Cone of the height h and radius of the base R, both with respect to the apex and to the center of mass.

c) Body of a box shape with sides a, b, and c.

Solution: a) By symmetry for the sphere

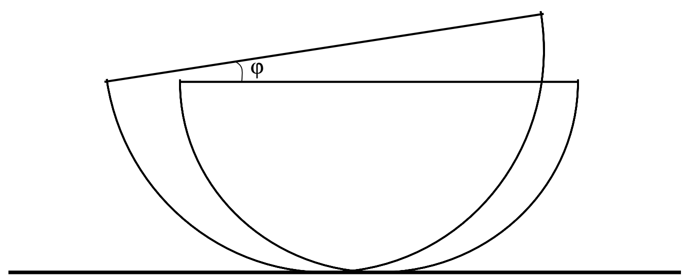
$$I_{\alpha\beta} = I\delta_{\alpha\beta}. \tag{1}$$

One can find I easily noticing that for the hollow sphere is

$$\begin{aligned} I &= \frac{1}{3}\left(I_{xx} + I_{yy} + I_{zz}\right) = \frac{1}{3}\sum_i m_i \left(y_i^2 + z_i^2 + z_i^2 + x_i^2 + x_i^2 + y_i^2\right) \\ &= \frac{2}{3}\sum_i m_i r_i^2 = \frac{2}{3}R^2 \sum_i m_i = \frac{2}{3}MR^2. \end{aligned} \tag{2}$$

b) and c): Standard solutions

2 Half-cylinder



(10 points) Consider a half-cylinder of mass M and radius R on a horizontal plane.

a) Find the position of its center of mass (CM) and the moment of inertia with respect to CM.

b) Write down the Lagrange function in terms of the angle φ (see Fig.)

c) Find the frequency of cylinder's oscillations in the linear regime, $\varphi \ll 1$.

Solution:

(a) First we find the distance a between the CM and the geometrical center of the cylinder. With σ being the density for the cross-sectional surface, so that

$$M = \sigma \frac{\pi R^2}{2}, \tag{3}$$

one obtains

$$
\begin{aligned}
a &= \frac{2\sigma}{M}\int_0^R dx\, x\sqrt{R^2-x^2} = \frac{\sigma}{M}\int_0^{R^2} dy\sqrt{R^2-y} \\
&= -\frac{\sigma}{M}\frac{2}{3}\left(R^2-y\right)^{3/2}\Big|_0^{R^2} = \frac{\sigma}{M}\frac{2}{3}R^3 = \frac{4}{3\pi}R. \qquad (4)
\end{aligned}
$$

The moment of inertia of the half-cylinder with respect to the geometrical center is the same as that of the cylinder,

$$I' = \frac{1}{2}MR^2. \qquad (5)$$

The moment of inertia with respect to the CM I can be found from the relation

$$I' = I + Ma^2 \qquad (6)$$

that yields

$$I = I' - Ma^2 = MR^2\left[\frac{1}{2} - \left(\frac{4}{3\pi}\right)^2\right] = \frac{1}{2}MR^2\left[1 - \frac{32}{(3\pi)^2}\right] \simeq 0.3199MR^2. \qquad (7)$$

(b) The Lagrange function $\mathcal{L}$ is given by

$$\mathcal{L}(\varphi, \dot{\varphi}) = T(\varphi, \dot{\varphi}) - U(\varphi). \qquad (8)$$

The potential energy U of the half-cylinder is due to the elevation of its CM resulting from the deviation of φ from zero. Taking into account that the geometrical center remains at the same height, one obtains the height of the CM

$$y = a(1 - \cos\varphi), \qquad (9)$$

so that

$$U(\varphi) = Mgy = Mga(1 - \cos\varphi). \qquad (10)$$

Kinetic energy of the half-cylinder depends on whether it freely slides on the surface (zero friction) or it rolls on the surface (large enough friction).

In the sliding case the horizontal position of the CM remains the same and only its height y changes. Thus one obtains

$$T = \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}M\dot{y}^2 = \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}Ma^2\sin^2\varphi\,\dot{\varphi}^2 = \frac{1}{2}\left(I + Ma^2\sin^2\varphi\right)\dot{\varphi}^2 \qquad (11)$$

and

$$\mathcal{L} = \frac{1}{2}\left(I + Ma^2\sin^2\varphi\right)\dot{\varphi}^2 - Mga(1 - \cos\varphi). \qquad (12)$$

In the rolling case the horizontal displacement of the geometrical center is given by

$$x_c = -R\varphi, \qquad (13)$$

where we have directed the x axis to the right. Thus the horizontal displacement of the CM is

$$x = x_c + a\sin\varphi = -R\varphi + a\sin\varphi. \qquad (14)$$

The kinetic energy is thus

$$
\begin{aligned}
T &= \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}M\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}M\left[(-R\dot{\varphi} + a\cos\varphi\,\dot{\varphi})^2 + a^2\sin^2\varphi\,\dot{\varphi}^2\right] \\
&= \frac{1}{2}\left[I + M\left(R^2 - 2Ra\cos\varphi + a^2\right)\right]\dot{\varphi}^2 \qquad (15)
\end{aligned}
$$

and the Lagrange function becomes

$$\mathcal{L} = \frac{1}{2}\left[I + M\left(R^2 - 2Ra\cos\varphi + a^2\right)\right]\dot{\varphi}^2 - Mga(1 - \cos\varphi). \qquad (16)$$

c) For $\varphi \ll 1$ the Lagrange function simplifies to

$$\mathcal{L} \cong \frac{1}{2} I \dot{\varphi}^2 - \frac{1}{2} M g a \varphi^2 \tag{17}$$

in the sliding case and to

$$\begin{aligned} \mathcal{L} &\cong \frac{1}{2} \left[I + M (R - a)^2 \right] \dot{\varphi}^2 - \frac{1}{2} M g a \varphi^2 \\ &= \frac{1}{2} I'' \dot{\varphi}^2 - \frac{1}{2} M g a \varphi^2 \end{aligned} \tag{18}$$

in the rolling case. With the help of Eq. (4) and (7) one obtains

$$\begin{aligned} I'' &= I + M (R - a)^2 = \frac{1}{2} M R^2 \left[1 - \frac{32}{(3\pi)^2} \right] + M \left(R - \frac{4}{3\pi} R \right)^2 \\ &= M R^2 \left[\frac{1}{2} - \frac{16}{(3\pi)^2} + 1 - \frac{8}{3\pi} + \frac{16}{(3\pi)^2} \right] \\ &= M R^2 \left[\frac{3}{2} - \frac{8}{3\pi} \right] \simeq 0.6512 M R^2. \end{aligned} \tag{19}$$

These results are easy to interpret. For $\varphi \ll 1$,in the sliding case the half-cylinder approximately rotates around its center of mass, whereas in the rolling case the half-cylinder approximately rotates around its contact point with the surface, $R - a$ being the distance between the CM and the contact point. Since in the latter case the moment of inertia is larger, the frequency of oscillations is smaller.

The Lagrange equation in our case has the form

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0 \tag{20}$$

that in the rolling case becomes

$$I'' \ddot{\varphi} + M g a \varphi = 0 \tag{21}$$

or

$$\ddot{\varphi} + \omega_0^2 \varphi = 0 \tag{22}$$

with the oscillation frequency

$$\omega_0 = \sqrt{\frac{M g a}{I''}} = \sqrt{\frac{g}{R} \frac{\frac{4}{3\pi}}{\frac{3}{2} - \frac{8}{3\pi}}} \simeq 0.807\,3 \sqrt{\frac{g}{R}} \tag{23}$$

that is close to that of a simple pendulum $\sqrt{g/R}$.

In the sliding case one obtains

$$\omega_0 = \sqrt{\frac{M g a}{I}} = \sqrt{\frac{g}{R} \frac{\frac{4}{3\pi}}{\frac{1}{2}\left(1 - \frac{32}{(3\pi)^2}\right)}} = \sqrt{\frac{g}{R} \frac{\frac{2}{3\pi}}{1 - \frac{32}{(3\pi)^2}}} \simeq 1.1519 \sqrt{\frac{g}{R}}. \tag{24}$$

3 Self-rotation

(15 points) How can a cat manage always to land on her feet? How can a system with zero angular momentum set itself into rotation? Consider a person standing on a rotating platform without friction, so that its angular momentum is conserved and is zero, $L_z = 0$. The person having together with the platform a moment of inertia $I_{zz} \equiv I$ moves a point mass m by (massless) hand around a closed contour in the x-y plane, defined in the frame of the platform. By which angle the person on the platform rotates as the mass m makes a full turn?

a) Write down the condition $L_z = 0$ in terms of the projections of the point-mass position and velocity on the axes of the body (platform) frame.

b) Change to the polar coordinates (r, φ) for the point mass and obtain a relation between $d\varphi$ and the infinitesimal rotation of the platform $d\theta$. Let $\Delta\theta$ be the angle of rotation of the platform corresponding to one full turn of the point mass. What do you expect for $\Delta\theta$ in the limits $I \to \infty$ and $I \to 0$?

c) Consider a particular case of rotation of the point mass around a circle with radius R and the center at the distance $l > R$ from the center of the platform and calculate $\Delta\theta$. What is the condition for $\Delta\theta$ to be maximal?

Solution:

a) The angular momentum of the system is given by

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot\boldsymbol{\omega} + [\mathbf{r} \times \mathbf{p}] = \overleftrightarrow{\mathbf{I}} \cdot\boldsymbol{\omega} + m [\mathbf{r} \times \mathbf{v}], \tag{25}$$

where $\boldsymbol{\omega}$ is the angular velocity of the body and plarform and $\mathbf{v}$ is the velocity of the point mass m. The latter consists of two contributions, one due to the rotation of the platform and the other, $\mathbf{u}$, due to the motion with respect to the platform,

$$\mathbf{v} = [\boldsymbol{\omega} \times \mathbf{r}] + \mathbf{u}. \tag{26}$$

Plugging Eq. (26) into Eq. (25) yields

$$\mathbf{L} = \overleftrightarrow{\mathbf{I}} \cdot\boldsymbol{\omega} + mr^2\boldsymbol{\omega} - m (\boldsymbol{\omega} \cdot \mathbf{r})\, \mathbf{r} + m [\mathbf{r} \times \mathbf{u}]. \tag{27}$$

We are interested in the z component of the above equation taking into account $L_z = 0$. Also we take into account that $\boldsymbol{\omega} \| \mathbf{e}_z$. Since the motion of the point mass is defined in the body frame, it is convenient to project $\mathbf{r}$ and $\mathbf{u}$ onto the rotating body-frame axes:

$$\mathbf{r} = r_\alpha \mathbf{e}^{(\alpha)}, \qquad \mathbf{u} = u_\alpha \mathbf{e}^{(\alpha)}. \tag{28}$$

Since for the motion in the x-y plane one has $\boldsymbol{\omega} \cdot \mathbf{r} = \mathbf{0}$, this yields

$$0 = L_z = \left(I + mr^2\right) \omega + m (r_x u_y - r_y u_x) \tag{29}$$

and thus

$$\omega = -\frac{m (r_x u_y - r_y u_x)}{I + mr^2} = -\frac{m (r_x \dot{r}_y - r_y \dot{r}_x)}{I + mr^2}. \tag{30}$$

This formula gives the back reaction of the moving mass m onto the body+platform subsystem.

b) Now we change to the polar coordinates

$$r_x = r \cos\varphi, \qquad r_y = r \sin\varphi. \tag{31}$$

It is clear that the rhs of Eq. (30) should be sensitive to $\dot{\varphi}$ but not to $\dot{r}$. Indeed, using

$$\dot{r}_x = \dot{r} \cos\varphi - r \sin\varphi\, \dot{\varphi}, \qquad \dot{r}_y = \dot{r} \sin\varphi + r \cos\varphi\, \dot{\varphi} \tag{32}$$

one can simplify Eq. (30) to

$$\omega \equiv \dot{\theta} = -\frac{mr^2}{I + mr^2} \dot{\varphi}, \tag{33}$$

or, in the differential form,

$$d\theta = -\frac{mr^2}{I + mr^2} d\varphi. \tag{34}$$

One can see that in the limit $I \to \infty$ the platform does not rotate, $d\theta = 0$. In the limit $I \to 0$ one has $d\theta = -d\varphi$, and the situation depends on whether the center of rotation of the platform is inside or outside the contour circomscribed by the mass m. If it is inside, then the full rotation of m corresponds to $\Delta\varphi = 2\pi$. In this case $\Delta\theta = -2\pi$. If, however, the the contour does not encircle the axis of the platform rotation, then the full turn corresponds to $\Delta\varphi = 0$ and thus one obtains $\Delta\theta = 0$. One can see that in the latter case $\Delta\theta = 0$ in both limits $I \to \infty$ and $I \to 0$, so that there should exist some relation between the parameters for which the person can rotate itself most efficiently.

c) In the particular case of the contour being a circle of radius R with the center at l from the axis of the platform rotation, it is convenient to parametrize the motion of m with the polar angle ψ with respect to the circle R. The relations between (r, φ) and ψ are the following

$$\begin{aligned} r_x &= r\cos\varphi = l + R\cos\psi \\ r_y &= r\sin\varphi = R\sin\psi. \end{aligned} \tag{35}$$

First, one obtains from here

$$r^2 = r_x^2 + r_y^2 = l^2 + 2lR\cos\psi + R^2. \tag{36}$$

Then $d\varphi$ can be found from

$$\begin{aligned} dr_x &= \cos\varphi\, dr - r\sin\varphi\, d\varphi = -R\sin\psi\, d\psi \\ dr_y &= \sin\varphi\, dr + r\cos\varphi\, d\varphi = R\cos\psi\, d\psi. \end{aligned} \tag{37}$$

Elimination of dr yields

$$rd\varphi = R\cos\varphi\cos\psi\, d\psi + R\sin\varphi\sin\psi\, d\psi \tag{38}$$

or

$$r^2 d\varphi = R\,(l + R\cos\psi)\cos\psi\, d\psi + R^2\sin^2\psi\, d\psi = R\,(R + l\cos\psi)\, d\psi. \tag{39}$$

Inserting this and Eq. (36) into Eq. (34) yields

$$d\theta = -\frac{mR\,(R + l\cos\psi)}{I + m\,(l^2 + 2lR\cos\psi + R^2)} d\psi. \tag{40}$$

Thus

$$\begin{aligned} \Delta\theta &= -\int_0^{2\pi} d\psi \frac{mR\,(R + l\cos\psi)}{I + m\,(l^2 + 2lR\cos\psi + R^2)} \\ &= -\pi\left[1 - \frac{I + m\,(l^2 - R^2)}{\sqrt{I^2 + 2Im\,(l^2 + R^2) + m^2\,(l^2 - R^2)^2}}\right]. \end{aligned} \tag{41}$$

One can see that $\Delta\theta \to 0$ for both $I \to 0$ and $I \to \infty$, if $l > R$, as stated above. Also for $l < R$ this formula gives expected results in these limits. One can also see that $\Delta\theta$ decreases with increasing l and vanishes in the limit $l \to \infty$. Obviously the greatest values of $\Delta\theta$ can be achieved at $l = 0$.

Let us consider particular cases of Eq. (41). For $l = 0$ one has

$$\begin{aligned} \Delta\theta &= -\pi\left[1 - \frac{I - mR^2}{\sqrt{I^2 + 2ImR^2 + (mR^2)^2}}\right] \\ &= -\pi\left[1 - \frac{I - mR^2}{I + mR^2}\right] = -2\pi\frac{mR^2}{I + mR^2}. \end{aligned} \tag{42}$$

For $l = R$ the result simplifies to

$$\Delta\theta = -\pi\left[1 - \sqrt{\frac{I}{I + 4mR^2}}\right]. \tag{43}$$

Note that this expression gives $\Delta\theta \to -\pi$ in the limit $I \to 0$. This means that the limits $I \to 0$ and $l \to R$ are not interchangeable.

For $I \gg m\,(l^2 + R^2)$ the expansion of Eq. (41) yields

$$\Delta\theta \cong -2\pi\frac{mR^2}{I} \tag{44}$$

independently of l. This is a plausible result that can be guessed. Note the similarity of this result with Eq. (42)

Finally, one can find the value of I that maximizes $\Delta\theta$ by differentiating Eq. (41) over I and setting the result to zero. This yields the condition

$$I = m\left(l^2 - R^2\right) \tag{45}$$

for the dumbell m that should be used for the maximal performance. Substituting it into Eq. (41) one obtains

$$\Delta\theta_{\max} = -\pi \left[1 - \sqrt{1 - \left(\frac{R}{l} \right)^2} \right]. \tag{46}$$

To make a $3d$ plot of Eq. (41), it is convenient to introduce reduced variables

$$\mu \equiv \frac{mR^2}{I}, \qquad \tilde{l} \equiv \frac{l}{R}, \tag{47}$$

so that

$$\Delta\theta = -\pi \left[1 - \frac{1 + \mu \left(\tilde{l}^2 - 1 \right)}{\sqrt{1 + 2\mu \left(\tilde{l}^2 + 1 \right) + \mu^2 \left(\tilde{l}^2 - 1 \right)^2}} \right]. \tag{48}$$

The $3d$ plot of $-\Delta\theta/\pi$ in the range $0 \leq \mu \leq 20$ and $0 \leq \tilde{l} \leq 2$ is shown below.