CLASSICAL MECHANICS

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Part I
Newtonian Mechanics

Part II
Lagrangian Mechanics

1 The least-action principle and Lagrange equations

Newtonian mechanics is fully sufficient practically. However, it is desirable to find a way to obtain equations of motion from some scalar generating function. For conservative systems, the complete information about the system is contained in the total energy $E = E_k + U$, expressed as a function of coordinates and velocities, or angles and their time derivatives for some systems considered above. However, there is no way to obtain equations of motion from the energy function directly. It turns out that generating function of equations of motion is Lagrange function or simply Lagrangian, in most cases

$$\mathcal{L} = E_k - U. \tag{1}$$

Lagrange formalism is build upon the so-called *Least-Action principle*, also called *Hamilton principle*. According to this principle, that can be put into the foundation of mechanics, the actual dynamics of the system, that is, the actual time dependence of its *generalized coordinates* $\{q_i(t)\}$ minimizes the action on the way from state 1 to state 2,

$$\mathcal{S} = \int_{t_1}^{t_2} dt \mathcal{L}\left(q(t), \dot{q}(t), t\right), \qquad q \equiv \{q_i\}. \tag{2}$$

Here q, $\dot{q}$ may be coordinates and velocities, angles of the polar or spherical coordinate systems and their time derivatives, or whatever other dynamical variables. Varying $\mathcal{S}$ with respect to an arbitrary small deviation $\delta q(t)$ from the actual solution minimizing the action, one obtains

$$\begin{aligned} 0 &= \delta\mathcal{S} = \int_{t_1}^{t_2} dt \left[\mathcal{L}\left(q + \delta q, \dot{q} + \delta\dot{q}, t\right) - \mathcal{L}\left(q, \dot{q}, t\right)\right] \\ &= \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial\mathcal{L}}{\partial q_i}\delta q_i + \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta\dot{q}_i\right] \\ &= \sum_i \frac{\partial\mathcal{L}}{\partial\dot{q}_i}\delta q_i\bigg|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt \sum_i \left[\frac{\partial\mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right]\delta q_i. \end{aligned} \tag{3}$$

The first term here vanishes, because of the boundary conditions fixing $q_i(t)$, i.e., $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The second term should vanish for different $\delta q_i(t)$, that requires that Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4}$$

for all i should be fulfilled. Introducing vectors

$$\mathbf{q} = \{q_i\}, \qquad \dot{\mathbf{q}} = \{\dot{q}_i\}, \tag{5}$$

one can write Lagrange equations in the vector form

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} - \frac{\partial \mathcal{L}}{\partial \mathbf{q}} = 0. \tag{6}$$

One can see that using Lagrange function of Eq. (1) and the above equation yields Newtonian equations of motion. For instance, for a point mass in a potential field

$$\mathcal{L} = \frac{m\mathbf{v}^2}{2} - U(\mathbf{r}). \tag{7}$$

Identifying $\mathbf{r} = \mathbf{q}$, $\mathbf{v} = \dot{\mathbf{q}}$, from Eq. (6) one obtains

$$m\dot{\mathbf{v}} + \frac{\partial U}{\partial \mathbf{r}} = 0, \tag{8}$$

Newton's second law for potential forces.

Lagrange formalism is especially convenient for systems with holonomic constraints, because in it, holonomic constraints are hidden and never appear explicitly. As an example, the Lagrange function of a pendulum considered in Newtonian mechanics above has the form

$$\mathcal{L} = \frac{ml^2\dot{\varphi}^2}{2} + mgl\cos\varphi, \tag{9}$$

where $\varphi = q$ and $\dot{\varphi} = \dot{q}$. Lagrange equation in this case reads

$$ml^2\ddot{\varphi} + mgl\sin\varphi = 0 \tag{10}$$

or

$$\ddot{\varphi} + \omega_0^2\sin\varphi = 0, \qquad \omega_0^2 = g/l \tag{11}$$

that is the Newtonian equation of motion after eliminating the holonomic constraint by using polar coordinate system. Thus, in the case of holonomic constraints using Lagrange formalism is a shorter way to equations of motion, although writing down the kinetic energy requires some habit and, in more complicated cases, a special calculation.

Lagrange formalism is also convenient in the case of using special coordinates systems, even in the absence of constraints. As an example one can consider a pendulum, in which the mass is attached to a spring and can move along the light rod. Kinetic energy in polar coordinates has the form

$$E_k = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right). \tag{12}$$

Potential energy is the sum of gravity energy and the energy of the spring,

$$U = -mgr\cos\varphi + \frac{k}{2}(r-l)^2, \tag{13}$$

where k is spring constant and l is equilibrium position of the mass on a spring. It is convenient instead of r use $x \equiv r - l$ as a dynamical variable. In terms of x and φ Lagrange function becomes

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}^2 + (l+x)^2\dot{\varphi}^2\right) + mg(l+x)\cos\varphi - \frac{k}{2}x^2. \tag{14}$$

Lagrange equation are

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} - \frac{\partial\mathcal{L}}{\partial\varphi} = m\frac{d}{dt}(l+x)^2\dot{\varphi} + mg(l+x)\sin\varphi = 0 \tag{15}$$

and

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} - \frac{\partial\mathcal{L}}{\partial x} = m\ddot{x} - m(l+x)\dot{\varphi}^2 - mg\cos\varphi + kx = 0. \tag{16}$$

They simplify to

$$\begin{aligned}(l+x)\ddot{\varphi} + 2\dot{x}\dot{\varphi} + g\sin\varphi &= 0 \qquad (17)\\ \ddot{x} - (l+x)\dot{\varphi}^2 - g\cos\varphi + \frac{k}{m}x &= 0. \qquad (18)\end{aligned}$$

In the linear approximation is x and φ (small oscillations) these equations decouple,

$$\ddot{\varphi} + \frac{g}{l}\varphi = 0, \qquad \ddot{x} - g + \frac{k}{m}x = 0. \tag{19}$$

One can see that multiplying Lagrange function by a constant does not change Lagrange equations of motion. Thus the choice of Lagrange function in Eq. (1) is not unique. Moreover, one can add a full time derivative of a function of coordinates and time $\partial_t f(q,t)$ to the Lagrange function,

$$\mathcal{L}' = \mathcal{L} + \frac{d}{dt}f(q,t). \tag{20}$$

This adds an invariable contribution to the action,

$$\mathcal{S}' = \mathcal{S} + f(q(t_2),t_2) - f(q(t_1),t_1) \tag{21}$$

that does not change equations of motion. Proof of irrelevance of this addition to the Lagrangian using Lagrange equations is a bit more cumbersome:

$$\begin{aligned}\frac{d}{dt}\frac{\partial}{\partial\dot{q}}\frac{d}{dt}f(q,t) - \frac{\partial}{\partial q}\frac{d}{dt}f(q,t) &= \frac{d}{dt}\frac{\partial}{\partial\dot{q}}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q}\right] - \frac{\partial}{\partial q}\left[\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q}\dot{q}\right]\\ = \frac{d}{dt}\frac{\partial f}{\partial q} - \frac{\partial^2 f}{\partial q\partial t} - \frac{\partial^2 f}{\partial q^2}\dot{q} &= \frac{\partial^2 f}{\partial q\partial t} + \frac{\partial^2 f}{\partial q^2}\dot{q} - \frac{\partial^2 f}{\partial q\partial t} - \frac{\partial^2 f}{\partial q^2}\dot{q} = 0. \qquad (22)\end{aligned}$$

In the presence of non-holonomic constraints Lagrange formalism loses its elegance. The general form of a system of different non-holonomic constraints labeled by the index α is

$$\sum_i c_{\alpha i}(q)\dot{q}_i = 0, \qquad \alpha = 1, 2, \ldots \tag{23}$$

If these constraints can be integrated, that is, represented as constraints on q, they are holonomic constraints. In general, the constraints above cannot be integrated. In this case the way to proceed is first to notice that the constraints above can be reformulated as constraints on possible variations of q, that is,

$$\sum_i c_{\alpha i}(q)\delta q_i = 0, \qquad \alpha = 1, 2, \ldots \tag{24}$$

Constrained variation of the action can be performed with the help of the method of Lagrange multipliers. One obtains

$$0 = \delta\mathcal{S}' = \delta\mathcal{S} + \sum_\alpha \lambda_\alpha \int_{t_1}^{t_2} dt \sum_i c_{\alpha i}(q)\delta q_i, \tag{25}$$

where λ_α are Lagrange multipliers. Using Eq. (3), one arrives at constrained Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_i} - \frac{\partial\mathcal{L}}{\partial q_i} - \sum_\alpha \lambda_\alpha c_{\alpha i}(q) = 0 \tag{26}$$

that have to be solved together with Eq. (23). The bottom line is that in the case of non-holonomic constraint Newtonian approach is better than Lagrangian, because reaction forces within the former are physically more transparent than Lagrange multipliers in the latter.

2 Lagrangian of a particle in electromagnetic field

Lorentz force curving trajectories of charged particles in a magnetic field is a special kind of force that is conservative but not potential. One can construct the Lagrangian for a particle in electromagnetic field that results in correct Newtonian equations of motion

$$m\dot{\mathbf{v}} = q\mathbf{E} + q\left[\mathbf{v}\times\mathbf{B}\right] \tag{27}$$

considered above. Here the electric and magnetic fields can be expressed via the scalar and vector potentials as follows

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} - \nabla\varphi, \qquad \mathbf{B} = \nabla\times\mathbf{A}. \tag{28}$$

Let us show that Lagrangian can be written as

$$\mathcal{L} = \frac{m\mathbf{v}^2}{2} + q\mathbf{v}\cdot\mathbf{A} - q\varphi. \tag{29}$$

Indeed, Lagrange equation becomes

$$0 = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}} - \frac{\partial\mathcal{L}}{\partial\mathbf{r}} = \frac{d}{dt}\left(m\mathbf{v} + q\mathbf{A}\right) + q\nabla\varphi - q\nabla\left(\mathbf{v}\cdot\mathbf{A}\right). \tag{30}$$

Using the formula for the gradient of a dot-product

$$\nabla\left(\mathbf{A}\cdot\mathbf{B}\right) = \left(\mathbf{A}\nabla\right)\mathbf{B} + \left(\mathbf{B}\nabla\right)\mathbf{A} + \mathbf{A}\times\left(\nabla\times\mathbf{B}\right) + \mathbf{B}\times\left(\nabla\times\mathbf{A}\right), \tag{31}$$

one obtains

$$0 = m\dot{\mathbf{v}} + q\dot{\mathbf{A}} + q\nabla\varphi - q\left(\mathbf{v}\nabla\right)\mathbf{A} - q\mathbf{v}\times\left(\nabla\times\mathbf{A}\right). \tag{32}$$

Finally, using the formula

$$\dot{\mathbf{A}} = \frac{d}{dt}\mathbf{A}(\mathbf{r}(t),t) = \frac{\partial\mathbf{A}}{\partial t} + \left(\mathbf{v}\nabla\right)\mathbf{A} \tag{33}$$

and Eq. (28), one arrives at Eq. (27).

3 Transformations and conservation laws

3.1 Galilean transformation

A free body is moving with a constant velocity by inertia, if the frame, in which the motion is considered, is not moving itself with acceleration. Such frame is called *inertial frame*. Any frame, moving with a constant velocity with respect to the inertial frame, is also an inertial frame. Thus, there is an infinite number of inertial frames. In each of these inertial frames a free body will be moving with a constant velocity.

Transformation between inertial frames is *Galilean transformation*,

$$\begin{aligned} \mathbf{v} &= \mathbf{v}' + \mathbf{V}, \qquad \mathbf{V} = \text{const} \\ \mathbf{r} &= \mathbf{r}' + \mathbf{V}t + \text{const}. \end{aligned} \tag{34}$$

Since all inertial frames are equivalent, equations of motion in these frames should be the same. This means that Lagrange function transformed from one frame to the other should have the same form up to the irrelevant full time derivative. For instance, Lagrangian of a free particle $\mathcal{L} = m\mathbf{v}^2/2$ can be transformed as follows

$$\begin{aligned} \mathcal{L} &= \frac{m\mathbf{v}^2}{2} = \frac{m(\mathbf{v}' + \mathbf{V})^2}{2} = \frac{m\mathbf{v}'^2}{2} + m\mathbf{v}' \cdot \mathbf{V} + \frac{m\mathbf{V}^2}{2} \\ &= \mathcal{L}' + \frac{d}{dt}\left(m\mathbf{r}' \cdot \mathbf{V} + \frac{m\mathbf{V}^2}{2}t\right), \end{aligned} \tag{35}$$

where the Lagrangian in primed frame $\mathcal{L}' = m\mathbf{v}'^2/2$ has the same functional form and the irrelevant full derivative can be discarded.

If one requires that all equations of (non-relativistic) physics are covariant (i.e., have the same form) with respect to Galilean transformations, the principle of Galilean invariance becomes non-trivial and productive. For instance, a charged particle in a magnetic field is circling. Galilean transformation into another frame with $\mathbf{V}\perp\mathbf{B}$ makes non-closed orbits in form of cycloids. What should be the corresponding equation of motion? We know that cycloidal orbits emerge in the case of electric field $\mathbf{E}\perp\mathbf{B}$. Thus, to keep equations covariant, one has to require that electromagnetic field is getting changed by Galilean transformations, so that in the moving frame there is an electric field making trajectories cycloidal.

3.2 Time invariance and energy conservation.

If the problem has no explicit time dependence (i.e., is invariant with respect to time shift) and the forces are potential, the total energy of the system $E = E_k + U$ is conserved, as we have seen in Newtonian mechanics above. Within Lagrangian formalism, one can define

$$E \equiv \sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L}. \tag{36}$$

With the help of Lagrange equations it can be shown that in the case of $\mathcal{L} = \mathcal{L}(q, \dot{q})$ one obtains $\dot{E} = 0$ from Lagrange equations. One proceeds as follows

$$\dot{E} = \sum_i \ddot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} + \sum_i \dot{q}_i \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i - \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i = \sum_i \dot{q}_i \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i}\right) = 0. \tag{37}$$

Since Lagrangian is bilinear function of $\dot{q}_i$, the above-defined E is indeed the total energy:

$$\sum_i \dot{q}_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \mathcal{L} = 2E_k - (E_k - U) = E_k + U = E. \tag{38}$$

3.3 Translational invariance and momentum conservation

Lagrangian of an isolated system is invariant under spatial translations

$$\mathbf{r}_i \Rightarrow \mathbf{r}_i + \boldsymbol{\varepsilon}. \tag{39}$$

This actually means that potential energy depends only on the differences $\mathbf{r}_i - \mathbf{r}_j$. That is, there is only potential energy due to interaction between the parts of the system but no external potential energy. Considering $\boldsymbol{\varepsilon}$ as infinitesimal and require $\delta\mathcal{L} = 0$ as the result of translation, one obtains

$$0 = \delta\mathcal{L} = \boldsymbol{\varepsilon} \cdot \sum_i \frac{\partial\mathcal{L}}{\partial\mathbf{r}_i} \quad \Rightarrow \quad \sum_i \frac{\partial\mathcal{L}}{\partial\mathbf{r}_i} = 0, \tag{40}$$

as the direction of $\boldsymbol{\varepsilon}$ is arbitrary. Further, using $\partial\mathcal{L}/\partial\mathbf{r}_i = -\partial U/\partial\mathbf{r}_i = \mathbf{f}_i$, one obtains Newton's third law $\sum_i \mathbf{f}_i = 0$. On the other hand, from Lagrange equations follows

$$0 = \sum_i \left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}_i} - \frac{\partial\mathcal{L}}{\partial\mathbf{r}_i} \right) = \frac{d}{dt}\sum_i \frac{\partial\mathcal{L}}{\partial\mathbf{v}_i} = \frac{d}{dt}\sum_i \mathbf{p}_i = \frac{d}{dt}\mathbf{P}, \tag{41}$$

thus the total momentum is conserved, $\mathbf{P} = \text{const}$.

3.4 Cyclic coordinates and generalized momenta

The results of preceding section can be generalized for the case when the Lagrangian is independent of one or several generalized coordinates q_i. In a similar way it then follows

$$\frac{\partial\mathcal{L}}{\partial q_i} = 0 \quad \Rightarrow \quad \frac{\partial\mathcal{L}}{\partial\dot{q}_i} = p_i = \text{const}, \tag{42}$$

where p_i are *generalized momenta*. In particular, when polar or spherical coordinate system is used and Lagrangian does not depend on the angle φ, e.g., pendulum of Eq. (9) without gravity, then conserved generalized momentum corresponding to φ is angular momentum,

$$\frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = p_\varphi = l_\varphi. \tag{43}$$

For a particle on a sphere with $U = U(\theta)$, considered in Newtonian mechanics, $l_\varphi = l_z$. For a charged particle in electromagnetic field, Lagrangian is given by Eq. (29), and generalized momentum becomes

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}. \tag{44}$$

3.5 Rotational invariance and angular momentum conservation

In the case of rotational invariance infinitesimal rotation by an angle $\delta\boldsymbol{\varphi}$,

$$\delta\mathbf{r}_i = \delta\boldsymbol{\varphi} \times \mathbf{r}_i, \qquad \delta\mathbf{v}_i = \delta\boldsymbol{\varphi} \times \mathbf{v}_i \tag{45}$$

leaves the Lagrangian unchanged. Using Lagrange equations and the definition of the generalized momentum, one proceeds as

$$\begin{aligned} 0 &= \delta\mathcal{L} = \sum_i \left(\frac{\partial\mathcal{L}}{\partial\mathbf{r}_i} \cdot \delta\mathbf{r}_i + \frac{\partial\mathcal{L}}{\partial\mathbf{v}_i} \cdot \delta\mathbf{v}_i \right) = \sum_i \left(\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\mathbf{v}_i} \cdot \delta\mathbf{r}_i + \frac{\partial\mathcal{L}}{\partial\mathbf{v}_i} \cdot \delta\mathbf{v}_i \right) \\ &= \sum_i (\dot{\mathbf{p}}_i \cdot \delta\mathbf{r}_i + \mathbf{p}_i \cdot \delta\mathbf{v}_i) = \sum_i (\dot{\mathbf{p}}_i \cdot [\delta\boldsymbol{\varphi} \times \mathbf{r}_i] + \mathbf{p}_i \cdot [\delta\boldsymbol{\varphi} \times \mathbf{v}_i]) \\ &= \delta\boldsymbol{\varphi} \cdot \sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i + \mathbf{v}_i \times \mathbf{p}_i) = \delta\boldsymbol{\varphi} \cdot \frac{d}{dt}\sum_i \mathbf{r}_i \times \mathbf{p}_i = \delta\boldsymbol{\varphi} \cdot \dot{\mathbf{L}}. \end{aligned} \tag{46}$$

As the direction of $\delta\boldsymbol{\varphi}$ is arbitrary, one concludes that the total angular momentum of the system is conserved, $\mathbf{L} = \sum_i [\mathbf{r}_i \times \mathbf{p}_i] = \text{const}$.

4 Small oscillations with many degrees of freedom

4.1 General formalism

Consider a dynamical system with N degrees of freedom near a minimum of the potential energy. The Lagrangian has a general form

$$\mathcal{L} = \frac{1}{2}\sum_{ij}\left(m_{ij}\dot{x}_i\dot{x}_j - k_{ij}x_i x_j\right), \tag{47}$$

where $x_i \equiv q_i - q_i(0)$ are deviations from the minimum and the mass and stiffness coefficients are symmetric, $m_{ij} = m_{ji}$ and $k_{ij} = k_{ji}$. The Lagrangian can be written in the matrix form as

$$\mathcal{L} = \frac{1}{2}\left(\dot{\mathbf{X}}^T \cdot \mathbb{M} \cdot \dot{\mathbf{X}} - \mathbf{X}^T \cdot \mathbb{K} \cdot \mathbf{X}\right), \tag{48}$$

where $\mathbb{M}$ is the mass matrix, $\mathbb{K}$ is the stiffness matrix, $\mathbf{X} = (x_1, x_2, \ldots, x_N)$ (a column) and $\mathbf{X}^T$ its transposition (a row). The Lagrange equations read

$$\sum_j \left(m_{ij}\ddot{x}_j + k_{ij}x_j\right) = 0, \qquad i = 1\ldots N \tag{49}$$

or, in the matrix form,

$$\mathbb{M} \cdot \ddot{\mathbf{X}} + \mathbb{K} \cdot \mathbf{X} = 0. \tag{50}$$

We search for the solution in the form

$$\mathbf{X} = \mathbf{a}\sin(\omega t) \tag{51}$$

[or the same with $\cos(\omega t)$] and obtain the system of linear algebraic equations

$$\left(-\omega^2\mathbb{M} + \mathbb{K}\right) \cdot \mathbf{a} = 0. \tag{52}$$

This system of equations has a nontrivial solution for a only if

$$\left|-\omega^2\mathbb{M} + \mathbb{K}\right| = 0 \tag{53}$$

that defines N eigenfrequencies ω_α^2, $\alpha = 1\ldots N$. One can prove that ω_α^2 are positive, if the potential energy is a positively defined bilinear form, as it is the case for the energy minimum. Then the vectors a_α are real. We will assume the simplest case of ω_α^2 nondegenerate, $\omega_\alpha^2 \neq \omega_\beta^2$ for $\alpha \neq \beta$. Technically, Eq. (6) is a generalized eigenvalue problem,

$$\mathbb{K} \cdot \mathbf{a} = \lambda\mathbb{M} \cdot \mathbf{a} \tag{54}$$

and $\mathbf{a}$ is eigenvector of $\mathbb{K}$ with respect to $\mathbb{M}$. Algorithm for solving this problem is implemented in Wolfram Mathematica.

The problem we are solving is resembling an eigenvalue problem. It is more complicated, however, since we have two matrices instead of one. Whereas in eigenvalue problems eigenvectors corresponding to different eigenvalues are orthogonal, here a generalized orthogonality relation takes place. To obtain it, write Eq. (52) with two different eigenvalues $\omega_\alpha^2$and ω_β^2:

$$\begin{aligned}\omega_\alpha^2\mathbb{M} \cdot \mathbf{a}_\alpha &= \mathbb{K} \cdot \mathbf{a}_\alpha \\ \omega_\beta^2\mathbb{M} \cdot \mathbf{a}_\beta &= \mathbb{K} \cdot \mathbf{a}_\beta.\end{aligned} \tag{55}$$

Now multiply the first equation by $\mathbf{a}_\beta^T$ from the left, multiply the second equation by $\mathbf{a}_\alpha^T$ from the left, and subtract them from each other:

$$\omega_\alpha^2\mathbf{a}_\beta^T \cdot \mathbb{M} \cdot \mathbf{a}_\alpha - \omega_\beta^2\mathbf{a}_\alpha^T \cdot \mathbb{M} \cdot \mathbf{a}_\beta = \mathbf{a}_\beta^T \cdot \mathbb{K} \cdot \mathbf{a}_\alpha - \mathbf{a}_\alpha^T \cdot \mathbb{K} \cdot \mathbf{a}_\beta. \tag{56}$$

Using symmetry of matrices $\mathbb{M}$ and $\mathbb{K}$, one can be easily show $\mathbf{a}_\beta^T \cdot \mathbb{M} \cdot \mathbf{a}_\alpha = \mathbf{a}_\alpha^T \cdot \mathbb{M} \cdot \mathbf{a}_\beta$ and $\mathbf{a}_\beta^T \cdot \mathbb{K} \cdot \mathbf{a}_\alpha = \mathbf{a}_\alpha^T \cdot \mathbb{K} \cdot \mathbf{a}_\beta$. Thus the rhs vanishes and one obtains

$$\left(\omega_\alpha^2 - \omega_\beta^2\right) \mathbf{a}_\beta^T \cdot \mathbb{M} \cdot \mathbf{a}_\alpha = 0. \tag{57}$$

This means that for $\alpha \neq \beta$ one has $\mathbf{a}_\beta^T \cdot \mathbb{M} \cdot \mathbf{a}_\alpha = 0$. It is convenient also to require$\mathbf{a}_\alpha^T \cdot \mathbb{M} \cdot \mathbf{a}_\alpha = 1$. This gives the generalized orthogonality condition

$$\mathbf{a}_\alpha^T \cdot \mathbb{M} \cdot \mathbf{a}_\beta = \delta_{\alpha\beta}. \tag{58}$$

Now from the second of equations (55) one obtains

$$\mathbf{a}_\alpha^T \cdot \mathbb{K} \cdot \mathbf{a}_\beta = \omega_\beta^2 \mathbf{a}_\alpha^T \cdot \mathbb{M} \cdot \mathbf{a}_\beta = \omega_\beta^2 \delta_{\alpha\beta}. \tag{59}$$

One can compose the $N \times N$ matrix of vectors $\mathbb{A}$ by stacking all $\mathbf{a}_\alpha$ together. In terms of $\mathbb{A}$ Eq. (58) takes the form

$$\mathbb{A}^T \cdot \mathbb{M} \cdot \mathbb{A} = \mathbb{I}, \tag{60}$$

where $\mathbb{I}$ is the unit matrix, whereas Eq. (59) becomes

$$\mathbb{A}^T \cdot \mathbb{K} \cdot \mathbb{A} = \mathbf{\Omega}^2, \tag{61}$$

where $\mathbf{\Omega} = \operatorname{diag}\{\omega_\alpha\}$.

Let us now introduce the normal-coordinate vector $\boldsymbol{\zeta}$ defined by

$$\mathbf{X} = \mathbb{A} \cdot \boldsymbol{\zeta}, \qquad \mathbf{X}^T = (\mathbb{A} \cdot \boldsymbol{\zeta})^T = \boldsymbol{\zeta}^T \cdot \mathbb{A}^T. \tag{62}$$

These equations can be resolved for $\boldsymbol{\zeta}$ and ζ_α:

$$\boldsymbol{\zeta} = \mathbb{A}^{-1} \cdot \mathbf{X}, \qquad \zeta_\alpha = \left(\mathbb{A}^{-1}\right)_{\alpha i} x_i. \tag{63}$$

Note that, according to Eq. (60), $\mathbb{A}^{-1} \neq \mathbb{A}^T$and $\left(\mathbb{A}^{-1}\right)_{\alpha i} \neq a_{\alpha i}^{-1}$. Inserting Eq. (62) into the Lagrangian, Eq. (48), and using Eqs. (60) and (61), one obtains

$$\mathcal{L} = \frac{1}{2}\left(\dot{\boldsymbol{\zeta}}^T \cdot \mathbb{A}^T \cdot \mathbb{M} \cdot \mathbb{A} \cdot \dot{\boldsymbol{\zeta}} - \boldsymbol{\zeta}^T \cdot \mathbb{A}^T \cdot \mathbb{K} \cdot \mathbb{A} \cdot \boldsymbol{\zeta}\right) = \frac{1}{2}\left(\dot{\boldsymbol{\zeta}}^T \cdot \dot{\boldsymbol{\zeta}} - \boldsymbol{\zeta}^T \cdot \mathbf{\Omega}^2 \cdot \boldsymbol{\zeta}\right) \tag{64}$$

or

$$\mathcal{L} = \frac{1}{2} \sum_{\alpha=1}^{N} \left(\dot{\zeta}_\alpha^2 - \omega_\alpha^2 \zeta_\alpha^2\right), \tag{65}$$

where ζ_α are components of the normal-coordinate vector $\boldsymbol{\zeta}$ or normal coordinates. One can see that the Lagrangian is separable in terms of the normal coordinates, so that every normal coordinate ζ_α oscillates with its own frequency ω_α:

$$\zeta_\alpha(t) = C_{s\alpha} \sin \omega_\alpha t + C_{c\alpha} \cos \omega_\alpha t. \tag{66}$$

From Eq. (62) one obtains

$$x_i(t) = \sum_{\alpha=1}^{N} a_{i\alpha} \zeta_\alpha(t) \tag{67}$$

that becomes

$$x_i(t) = \sum_{\alpha=1}^{N} a_{i\alpha} \left(C_{s\alpha} \sin \omega_\alpha t + C_{c\alpha} \cos \omega_\alpha t\right). \tag{68}$$

To obtain the coefficients from the initial conditions, one writes

$$\dot{\zeta}_\alpha(t) = C_{s\alpha} \omega_\alpha \cos \omega_\alpha t - C_{c\alpha} \omega_\alpha \sin \omega_\alpha t \tag{69}$$

and at $t = 0$ using Eq. (63) obtains

$$C_{c\alpha} = \left(\mathbb{A}^{-1}\right)_{\alpha i} x_i(0), \qquad C_{s\alpha} = \left(\mathbb{A}^{-1}\right)_{\alpha i} \frac{\dot{x}_i(0)}{\omega_\alpha}. \tag{70}$$

4.2 Double pendulum

Consider a double pendulum that consists of the pendulum 1 attached in a standard way and the pendulum 2 attached to the end of pendulum 1. For simplicity we consider the case $m_1 = m_2 = m$ and $l_1 = l_2 = l$. Describing the pendula with the angles θ_1 and θ_2, one writes

$$\begin{aligned} x_1 &= l\sin\theta_1, \qquad x_2 = l\sin\theta_1 + l\sin\theta_2 \\ y_1 &= -l\cos\theta_1, \qquad y_2 = -l\cos\theta_1 - l\cos\theta_2. \end{aligned} \tag{71}$$

Potential energy reads

$$U = mg\,(y_1 + y_2) = -mgl\,(2\cos\theta_2 + \cos\theta_1) \tag{72}$$

and the kinetic energy is given by

$$\begin{aligned} E_k &= \frac{m}{2}\left[\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right] = \frac{ml^2}{2}\left[\dot{\theta}_1^2 + \left(\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2\right)^2 + \left(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2\right)^2\right] \\ &= \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\dot{\theta}_1\dot{\theta}_2\right]. \end{aligned} \tag{73}$$

Near the ground state $\theta_1 = \theta_2 = 0$ one leaves only quadratic terms that results in

$$\mathcal{L} = E_k - U = \frac{ml^2}{2}\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right) - \frac{mgl}{2}\left(2\theta_1^2 + \theta_2^2\right). \tag{74}$$

Identifying this with Eq. (48) yields

$$\mathbb{M} = ml^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbb{K} = mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}. \tag{75}$$

The generalized eigenvalue problem, Eq. (52), has the form

$$\begin{pmatrix} 2\left(-\omega^2 + \omega_0^2\right) & -\omega^2 \\ -\omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0, \tag{76}$$

where we have introduced

$$\omega_0^2 = g/l. \tag{77}$$

Secular equation corrresponding to Eq. (76) has the form

$$\begin{aligned} 0 &= 2\left(-\omega^2 + \omega_0^2\right)^2 - \omega^4 = \left[\sqrt{2}\left(\omega^2 - \omega_0^2\right) + \omega^2\right]\left[\sqrt{2}\left(\omega^2 - \omega_0^2\right) - \omega^2\right] \\ &= \left[\left(\sqrt{2} + 1\right)\omega^2 - \sqrt{2}\omega_0^2\right]\left[\left(\sqrt{2} - 1\right)\omega^2 - \sqrt{2}\omega_0^2\right]. \end{aligned} \tag{78}$$

Thus the eigenfrequencies are given by

$$\omega_1^2 = \frac{\sqrt{2}\omega_0^2}{\sqrt{2} + 1} = \left(2 - \sqrt{2}\right)\omega_0^2, \qquad \omega_2^2 = \frac{\sqrt{2}\omega_0^2}{\sqrt{2} - 1} = \left(2 + \sqrt{2}\right)\omega_0^2. \tag{79}$$

Let us find now generalized eigenvectors and normal modes of the system. Eq. (76) becomes

$$\begin{aligned} 2\left(\omega_\alpha^2 - \omega_0^2\right)a_{1\alpha} + \omega_\alpha^2 a_{2\alpha} &= 0 \\ \omega_\alpha^2 a_{1\alpha} + \left(\omega_\alpha^2 - \omega_0^2\right)a_{2\alpha} &= 0. \end{aligned} \tag{80}$$

For instance, from the second equation one obtains

$$\mathbf{a}_\alpha = \mu_\alpha\begin{pmatrix} -\omega_\alpha^2 + \omega_0^2 \\ \omega_\alpha^2 \end{pmatrix}, \tag{81}$$

where μ_α are normalization coefficients. In particular,

$$\begin{aligned}
\mathbf{a}_1 &= \mu_1 \begin{pmatrix} -\omega_1^2+\omega_0^2 \\ \omega_1^2 \end{pmatrix} = \mu_1\omega_0^2 \begin{pmatrix} \sqrt{2}-1 \\ 2-\sqrt{2} \end{pmatrix} = \mu_1\omega_0^2\left(\sqrt{2}-1\right)\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\
\mathbf{a}_2 &= \mu_2 \begin{pmatrix} -\omega_2^2+\omega_0^2 \\ \omega_2^2 \end{pmatrix} = \mu_2\omega_0^2 \begin{pmatrix} -\sqrt{2}-1 \\ 2+\sqrt{2} \end{pmatrix} = \mu_2\omega_0^2\left(\sqrt{2}+1\right)\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}.
\end{aligned} \tag{82}$$

Let us check orthogonality of these eigenvectors, Eq. (60):

$$\mathbf{a}_1^T\cdot\mathbb{M}\cdot\mathbf{a}_2 = \mu_1\mu_2 ml^2\omega_0^4 \begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} = 0, \tag{83}$$

as expected. Now calculate normalization factors:

$$\begin{aligned}
1 &= \mathbf{a}_1^T\cdot\mathbb{M}\cdot\mathbf{a}_1 = \mu_1^2 ml^2\omega_0^4\left(\sqrt{2}-1\right)^2\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} = \mu_1^2 ml^2\omega_0^4\left(\sqrt{2}-1\right)2\sqrt{2} \\
1 &= \mathbf{a}_2^T\cdot\mathbb{M}\cdot\mathbf{a}_2 = \mu_2^2 ml^2\omega_0^4\left(\sqrt{2}+1\right)^2\begin{pmatrix} -1 & \sqrt{2} \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\cdot\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} = \mu_2^2 ml^2\omega_0^4\left(\sqrt{2}+1\right)2\sqrt{2}
\end{aligned} \tag{84}$$

Thus

$$\begin{aligned}
\mu_1\omega_0^2 &= \frac{1}{\sqrt{ml^2}}\left(\frac{1}{\left(\sqrt{2}-1\right)2\sqrt{2}}\right)^{1/2} = \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2} \\
\mu_2\omega_0^2 &= \frac{1}{\sqrt{ml^2}}\left(\frac{1}{\left(\sqrt{2}+1\right)2\sqrt{2}}\right)^{1/2} = \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}
\end{aligned} \tag{85}$$

and, finally,

$$\begin{aligned}
\mathbf{a}_1 &= \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\
\mathbf{a}_2 &= \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}.
\end{aligned} \tag{86}$$

Now, the angles expressed through the eigenmodes are given by Eq. (62) in matrix form or by Eq. (67) in components. The latter yields

$$\begin{aligned}
\theta_1 &= a_{11}\zeta_1 + a_{12}\zeta_2 = \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\zeta_1 - \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\zeta_2 \\
\theta_2 &= a_{21}\zeta_1 + a_{22}\zeta_2 = \frac{\sqrt{2}}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\zeta_1 + \frac{\sqrt{2}}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\zeta_2.
\end{aligned}$$

Inverse transformation with the help of

$$\det \equiv a_{11}a_{22} - a_{21}a_{12} = \frac{1}{ml^2} \tag{87}$$

takes the form

$$\begin{aligned}
\zeta_1 &= \frac{\det_1}{\det} = \left(\theta_1 a_{22} - \theta_2 a_{12}\right) ml^2 = \sqrt{ml^2}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\left(\sqrt{2}\theta_1 + \theta_2\right) \\
\zeta_2 &= \frac{\det_2}{\det} = \left(a_{11}\theta_2 - a_{21}\theta_1\right) ml^2 = \sqrt{ml^2}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\left(-\sqrt{2}\theta_1 + \theta_2\right).
\end{aligned} \tag{88}$$

One can see that in the normal mode 1 both pendula are swinging in-phase and in the mode 2 they are swinging antiphase. The frequency of the mode 1 is lower.

4.3 Two coupled oscillators

Another model that illustrates general principles and has an interesting physical behavior is the model of two coupled oscillators with the Lagrangian

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 - \omega_1^2 x_1^2 + \dot{x}_2^2 - \omega_2^2 x_2^2 - \Delta^2 x_1 x_2\right) \tag{89}$$

with the positive coupling, $\Delta > 0$. We will show that if $|\omega_1 - \omega_2| \gg \Delta$ (off-resonance case), the oscillators behave nearly independently from each other and oscillate with their own frequencies $\omega_{1,2}$, whereas for $|\omega_1 - \omega_2| \sim \Delta$ (resonance case) they strongly hybridize and each normal mode involves both coordinates. Here the matrices $\mathbb{M}$ and $\mathbb{K}$ are given by

$$\mathbb{M} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbb{K} = m\begin{pmatrix} \omega_1^2 & \Delta^2/2 \\ \Delta^2/2 & \omega_2^2 \end{pmatrix} \tag{90}$$

and the eigenvalue problem, Eq. (52), reads

$$\begin{pmatrix} -\omega^2 + \omega_1^2 & \Delta^2/2 \\ \Delta^2/2 & -\omega^2 + \omega_2^2 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0. \tag{91}$$

Note that this is a standard eigenvalue problem, not a generalized one. The secular equation has the form

$$\left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right) - \frac{\Delta^4}{4} = 0 \tag{92}$$

and eigenfrequencies are given by

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 \pm \sqrt{\left(\omega_1^2 - \omega_2^2\right) + \Delta^4}\right). \tag{93}$$

One can see that, indeed, for $|\omega_1 - \omega_2| \gg \Delta$ the eigenfrequencies $\omega_\pm$ coincide with $\omega_{1,2}$. On the other hand, at resonance, $\omega_1 = \omega_2 = \omega_0$, one obtains frequencies of two modes split by coupling.

$$\omega_\pm^2 = \omega_0^2 \pm \frac{1}{2}\Delta^2. \tag{94}$$

Let us now find the eigenvectors and normal modes. From the equation

$$\left(-\omega_\pm^2 + \omega_1^2\right) a_{1\pm} + \frac{\Delta^2}{2} a_{2\pm} = 0 \tag{95}$$

one obtains

$$\mathbf{a}_\pm = \mu_\pm \begin{pmatrix} \Delta^2/2 \\ \omega_\pm^2 - \omega_1^2 \end{pmatrix}, \tag{96}$$

where μ_α $(\alpha = \pm)$ are normalization coefficients. This form is nonsymmetric, and it is convenient to symmetrize it. To this end, consider the case $\omega_1 < \omega_2$. One can see from Eq. (25) that the frequencies are ordered as $\omega_- < \omega_1 < \omega_2 < \omega_+$. Keeping these relations in mind, one can symmetrize $\mathbf{a}_\pm$ as

$$\mathbf{a}_+ = \mu_+ \begin{pmatrix} \sqrt{\frac{\Delta^2}{2\left(\omega_+^2 - \omega_1^2\right)}} \\ \sqrt{\frac{2\left(\omega_+^2 - \omega_1^2\right)}{\Delta^2}} \end{pmatrix}, \qquad \mathbf{a}_- = \mu_- \begin{pmatrix} \sqrt{\frac{\Delta^2}{2\left(\omega_1^2 - \omega_-^2\right)}} \\ -\sqrt{\frac{2\left(\omega_1^2 - \omega_-^2\right)}{\Delta^2}} \end{pmatrix}, \tag{97}$$

where μ_α is another set of normalization coefficients. One can check orthogonality of these eigenvectors, $\mathbf{a}_+^T \cdot \mathbf{a}_- = 0$. From the normalization condition $\mathbf{a}_\alpha^T \cdot \mathbf{a}_\alpha = 1$ one finds μ_α that turn out to be the same,

$$\mu_\alpha \equiv \mu_\pm = \sqrt{\frac{\Delta^2}{2\sqrt{\left(\omega_1^2 - \omega_2^2\right) + \Delta^4}}}. \tag{98}$$

Substituting this into the formulas above and simplifying, one finally obtains a symmetric expression

$$\mathbf{a}_{\pm} = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \sqrt{1 \pm \frac{\omega_1^2 - \omega_2^2}{\sqrt{\left(\omega_1^2 - \omega_2^2\right) + \Delta^4}}} \\ \pm \sqrt{1 \mp \frac{\omega_1^2 - \omega_2^2}{\sqrt{\left(\omega_1^2 - \omega_2^2\right) + \Delta^4}}} \end{array} \right). \tag{99}$$

Actually this result is valid for an arbitrary relation between $\omega_1$and ω_2. Off resonance for $\omega_1 < \omega_2$ one has

$$\mathbf{a}_{-} \cong \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \qquad \mathbf{a}_{+} \cong \left(\begin{array}{c} 0 \\ 1 \end{array} \right) \tag{100}$$

that means that $\omega_- \cong \omega_1$ corresponds to the first oscillator and $\omega_+ \cong \omega_2$ corresponds to the second oscillator. Off resonance for $\omega_1 > \omega_2$ it is vice versa,

$$\mathbf{a}_{-} \cong \left(\begin{array}{c} 0 \\ 1 \end{array} \right), \qquad \mathbf{a}_{+} \cong \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \tag{101}$$

thus $\omega_- \cong \omega_2$ corresponds to the second oscillator and $\omega_+ \cong \omega_1$ corresponds to the first one. At resonance, $\omega_1 = \omega_2 = \omega_0$, one obtains

$$\mathbf{a}_{\pm} = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 1 \\ \pm 1 \end{array} \right) \tag{102}$$

that means that the modes are strongly mixed.

The original coordinates expressed through the normal modes are given by the formula similar to Eq. (67), i.e.,

$$x_1 = a_{1+}\zeta_+ + a_{1-}\zeta_-, \qquad x_2 = a_{2+}\zeta_+ + a_{2-}\zeta_-. \tag{103}$$

Normal modes behave as independent harmonic oscillators with frequencies $\omega_{\pm}$. Thus the time dependence of both x_1 and x_2 is a superposition of these two oscillations.

Now consider, as an important illustration, the resonance case $\omega_1 = \omega_2 = \omega_0$ with a small coupling $\Delta \ll \omega_0$. Here

$$x_1 = \frac{1}{\sqrt{2}} \left(\zeta_+ + \zeta_- \right), \qquad x_2 = \frac{1}{\sqrt{2}} \left(\zeta_+ - \zeta_- \right) \tag{104}$$

and, inversely,

$$\zeta_+ = \frac{1}{\sqrt{2}} \left(x_1 + x_2 \right), \qquad \zeta_- = \frac{1}{\sqrt{2}} \left(x_1 - x_2 \right). \tag{105}$$

One can see that ζ_+ is in-phase oscillation while ζ_- is anti-phase oscillation. The latter has a lower frequency that can be inferred from Eq. (89). Suppose at $t = 0$ oscillator 1 was in a general state while oscillator 2 was in its ground state. Then the initial condition for the normal modes is

$$\zeta_{\pm}(0) = \frac{1}{\sqrt{2}} x_1(0), \qquad \dot{\zeta}_{\pm}(0) = \frac{1}{\sqrt{2}} \dot{x}_1(0). \tag{106}$$

Thus the time dependence of the normal-mode coordinates is given by

$$\zeta_{\pm}(t) = \frac{1}{\sqrt{2}} x_1(0) \cos\left(\omega_{\pm} t\right) + \frac{1}{\sqrt{2}} \dot{x}_1(0) \frac{\sin\left(\omega_{\pm} t\right)}{\omega_{\pm}}. \tag{107}$$

Inserting this into Eq. (104) one obtains

$$\begin{aligned} x_1(t) &= \frac{1}{2} \left\{ x_1(0) \left[\cos\left(\omega_+ t\right) + \cos\left(\omega_- t\right) \right] + \dot{x}_1(0) \left[\frac{\sin\left(\omega_+ t\right)}{\omega_+} + \frac{\sin\left(\omega_- t\right)}{\omega_-} \right] \right\} \\ x_2(t) &= \frac{1}{2} \left\{ x_1(0) \left[\cos\left(\omega_+ t\right) - \cos\left(\omega_- t\right) \right] + \dot{x}_1(0) \left[\frac{\sin\left(\omega_+ t\right)}{\omega_+} - \frac{\sin\left(\omega_- t\right)}{\omega_-} \right] \right\}. \end{aligned}$$

In the small-coupling case the eigenfrequencies are close to each other, so that one can approximate

$$\frac{1}{\omega_+} \cong \frac{1}{\omega_-} \cong \frac{1}{\omega_0}. \tag{108}$$

However, one cannot make this approximation in the arguments of sin and cos since it will break down at large times. From Eq. (94) in the weak-coupling case follows

$$\omega_\pm = \omega_0 \sqrt{1 \pm \frac{\Delta^2}{2\omega_0^2}} \cong \omega_0 \pm \frac{\Delta^2}{4\omega_0}. \tag{109}$$

Usung this and trigonometric relations, one obtains

$$\begin{aligned} x_1(t) &= \left[x_1(0)\cos(\omega_0 t) + \frac{\dot{x}_1(0)}{\omega_0}\sin(\omega_0 t)\right]\cos\left(\frac{\Delta^2}{4\omega_0}t\right) \\ x_2(t) &= \left[-x_1(0)\sin(\omega_0 t) + \frac{\dot{x}_1(0)}{\omega_0}\cos(\omega_0 t)\right]\sin\left(\frac{\Delta^2}{4\omega_0}t\right). \end{aligned} \tag{110}$$

This is a standard time dependence for a harmonic oscillator (square brackets) multiplied by a slowly oscillating function of time. Using

$$\dot{x}_1(t) = \left[-x_1(0)\omega_0 \sin(\omega_0 t) + \dot{x}_1(0)\cos(\omega_0 t)\right]\cos\left(\frac{\Delta^2}{4\omega_0}t\right) \tag{111}$$

(we do not differentiate the slow function), one can compute the time dependence of the energy of the first oscillator as

$$E_1(t) = \frac{m}{2}\left(\dot{x}_1^2(t) + \omega_0^2 x_1^2(0)\right) = E_1(0)\cos^2\left(\frac{\Delta^2}{4\omega_0}t\right). \tag{112}$$

Similarly for the second oscillator one obtains

$$E_2(t) = E_1(0)\sin^2\left(\frac{\Delta^2}{4\omega_0}t\right). \tag{113}$$

One can see that the total energy $E(t) = E_1(t) + E_2(t)$ is conserved and slowly migrating between the two oscillators.