Part I

Hamiltonian Mechanics

Both Newtonian and Lagrangian formalisms operate with systems of second-order differential equations for time-dependent generalized coordinates, $\ddot{q}_i = \ldots$. For a system with N degrees of freedom, N such equations can be reformulated as systems of $2N$ first-order differential equations if one considers velocities $v_i = \dot{q}_i$ as additional dynamical variables. This system of equations has the form $\dot{q}_i = v_i$, $\dot{v}_i = \ldots$ that is non-symmetric with respect to q_i and v_i.

Hamiltonian formalism uses q_i and p_i as dynamical variables, where p_i are generalized momenta defined by

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}. \tag{0.1}$$

The resulting $2N$ Hamiltonian equations of motion for q_i and p_i have an elegant symmetric form that is the reason for calling them *canonical equations*. Although for most of mechanical problems Hamiltonian formalism is of no practical advantage, it is worth studying because of the similarity between its mathematical structure and that of quantum mechanics. In fact, a significant part of quantum mechanics using matrix and operator algebra grew out of Hamiltonian mechanics. The latter is invoked in constructing new field theories. Hamiltonian formalism finds application in statistical physics, too.

1 Hamiltonian function and equations

Hamiltonian equations can be obtained from Lagrange equations that can be written in the form

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i} \tag{1.1}$$

using a Legendre transformation. The differential of the Lagrange function has the form

$$d\mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial q_i} dq_i + \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right) \tag{1.2}$$

that with the help of Eqs. (0.1) and (1.1) can be rewritten as

$$d\mathcal{L} = \sum_i \left(\dot{p}_i dq_i + p_i d\dot{q}_i \right). \tag{1.3}$$

Let us intoduce the Hamiltonian function, or the Hamiltonian, defined by

$$\mathcal{H} = \sum_i p_i \dot{q}_i - \mathcal{L}. \tag{1.4}$$

With the help of the above its differential becomes

$$d\mathcal{H} = \sum_i \left(\dot{q}_i dp_i - \dot{p}_i dq_i \right). \tag{1.5}$$

One can see that the natural variables of $\mathcal{H}$ are q and p and

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{1.6}$$

that are Hamiltonian equations of motion for q and p.

As the Lagrange function is bilinear in $\dot{q}_i$,

$$\mathcal{L} = E_k(q, \dot{q}) - U(q), \qquad E_k = \frac{1}{2}\sum_{ij} a_{ij}(q)\dot{q}_i\dot{q}_j, \tag{1.7}$$

the first term in Eq. (1.4) is the double kinetic energy,

$$\sum_i p_i \dot{q}_i = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\dot{q}_i = 2E_k. \tag{1.8}$$

Now one can see that $\mathcal{H}$ is the full energy,

$$\mathcal{H} = E_k + U. \tag{1.9}$$

In most cases the kinetic energy is written in a natural way in terms of $\dot{q}_i$, so that expressing it in terms of p_i may require work. For unconstrained particles there is the linear relation $\mathbf{p}_i = m\mathbf{v}_i$ and reexpressing $\mathcal{H}$ in terms of p_i is not a problem. In particular, for a system of particles one obtains

$$\mathcal{H} = \sum_i \frac{\mathbf{p}_i^2}{2m_i} + U. \tag{1.10}$$

On the other hand, for systems with constraints, such as double pendulum, different generalized velocities $\dot{q}_i$ couple since $a_{ij} \equiv (\mathbb{A})_{ij}$ in Eq. (1.7) is a non-diagonal matrix. Eq. (1.7) can be written in the matrix-vector form

$$E_k = \frac{1}{2}\dot{\mathbf{q}}^T \cdot \mathbb{A} \cdot \dot{\mathbf{q}}, \tag{1.11}$$

where $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \ldots)$ is a column and the transposed $\dot{\mathbf{q}}^T$ is a row. Further, from Eqs. (0.1) and (1.7) one obtains

$$\mathbf{p} = \mathbb{A} \cdot \dot{\mathbf{q}} \tag{1.12}$$

and thus

$$\dot{\mathbf{q}} = \mathbb{A}^{-1} \cdot \mathbf{p}, \qquad \dot{\mathbf{q}}^T = \left(\mathbb{A}^{-1} \cdot \mathbf{p}\right)^T = \mathbf{p}^T \cdot \left(\mathbb{A}^{-1}\right)^T = \mathbf{p}^T \cdot \mathbb{A}^{-1}, \tag{1.13}$$

where we have taken into account that $\mathbb{A}^{-1}$, as well as $\mathbb{A}$, is a symmetric matrix. Plugging this result into Eq. (1.11) one obtains

$$\mathcal{H} = \frac{1}{2}\mathbf{p}^T \cdot \mathbb{A}^{-1} \cdot \mathbf{p} + U. \tag{1.14}$$

Although inverting kinetic-energy matrix $\mathbb{A}$ is possible, at least numerically, it makes Hamiltonian formalism less appropriate. For this reason it is never used for systems with constraints relevant in engineering, exactly where Lagrangian approach shows its strength.

Let us, as an illustration, find the Hamiltonian of a particle in spherical coordinates. Kinetic energy has the form

$$E_k = \frac{mv^2}{2} = \frac{m}{2}\left(v_r^2 + v_\theta^2 + v_\phi^2\right) = \frac{m}{2}\left[\dot{r}^2 + \left(r\dot{\theta}\right)^2 + \left(r\sin\theta\,\dot{\phi}\right)^2\right]. \tag{1.15}$$

Thus the generalized momenta (0.1) are given by

$$\begin{aligned} p_r &= \frac{\partial E_k}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial E_k}{\partial \dot{\theta}} = mr^2\dot{\theta} \\ p_\phi &= \frac{\partial E_k}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}. \end{aligned} \tag{1.16}$$

Solving these equations for $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$ yields

$$\dot{r} = \frac{p_r}{m}, \qquad \dot{\theta} = \frac{p_\theta}{mr^2}, \qquad \dot{\phi} = \frac{p_\phi}{mr^2\sin^2\theta}. \tag{1.17}$$

Inserting the results into Eq. (1.15) one obtains

$$\mathcal{H} = E_k + U = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2\theta}\right) + U(r,\theta,\phi). \tag{1.18}$$

One can see that Eqs. (1.17) are Hamilton equations

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r}, \qquad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta}, \qquad \dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi}. \tag{1.19}$$

2 Hamiltonian equations from the least-action principle

As Lagrange equations follow from the least-action principle and Hamiltonian equations can be derived from Lagrange equations, one can obtain Hamiltonian equations from the least-action principle directly. The calculation starts from the action written with the help of Eq. (1.4) in the form

$$\mathcal{S} = \int_{t_1}^{t_2} dt \mathcal{L}(q,\dot{q},t) = \int_{t_1}^{t_2} dt \left(\sum_i p_i \dot{q}_i - \mathcal{H}\right) = \int_{t_1}^{t_2} \left(\sum_i p_i dq_i - \mathcal{H} dt\right). \tag{2.1}$$

Here, in contrast to Lagrange formalism, one considers both p_i and q_i as independent dynamical variables and makes a variation

$$q_i \Rightarrow q_i + \delta q_i, \qquad p_i \Rightarrow p_i + \delta p_i. \tag{2.2}$$

The corresponding variation of the action has the form

$$\delta\mathcal{S} = \sum_i \int_{t_1}^{t_2} \left(\delta p_i dq_i + p_i d\delta q_i - \frac{\partial \mathcal{H}}{\partial q_i}\delta q_i dt - \frac{\partial \mathcal{H}}{\partial p_i}\delta p_i dt\right). \tag{2.3}$$

Integrating the second term by parts and using that $\delta q_i = 0$ at the begining and at the end, one can rewrite this as

$$\delta\mathcal{S} = \sum_i \int_{t_1}^{t_2} \left[\delta p_i \left(dq_i - \frac{\partial \mathcal{H}}{\partial p_i} dt\right) + \delta q_i \left(-dp_i - \frac{\partial \mathcal{H}}{\partial q_i} dt\right)\right]. \tag{2.4}$$

Since for the actual trajectories $\delta\mathcal{S} = 0$ for arbitrary and independent δq_i and δp_i, one concludes that both expressions in round brackets are zero that leads to Hamilton equations (1.6).

3 Poisson brackets and conservation

Time derivative of any quantity $f = f(q,p,t)$ has the form

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \left(\frac{\partial f}{\partial q_i}\dot{q}_i + \frac{\partial f}{\partial p_i}\dot{p}_i\right). \tag{3.1}$$

With the help of Eqs. (1.6) it can be rewritten as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{f, \mathcal{H}\}, \tag{3.2}$$

where $\{f, \mathcal{H}\}$ is a Poisson bracket defined by

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right). \tag{3.3}$$

One can see that if f does not depend on time explicitly and its Poisson bracket with the Hamiltonian is zero, f is an integral of motion. In particular, obviously $\{\mathcal{H},\mathcal{H}\}=0$, so that in the absence of explicit time dependence the energy is an integral of motion, $\mathcal{H}=E=\text{const}$. Note the similarity between Eq. (3.2) and the quantum mechanical equation of motion for an operator in Heisenberg representation where the commutator $\left[f,\hat{H}\right]$ replaces Poisson bracket.

Poisson brackets satisfy obvious relations such as

$$\{g,f\}=-\{f,g\}\,,\qquad \{f,g_1+g_2\}=\{f,g_1\}+\{f,g_2\}\,, \tag{3.4}$$

etc. There is also the nontrivial Jacobi identity

$$\{f,\{g,h\}\}+\{h,\{f,g\}\}+\{g,\{h,f\}\}=0 \tag{3.5}$$

that we quote without proof here. It follows then that if both f and g are integrals of motion, then $\{f,g\}$ is also an integral of motion. For the proof we take $h=\mathcal{H}$, then Jacobi identity takes the form

$$\left\{f,\ \underbrace{\{g,\mathcal{H}\}}_{0}\right\}+\{\mathcal{H},\{f,g\}\}+\left\{g,\ \underbrace{\{\mathcal{H},f\}}_{0}\right\}=\{\mathcal{H},\{f,g\}\}=0, \tag{3.6}$$

thus $\{f,g\}=\text{const}$. In principle, this could be used to obtain new integrals of motion from already known "old" integrals of motion. However, in most cases new integrals of motion $\{f,g\}$ are trivial, such as functions of the old integrals of motion.

The straightforwardly obtainable Poisson brackets

$$\begin{aligned}\{q_i,q_j\} &= 0, \qquad \{q_i,p_j\}=\delta_{ij}\\ \{p_i,q_j\} &= -\delta_{ij}, \qquad \{p_i,p_j\}=0\end{aligned} \tag{3.7}$$

are called *fundamental* Poisson brackets.

If the Hamiltonian depends on one generalized coordinate and the corresponding generalized momentum as a combination that does not contain other variables, this combination is an integral of motion. That is, for

$$\mathcal{H}\left(\{q_i,p_i\}\right)=\mathcal{H}\left(\{q_i,p_i\}_{i\neq 1};F(q_1,p_1)\right) \tag{3.8}$$

$F(q_1,p_1)$ is an integral of motion. This can be proven with the help of Poisson brackets. Since differentiation with respect to the variables $i\neq 1$ gives zero, one obtains

$$\dot{F}=\{F,\mathcal{H}\}=\frac{\partial F}{\partial q_1}\frac{\partial \mathcal{H}}{\partial p_1}-\frac{\partial F}{\partial p_1}\frac{\partial \mathcal{H}}{\partial q_1}=\frac{\partial \mathcal{H}}{\partial F}\left(\frac{\partial F}{\partial q_1}\frac{\partial F}{\partial p_1}-\frac{\partial F}{\partial p_1}\frac{\partial F}{\partial q_1}\right)=\frac{\partial \mathcal{H}}{\partial F}\{F,F\}=0. \tag{3.9}$$

This result can be generalized for *nested* dependences of the type

$$\mathcal{H}\left(\{q_i,p_i\}\right)=\mathcal{H}\left(\{q_i,p_i\}_{i\neq 1,2};F_2(q_2,p_2;F_1(q_1,p_1))\right). \tag{3.10}$$

Here

$$F_1(q_1,p_1)=\alpha_1,\qquad F_2(q_2,p_2;\alpha_1)=\alpha_2 \tag{3.11}$$

are integrals of motion. For F_2 one obtains

$$\begin{aligned}\dot{F}_2 &= \{F_2,\mathcal{H}\}=\frac{\partial F_2}{\partial q_1}\frac{\partial \mathcal{H}}{\partial p_1}-\frac{\partial F_2}{\partial p_1}\frac{\partial \mathcal{H}}{\partial q_1}+\frac{\partial F_2}{\partial q_2}\frac{\partial \mathcal{H}}{\partial p_2}-\frac{\partial F_2}{\partial p_2}\frac{\partial \mathcal{H}}{\partial q_2}\\ &= \left(\frac{\partial F_2}{\partial F_1}\right)^2\frac{\partial \mathcal{H}}{\partial F_2}\{F_1,F_1\}+\frac{\partial \mathcal{H}}{\partial F_2}\{F_2,F_2\}=0.\end{aligned} \tag{3.12}$$

3.1 Example: Particle's motion in spherical coordinates

As an example of nontrivial integrals of motion detectable with the help of Poisson brackets consider a particle moving in the potential

$$U(r,\theta,\phi) = a(r) + \frac{b(\theta)}{r^2} + \frac{c(\phi)}{r^2\sin^2\theta} \tag{3.13}$$

in the spherical coordinates. In this case with the help of Eq. (1.18) one obtains the Hamiltonian of the *nested* form

$$\mathcal{H} = \frac{1}{2m}\left[p_r^2 + 2ma(r) + \frac{p_\theta^2 + 2mb(\theta)}{r^2} + \frac{p_\phi^2 + 2mc(\phi)}{r^2\sin^2\theta}\right]. \tag{3.14}$$

In accordance with the above, one has integrals of motion

$$p_\phi^2 + 2mc(\phi) = \alpha_\phi \tag{3.15}$$

and

$$p_\theta^2 + 2mb(\theta) + \frac{\alpha_\phi}{\sin^2\theta} = \alpha_\theta. \tag{3.16}$$

If $c(\phi) = 0$, ϕ is cyclic variable and the first integral of motion simplifies to $p_\phi = \text{const}$. With these two integrals of motion, the problem becomes effectively one-dimensional with respect to r,

$$\mathcal{H} = \frac{1}{2m}\left[p_r^2 + 2ma(r) + \frac{\alpha_\theta}{r^2}\right], \tag{3.17}$$

and can be integrated directly using energy conservation, $\mathcal{H} = E = \text{const}$.

Integration of the whole problem can be done in three steps. From Eq. (3.14) one obtains

$$p_r = m\dot{r} = \sqrt{2mE - 2ma(r) - \alpha_\theta/r^2} \tag{3.18}$$

that can be integrated

$$t = m\int \frac{dr}{\sqrt{2mE - 2ma(r) - \alpha_\theta/r^2}} \tag{3.19}$$

to implicitly find $r(t)$.

Second, $\theta(t)$ can be defined using the integral of motion α_θ, Eq. (3.16):

$$p_\theta = mr^2\dot{\theta} = \sqrt{\alpha_\theta - 2mb(\theta) - \frac{\alpha_\phi}{\sin^2\theta}}. \tag{3.20}$$

This equation can be integrated to define $\theta(t)$ implicitly via

$$\int \frac{d\theta}{\sqrt{\alpha_\theta - 2mb(\theta) - \alpha_\phi/\sin^2\theta}} = \int \frac{dt}{mr^2(t)}. \tag{3.21}$$

Note that to work out the integral in the rhs, one at first has to find $r(t)$ from Eq. (3.19).

Finally, $\phi(t)$ can be defined using the integral of motion α_ϕ

$$p_\phi = mr^2\sin^2\theta\,\dot{\phi} = \sqrt{\alpha_\phi - 2mc(\phi)}. \tag{3.22}$$

This equation can be integrated to define $\phi(t)$ implicitly via

$$\int \frac{d\phi}{\sqrt{\alpha_\phi - 2mc(\phi)}} = \int \frac{dt}{mr^2(t)\sin^2\theta(t)}. \tag{3.23}$$

Note that previous two steps are needed to work out the integral on the right.

The situation allowing to obtain the solution of this and similar problems in quadratures is called *separation of variables*. Note that building integrals of motion goes in the direction inside→outside, whereas integrating the resulting separated equations goes in the direction outside→inside.

3.2 Poisson brackets and commutators

Let us finally work out the correspondence between Poisson brackets and commutators. In quantum mechanics the fundamental commutators

$$\hat{q}_i = q_i, \qquad \hat{p}_i = -i\hbar \frac{d}{dq_i}, \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{3.24}$$

are similar to the fundamental Poisson brackets, Eq. (3.7). Operator functions like $f(\hat{q}, \hat{p})$ that correspond to classical functions $f(q,p)$ are interpreted as Taylor series with full symmetrization over permutations of terms. The commutator $[f,g]$ can be calculated, for instance, by expanding both f and g in Taylor series in $\hat{q}$ and $\hat{p}$, commuting term by term, and then summing the series back. The final result of this procedure can be obtained much easier by formally substituting

$$[f,g] \Rightarrow \left[\sum_i \left(\frac{\partial f}{\partial \hat{q}_i} \hat{q}_i + \frac{\partial f}{\partial \hat{p}_i} \hat{p}_i \right), \sum_j \left(\frac{\partial g}{\partial \hat{q}_j} \hat{q}_j + \frac{\partial g}{\partial \hat{p}_j} \hat{p}_j \right) \right] \tag{3.25}$$

and then commuting the momentum and coordinate operators considering partial derivatives as numbers. With the help of Eq. (3.24) this yields the relation

$$[f,g] = i\hbar \{f,g\}. \tag{3.26}$$

In this formula the Poisson bracket should be symmetrized over permutations, $\partial_{\hat{q}} f \, \partial_{\hat{p}} g \Rightarrow (\partial_{\hat{q}} f \, \partial_{\hat{p}} g + \partial_{\hat{p}} g \, \partial_{\hat{q}} f)/2$ etc., since $\hat{q}$ and $\hat{p}$ are operators. If $\hbar$ is formally considered as small, as is sometimes done in the analysis of the semiclassical case, the symmetrization is irrelevant. This is because each commutation introduces a factor $\hbar$, so that changing the order of terms changes the result by terms starting with $\hbar^2$.

4 Canonical transformations

As any system of differential equations, Hamiltonian equations (1.6) allow change of variables

$$Q_i = Q_i(q,p,t), \qquad P_i = P_i(q,p,t), \tag{4.1}$$

where q,p are "old" variables and Q,P are "new" variables. The transformation of variables above is called *canonical* if the transformed Hamiltonian equations also have a Hamiltonian (canonical) form

$$\dot{Q}_i = \frac{\partial \mathcal{H}'}{\partial P_i}, \qquad \dot{P}_i = -\frac{\partial \mathcal{H}'}{\partial Q_i}, \tag{4.2}$$

where, in principle, $\mathcal{H}'$ can differ from $\mathcal{H}$. Sometimes one can find a canonical transformation that results in a simple Hamiltonian function allowing for an easy solution. In particular, if $\mathcal{H}'$ does not depend on Q_i, the variable Q_i is called *cyclic*. The second of the above equations then yields $\dot{P}_i = 0$ and $P_i = \text{const}$. Now, in the case of one degree of freedom, the rhs of the first equation above is an integral of motion, $\dot{Q} = \alpha = \text{const}$. This equation can be easily integrated, $Q = \alpha t + \text{const}$.

4.1 Least-action principle and generating functions

Possible relations between old and new variables can be obtained in the elegant and powerful form from the least-action principle. Validity of both Eqs. (1.6) and (4.2) implies that action $\mathcal{S}$ in terms of the old variables, Eq. (2.1), is equivalent to the action in terms of the new variables. It is sufficient to require that the integrands of the two actions differ by a full differential of some function F. Then variations of both actions coincide, so that $\delta\mathcal{S}=0$ for the old variables entails $\delta\mathcal{S}=0$ for the new variables. Then validity of Eq. (1.6) entails validity of Eq. (4.2). The difference between the two action integrands having the form

$$\sum_i p_i dq_i - \sum_i P_i dQ_i + \left(\mathcal{H}' - \mathcal{H}\right) dt = dF \tag{4.3}$$

suggests that $F = F(q, Q, t)$ and

$$p_i = \frac{\partial F}{\partial q_i}, \qquad P_i = -\frac{\partial F}{\partial Q_i}, \qquad \mathcal{H}' - \mathcal{H} = \frac{\partial F}{\partial t}. \tag{4.4}$$

These formulas establish relations between new and old variables. Function $F(q, Q, t)$ is *generating* function of the canonical transformation. In the absence of explicit time dependence of the canonical transformation, one has $\mathcal{H}' = \mathcal{H}$.

Canonical transformation specified by Eq. (4.4) can be put in other equvalent forms with the help of different Legendre transformations. For instance, differential of the function

$$\Phi = F + \sum_i Q_i P_i \tag{4.5}$$

with the help of Eq. (4.3) takes the form

$$d\Phi = \sum_i p_i dq_i + \sum_i Q_i dP_i + \left(\mathcal{H}' - \mathcal{H}\right) dt. \tag{4.6}$$

thus $\Phi = \Phi\left(q, P, t\right)$ and

$$p_i = \frac{\partial \Phi}{\partial q_i}, \qquad Q_i = \frac{\partial \Phi}{\partial P_i}, \qquad \mathcal{H}' - \mathcal{H} = \frac{\partial \Phi}{\partial t}. \tag{4.7}$$

One can introduce other types of generating functions such as $F - \sum_i q_i p_i$, $F - \sum_i q_i p_i + \sum_i Q_i P_i$, and many functions in which Legendre transformation is done with respect to some select canonical pairs $q_i p_i$ or $Q_i P_i$ with particular values of i. In all cases generating functions contain old and new variables for each degree of freedom i. It should be stressed that Legendre transformations lead to other expressions for the same canonical transformation specified by the generating function F rather than to new canonical transformations.

4.2 Trivial examples of canonical transformations

Transformation defined by the generating function

$$\Phi(q, P) = \sum_i q_i P_i \tag{4.8}$$

has the form

$$p_i = \frac{\partial \Phi}{\partial q_i} = P_i, \qquad Q_i = \frac{\partial \Phi}{\partial P_i} = q_i \tag{4.9}$$

or

$$Q_i = q_i, \qquad P_i = p_i. \tag{4.10}$$

This is the identity transformation.

Transformation with

$$F(q, Q) = \sum_i q_i Q_i \tag{4.11}$$

has the form

$$p_i = \frac{\partial F}{\partial q_i} = Q_i, \qquad P_i = -\frac{\partial F}{\partial Q_i} = -q_i \tag{4.12}$$

or

$$Q_i = p_i, \qquad P_i = -q_i. \tag{4.13}$$

This transformation interchanges generalized coordinates and momenta. The above example shows that there is no essential difference between generalized coordinates and momenta in the Hamiltonian formalism. One cannot say that generalized momenta are related to velocities while generalized coordinates not.

It should be noted that for the above two transformations that are obviously canonical, an attempt to make Legendre transformation (4.5) does not work. For instance, one cannot find the primary form $F(q,Q)$ of the transformation, the secondary form $\Phi(q,P)$ of which is given by Eq. (4.8). Since only $F(q,Q)$ follows from the least-action principle and it does not exist here, $\Phi(q,P)$ loses its relation to the general formalism. It looks like there are more canonical transformations than those following from the least-action principle.

4.3 Poisson-brackets criterion of canonicity

Canonicity of transformations that do not explicitly depend on time can be checked with the help of the Poisson-brackets criterion. This criterion can be obtained directly by changing variables in Hamilton equations. The key point here is that, according to Eq. (4.7), the Hamiltonian is the same for both sets of dynamic variables, $\mathcal{H}' = \mathcal{H}$. For instance, equations for Q_i follow as

$$\begin{aligned}
\dot{Q}_i &= \frac{\partial Q_i}{\partial q_j}\dot{q}_j + \frac{\partial Q_i}{\partial p_j}\dot{p}_j = \frac{\partial Q_i}{\partial q_j}\frac{\partial \mathcal{H}}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial \mathcal{H}}{\partial q_j} = \frac{\partial Q_i}{\partial q_j}\left(\frac{\partial \mathcal{H}}{\partial Q_l}\frac{\partial Q_l}{\partial p_j} + \frac{\partial \mathcal{H}}{\partial P_l}\frac{\partial P_l}{\partial p_j}\right) - \frac{\partial Q_i}{\partial p_j}\left(\frac{\partial \mathcal{H}}{\partial Q_l}\frac{\partial Q_l}{\partial q_j} + \frac{\partial \mathcal{H}}{\partial P_l}\frac{\partial P_l}{\partial q_j}\right) \\
&= \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial Q_l}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial Q_l}{\partial q_j}\right)\frac{\partial \mathcal{H}}{\partial Q_l} + \left(\frac{\partial Q_i}{\partial q_j}\frac{\partial P_l}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial P_l}{\partial q_j}\right)\frac{\partial \mathcal{H}}{\partial P_l} = \{Q_i, Q_l\}\frac{\partial \mathcal{H}}{\partial Q_l} + \{Q_i, P_l\}\frac{\partial \mathcal{H}}{\partial P_l}.
\end{aligned} \tag{4.14}$$

Here summation over repeated indices is assumed. Eqiations for Q_i are canonical, if

$$\{Q_i, Q_l\}_{qp} = 0, \qquad \{Q_i, P_l\}_{qp} = \delta_{il}. \tag{4.15}$$

Similarly one can obtain that equations for P_i are canonical if

$$\{P_i, Q_l\}_{qp} = -\delta_{il}, \qquad \{P_i, P_l\}_{qp} = 0. \tag{4.16}$$

Index qp shows that Poisson brackets are calculated with respect to the old (original) variables. The above means that fundamental Poisson brackets have the same form as those for the old variables, Eq. (3.7). However, Eq. (3.7) is trivial, whereas Eqs. (4.15) and (4.16) are not. As transformation can be done also in the opposite direction, $(Q,P) \Rightarrow (q,p)$, by the same method one obtains

$$\begin{aligned}
\{q_i, q_j\}_{QP} &= 0, \qquad \{q_i, p_j\}_{QP} = \delta_{ij} \\
\{p_i, q_j\}_{QP} &= -\delta_{ij}, \qquad \{p_i, p_j\}_{QP} = 0
\end{aligned} \tag{4.17}$$

as canonicity criterion.

4.4 Harmonic oscillator via canonical transformation

Another example of canonical transformations is the harmonic oscillator having the Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + \frac{kq^2}{2} = \frac{1}{2m}\left(p^2 + m^2\omega^2 q^2\right), \tag{4.18}$$

where $\omega = \sqrt{k/m}$ is the oscillator's eigenfrequency. Quadratic dependence on q and p suggests to use a transformation of the type

$$q = \frac{f(P)}{m\omega}\sin Q, \qquad p = f(P)\cos Q \tag{4.19}$$

that leads to the transformed Hamiltonian

$$\mathcal{H} = \frac{f^2(P)}{2m}\left(\cos^2 Q + \sin^2 Q\right) = \frac{f^2(P)}{2m} \tag{4.20}$$

that is independent of Q. Here function $f(P)$ can be obtained from the Poisson-brackets canonicity criterion, Eq. (4.17). In particular, one should have

$$\begin{aligned} 1 &= \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} = \frac{f(P)}{m\omega}\cos Q \times \frac{df(P)}{dP}\cos Q + \frac{1}{m\omega}\frac{df(P)}{dP}\sin Q \times f(P)\sin Q \\ &= \frac{f(P)}{m\omega}\frac{df(P)}{dP} = \frac{1}{2m\omega}\frac{d}{dP}f^2(P). \end{aligned} \tag{4.21}$$

Integrating this equation, one obtains

$$f(P) = \sqrt{2Pm\omega}. \tag{4.22}$$

This yields

$$q = \sqrt{\frac{2P}{m\omega}}\sin Q, \qquad p = \sqrt{2Pm\omega}\cos Q. \tag{4.23}$$

Now the new Hamiltonian has the form

$$\mathcal{H} = \omega P. \tag{4.24}$$

The generalized momentum

$$P = \frac{\mathcal{H}}{\omega} = \frac{E}{\omega} \tag{4.25}$$

is conserved. The interpretation of P is action over the period of motion I divided by 2π, see Eq. (5.37). The equation of motion for the cyclic variable Q is

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = \omega. \tag{4.26}$$

Its solution reads

$$Q = \omega t + \varphi_0, \tag{4.27}$$

where $\varphi_0 = \text{const}$. One can see that Q is the oscillator's phase angle. Inserting the above results into Eq. (4.23), one finally obtains the solution

$$q = \sqrt{\frac{2E}{m\omega^2}}\sin\left(\omega t + \varphi_0\right), \qquad p = \sqrt{2mE}\cos\left(\omega t + \varphi_0\right). \tag{4.28}$$

Generating function of the above canonical transformation can be found by integrating the equations

$$p = \frac{\partial F}{\partial q}, \qquad P = -\frac{\partial F}{\partial Q}. \tag{4.29}$$

Before this, one has to express p or P via q and Q with the help of Eq. (4.23). For instance,

$$m\omega q^2 + \frac{p^2}{m\omega} = m\omega q^2 + 2P\cos^2 Q = 2P, \tag{4.30}$$

thus

$$P = \frac{m\omega q^2}{2}\frac{1}{\sin^2 Q}. \tag{4.31}$$

Integrating this equation and discarding the integration constant one obtains

$$F(q, Q) = \frac{m\omega q^2}{2}\cot Q. \tag{4.32}$$

From this follows

$$p = \frac{\partial F}{\partial q} = m\omega q\cot Q, \qquad P = -\frac{\partial F}{\partial Q} = \frac{m\omega q^2}{2}\frac{1}{\sin^2 Q}. \tag{4.33}$$

Resolving the second of these equations for q and then substituting the result into the first equation, one obtains Eq. (4.23).

4.5 Symplectic formalism

Hamiltonian equations can be put into a compact and elegant symplectic form. Introducing the dynamical vector

$$\mathbf{x} = \{x_1, x_2, \ldots, x_{2N}\} = \{q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N\} \tag{4.34}$$

one can write Hamiltonian equations (1.6) in the form

$$\dot{\mathbf{x}} = \mathbb{J} \cdot \frac{\partial \mathcal{H}}{\partial \mathbf{x}}, \qquad \dot{x}_i = J_{ij} \frac{\partial \mathcal{H}}{\partial x_j}, \tag{4.35}$$

where summation over repeated indices is assumed and matrix $\mathbb{J}$ is given by

$$J_{ij} = \delta_{i+N,j} - \delta_{i-N,j}. \tag{4.36}$$

Let us now perform a canonical transformation (4.1) without time dependence and introduce the new dynamical vector

$$\mathbf{y} = \{y_1, y_2, \ldots, y_{2N}\} = \{Q_1, Q_2, \ldots, Q_N, P_1, P_2, \ldots, P_N\}. \tag{4.37}$$

The equation of motion for $\mathbf{y}$ follows from Eq. (4.35)

$$\dot{y}_i = \frac{\partial y_i}{\partial x_j} \dot{x}_j = \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial \mathcal{H}}{\partial x_k} = \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial y_l}{\partial x_k} \frac{\partial \mathcal{H}}{\partial y_l}. \tag{4.38}$$

The condition that the resulting equation is Hamiltonian and thus the transformation is canonical in the vector and component forms reads

$$\mathbb{M} \cdot \mathbb{J} \cdot \mathbb{M}^T = \mathbb{J}, \qquad \frac{\partial y_i}{\partial x_j} J_{jk} \frac{\partial y_l}{\partial x_k} = J_{il}, \tag{4.39}$$

where $M_{ij} \equiv \partial y_i / \partial x_j$ is the Jacobian matrix of the transformation. The condition above contains all four conditions of the standard formalism, Eqs. (4.15) and (4.16). One can prove the inverse and more general statement: If the transformation is canonical, Poisson brackets of *any* two variables A and B are invariant with respect to the transformation. The proof uses Eq. (4.39):

$$\{A, B\}_x = \frac{\partial A}{\partial x_i} J_{ij} \frac{\partial B}{\partial x_j} = \frac{\partial A}{\partial y_k} \frac{\partial y_k}{\partial x_i} J_{ij} \frac{\partial y_l}{\partial x_j} \frac{\partial B}{\partial y_l} = \frac{\partial A}{\partial y_l} J_{kl} \frac{\partial B}{\partial y_l} = \{A, B\}_y. \tag{4.40}$$

4.6 Relation between generating-function and Poisson-brackets canonicity criteria.

It is of interest to perform the direct canonicity check provided by Eqs. (4.15) and (4.16) for canonical transformations via generating functions obtained from the least-action principle in Sec. 4.1. The resulting equivalence between the two seemingly unrelated approaches is referred to as Carathéodory theorem. For simplicity, we will produce the proof for one degree of freedom. Using the second of relations (4.4), one can write

$$\{Q, P\}_{qp} = \frac{\partial Q}{\partial q} \frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p} \frac{\partial P}{\partial q} = -\frac{\partial Q}{\partial q} \left. \frac{\partial^2 F(q,Q)}{\partial Q \partial p} \right|_q + \frac{\partial Q}{\partial p} \left. \frac{\partial^2 F(q,Q)}{\partial Q \partial q} \right|_p. \tag{4.41}$$

Here differentiation of F over q and p should be done taking into account $F = F(q, Q(q, p))$:

$$\{Q, P\}_{qp} = \left(-\frac{\partial Q}{\partial q} \frac{\partial^2 F(q,Q)}{\partial^2 Q} \frac{\partial Q}{\partial p} + \frac{\partial Q}{\partial p} \frac{\partial^2 F(q,Q)}{\partial Q \partial q} + \frac{\partial Q}{\partial p} \frac{\partial^2 F(q,Q)}{\partial Q^2} \frac{\partial Q}{\partial q} \right) = \frac{\partial Q}{\partial p} \frac{\partial p}{\partial Q} = 1. \tag{4.42}$$

This proves canonicity. In the remaining term we used the first of relations (4.4) and simplified this term as

$$\frac{\partial Q(q, p(q,Q))}{\partial p(q,Q)} \frac{\partial p(q,Q)}{\partial Q} = \frac{\partial Q}{\partial Q} = 1. \tag{4.43}$$

For more than one degree of freedom the proof seems to be much more involved.

5 Action as function of coordinates and Hamilton-Jacobi equation

5.1 General formulation with a simple example

Action $\mathcal{S}$ given by Eq. (2.1) was used to derive Lagrangian and Hamiltonian equations of motion from the least-action principle, $\delta\mathcal{S} = 0$. This condition singles out the real physical trajectory from all other competing trajectories. Here we will consider the action for the real trajectory as function of the upper-limit variables $t_2 \Rightarrow t$ and $q_2 \Rightarrow q$. The expression for $\mathcal{S}$ in the Hamiltonian form, the last term in Eq. (2.1), shows that as q_i and t change in the course of motion, action acquires corresponding increments, so that infinitesimally one has

$$d\mathcal{S} = \sum_i p_i dq_i - \mathcal{H}dt. \tag{5.1}$$

This implies that $\mathcal{S} = \mathcal{S}(q,t)$ and

$$\frac{\partial \mathcal{S}}{\partial q_i} = p_i, \qquad \frac{\partial \mathcal{S}}{\partial t} = -\mathcal{H}. \tag{5.2}$$

Let us illustrate this on a free particle in one dimension. The Lagrangian has the form $\mathcal{L} = T = mv^2/2$, and the trajectory is described by $q = vt$. Thus the action reads

$$\mathcal{S} = \int^t dt\, \mathcal{L} = \frac{mv^2}{2}t + \text{const} = \frac{mq^2}{2t} + \text{const}. \tag{5.3}$$

Now one can check that

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial q} &= \frac{mq}{t} = mv = p \\ \frac{\partial \mathcal{S}}{\partial t} &= -\frac{mq^2}{2t^2} = -\frac{mv^2}{2} = -\frac{p^2}{2m} = -\mathcal{H}, \end{aligned} \tag{5.4}$$

as it should be.

If the problem does not explicitly depend on time, energy is conserved, $\mathcal{H} = \text{const} = E$. Then the second of equations (5.2) can be integrated that yields

$$\mathcal{S}(q,t) = \mathcal{S}_0(q) - Et. \tag{5.5}$$

Here $\mathcal{S}_0(q)$ is the so-called short or abbreviated action that can be obtained by integrating the first of equations (5.2). Alternatively, using Eq. (2.1) with $\int_{t_1}^{t_2} \Rightarrow \int^t$and $\mathcal{H} \Rightarrow E$, one obtains Eq. (5.5) with

$$\mathcal{S}_0(q) = \sum_i \int^{q_i} p_i dq_i. \tag{5.6}$$

Equations (5.2) can be used to set up the famous Hamilton-Jacobi equation that together with canonical transformations is an efficient tool for finding analytical solutions of mechanical problems. Hamilton-Jacobi equation

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}\left(q, \frac{\partial \mathcal{S}}{\partial q}, t\right) = 0 \tag{5.7}$$

is a nonlinear first-order partial differential equation (PDE) for the function $\mathcal{S}(q,t)$. As usual, q and $\partial\mathcal{S}/\partial q$ in the arguments stand for the whole sets of q_i and $\partial\mathcal{S}/\partial q_i = p_i$. For practical purposes it is sufficient to find just some solution of Hamilton-Jacobi equation rather than its most general solution.

In particular, for the free particle considered above, Eq. (5.7) becomes

$$\frac{\partial \mathcal{S}}{\partial t} + \frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial q}\right)^2 = 0. \tag{5.8}$$

The solution of this equation can be searched for in the form

$$\mathcal{S}\left(q,t\right)=\mathcal{S}_{0}\left(q\right)+\mathcal{S}^{\prime}\left(t\right). \tag{5.9}$$

Then Eq. (5.8) becomes

$$\frac{\partial\mathcal{S}^{\prime}}{\partial t}+\frac{1}{2m}\left(\frac{\partial\mathcal{S}_{0}}{\partial q}\right)^{2}=0. \tag{5.10}$$

To satisfy this equation, both terms should be constants compensating each other, so that

$$\frac{\partial\mathcal{S}^{\prime}}{\partial t}=\mathrm{const}=-E,\qquad\frac{\partial\mathcal{S}_{0}}{\partial q}=\mathrm{const}=p \tag{5.11}$$

and the constants satisfy

$$E=\frac{p^{2}}{2m},\qquad p=\sqrt{2mE}. \tag{5.12}$$

Integrating Eq. (5.11), one obtains $\mathcal{S}\left(q,t\right)$ in the two alternative forms

$$\mathcal{S}\left(q,t\right)=pq-\frac{p^{2}}{2m}t=\sqrt{2mE}q-Et, \tag{5.13}$$

up to an irrelevant constant. One can check that this result coincides with Eq. (5.3):

$$\mathcal{S}\left(q,t\right)=pq-\frac{p^{2}}{2m}t=\frac{mq}{t}q-\left(\frac{mq}{t}\right)^{2}\frac{t}{2m}=\frac{mq^{2}}{2t}. \tag{5.14}$$

Whereas the solution of Hamilton-Jacobi equation for one degree of freedom such as Eq. (5.13) depends on one constant, the so-called *complete integral* of Hamilton-Jacobi equation for N degrees of freedom depends on N constants that we call P_i. On the top of it, one can always add an irrelevant constant to $\mathcal{S}\left(q,t\right)$ that has been suppressed in Eq. (5.13). The complete integral yields the solution for the system's dynamics if one uses it as the generating function of a canonical transformation in terms of the old coordinates and new momenta, $\Phi\left(q,P,t\right)$ of Eq. (4.5). The new momenta are constants P_i in the complete integral. The new Hamiltonian $\mathcal{H}^{\prime}$ given by Eq. (4.7) vanishes according to Hamilton-Jacobi equation:

$$\mathcal{H}^{\prime}=\mathcal{H}+\frac{\partial\Phi}{\partial t}=\mathcal{H}+\frac{\partial\mathcal{S}}{\partial t}=\mathcal{H}-\mathcal{H}=0. \tag{5.15}$$

Thus Hamiltonian equations for the new dynamic variables Q_i and P_i become trivial:

$$\begin{aligned}\dot{Q}_{i}&=0\qquad\Longrightarrow\qquad Q_{i}=\mathrm{const}\\ \dot{P}_{i}&=0\qquad\Longrightarrow\qquad P_{i}=\mathrm{const}.\end{aligned} \tag{5.16}$$

Time dependences of the old variables q_i and p_i can be obtained from the first two equations (4.7). At first q_i are found resolving the equations

$$Q_{i}=\frac{\partial\mathcal{S}}{\partial P_{i}}. \tag{5.17}$$

Then p_i are given by the formulas

$$p_{i}=\frac{\partial\mathcal{S}}{\partial q_{i}}. \tag{5.18}$$

Some literature uses α_i and β_i as new momenta and coordinates within Hamilton-Jacobi formalism, $\alpha_i\equiv P_i$ and $\beta_i\equiv Q_i$.

Let us illustrate finding dynamical solution for the free particle. One can choose constant p in the first expression in Eq. (5.13) as the new momentum P,

$$\mathcal{S}\left(q,P,t\right)=Pq-\frac{P^{2}}{2m}t. \tag{5.19}$$

Equation (5.17) takes the form

$$Q=\frac{\partial\mathcal{S}}{\partial P}=q-\frac{P}{m}t \tag{5.20}$$

that yields the solution

$$q=\frac{P}{m}t+Q=\frac{P}{m}t+\text{const.} \tag{5.21}$$

Then momentum is defined by Eq. (5.18):

$$p=\frac{\partial\mathcal{S}}{\partial q}=P=\text{const.} \tag{5.22}$$

Thus Eq. (5.21) reproduces the well-known solution $q=(p/m)\,t+\text{const}$ for a free particle.

Alternatively one can use the second expression in Eq. (5.13) and choose energy E as conserved new momentum P. Now instead of Eq. (5.20) one obtains

$$Q=\frac{\partial\mathcal{S}}{\partial E}=\sqrt{\frac{m}{2E}}q-t \tag{5.23}$$

that yields the familiar solution

$$q=\sqrt{\frac{2E}{m}}t+\sqrt{\frac{2E}{m}}Q=\sqrt{\frac{2E}{m}}t+\text{const.} \tag{5.24}$$

The old momentum is given by the familiar formula

$$p=\frac{\partial\mathcal{S}}{\partial q}=\sqrt{2mE}. \tag{5.25}$$

This alternative solution using energy rather than momentum as conserved new momentum is preferred because it survives for systems with nontrivial potential energy where momentum is not conserved. As an example we will consider the harmonic oscillator in the next section.

5.2 Harmonic oscillator by Hamilton-Jacobi method

With the harmonic-oscillator Hamiltonian of Eq. (4.18), Hamilton-Jacobi equation (5.7) becomes

$$\frac{\partial\mathcal{S}}{\partial t}+\frac{1}{2m}\left(\frac{\partial\mathcal{S}}{\partial q}\right)^2+U(q)=0,\qquad U(q)=\frac{m\omega^2q^2}{2}. \tag{5.26}$$

Action has the form of Eq. (5.5) with $\mathcal{S}_0$ satisfying the equation

$$\frac{1}{2m}\left(\frac{\partial\mathcal{S}_0}{\partial q}\right)^2+U(q)=E. \tag{5.27}$$

Resolving this equation and integrating, one obtains

$$\mathcal{S}\left(q,E,t\right)=\int^q dq'\sqrt{2m\left[E-U(q')\right]}-Et. \tag{5.28}$$

Using this as generating function $\Phi\left(q,P,t\right)$ with $P=E$, one obtains the implicit formula for $q(t)$

$$Q=\frac{\partial\mathcal{S}}{\partial E}=\int^q dq'\sqrt{\frac{m}{2\left[E-U(q')\right]}}-t. \tag{5.29}$$

For the harmonic oscillator one can calculate the integral analytically as follows

$$Q=\int^q dq'\sqrt{\frac{m}{2E-m\omega^2q'^2}}-t=\frac{1}{\omega}\int^{\tilde{q}}\frac{d\tilde{q}'}{\sqrt{1-\tilde{q}'^2}}-t=\frac{1}{\omega}\arcsin\tilde{q}-t, \tag{5.30}$$

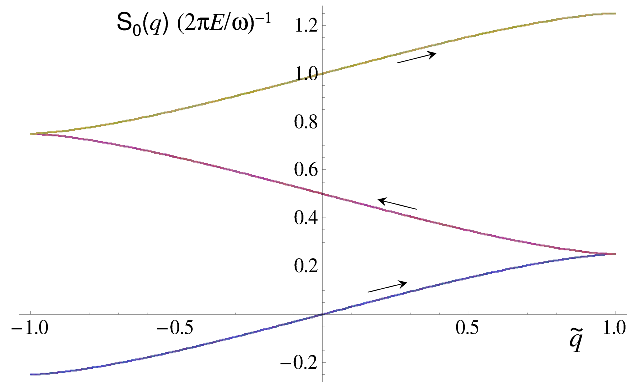

Figure 5.1: Short action of the harmonic oscillator

where

$$\tilde{q} \equiv \sqrt{\frac{m\omega^2}{2E}} q. \tag{5.31}$$

Inverting Eq. (5.30), one obtains the well-known solution

$$\tilde{q} = \sin(\omega t + \omega Q) = \sin(\omega t + \varphi_0) \tag{5.32}$$

or

$$q = \sqrt{\frac{2E}{m\omega^2}} \sin(\omega t + \varphi_0). \tag{5.33}$$

After that one finds p as

$$p = \frac{\partial \mathcal{S}}{\partial q} = \sqrt{2m [E - U(q)]} \tag{5.34}$$

that for the harmonic oscillator yields the well-known expression

$$p = \sqrt{2mE} \sqrt{1 - \tilde{q}^2} = \sqrt{2mE} \cos(\omega t + \varphi_0). \tag{5.35}$$

Finally we calculate the short action $\mathcal{S}_0$, the integral term in Eq. (5.28). The result has the form

$$\mathcal{S}_0(q) = I \left(\tilde{q} \sqrt{1 - \tilde{q}^2} + \arcsin \tilde{q} \right), \qquad I = \frac{E}{\omega}. \tag{5.36}$$

Here I is the harmonic-oscillator form of the so-called action variable that will be used below. $\mathcal{S}_0(q)$ is a multivalued function of q, taking into account different branches of the $\sqrt{\ldots}$ and $\arcsin(\ldots)$ functions, see Fig. 5.1. As $\tilde{q}$ changes from 0 to 1 that is the quarter of the oscillation period, the expression in brackets changes from 0 to $\pi/2$. The same happens each quarter period, so that the change over the period makes up 2π. Thus short action over the period is given by

$$\Delta \mathcal{S}_0^{(\text{Period})} = 2\pi I. \tag{5.37}$$

Unlike short action $\mathcal{S}_0$, the full action $\mathcal{S}$ does not increase with time on average. Since $\mathcal{L} = E_k - U$ and both kinetic and potential energies oscillate having the same average values, $\mathcal{S} = \int dt \mathcal{L}$ oscillates without growing.

5.3 Separation of variables

The method of solving Hamilton-Jacobi equation applied to the harmonic oscillator in the preceding section works for any system with one degree of freedom, described by a potential energy $U(q)$. This method can be generalized for systems with N degrees of freedom than allow separation of variables. In such systems one or more canonical pairs (q_i, p_i) enter the Hamiltonian $\mathcal{H}$ as combinations that do not contain other dynamical variables. In the case of one such pair the Hamilton-Jacobi equation has the form

$$\frac{\partial \mathcal{S}}{\partial t} + \mathcal{H}\left(q_{i\neq 1}, \frac{\partial \mathcal{S}}{\partial q_{i\neq 1}}, F_1\left(q_1, \frac{\partial \mathcal{S}}{\partial q_1}\right), t\right) = 0. \tag{5.38}$$

The solution can be searched for in the form of the sum

$$\mathcal{S} = \mathcal{S}^{(N-1)}\left(q_{i\neq 1}, t\right) + \mathcal{S}_0^{(1)}\left(q_1\right). \tag{5.39}$$

With this *Ansatz* Eq. (5.38) becomes

$$\frac{\partial \mathcal{S}^{(N-1)}}{\partial t} + \mathcal{H}\left(q_{i\neq 1}, \frac{\partial \mathcal{S}^{(N-1)}}{\partial q_{i\neq 1}}, F_1\left(q_1, \frac{d\mathcal{S}_0^{(1)}}{dq_1}\right), t\right) = 0. \tag{5.40}$$

Since this equation has to be valid for any value of q_1, condition $F_1 = \text{const} = \alpha_1$ should be fulfilled. Thus Eq. (5.40) splits up into two equations

$$\begin{aligned} F_1\left(q_1, \frac{d\mathcal{S}_0^{(1)}}{dq_1}\right) &= \alpha_1 \\ \frac{\partial \mathcal{S}^{(N-1)}}{\partial t} + \mathcal{H}\left(q_{i\neq 1}, \frac{\partial \mathcal{S}^{(N-1)}}{\partial q_{i\neq 1}}, \alpha_1, t\right) &= 0. \end{aligned} \tag{5.41}$$

This is why this situation is called separation of variables. The first equation here is an ordinary differential equation that allows for a solution in quadratures. The second equation is a Hamilton-Jacobi equation with $N-1$ degrees of freedom.

If the problem is time independent, one can search for the solution in the form

$$\mathcal{S} = -Et + \mathcal{S}_0^{(N-1)}\left(q_{i\neq 1}\right) + \mathcal{S}_0^{(1)}\left(q_1\right). \tag{5.42}$$

This results in simpler equations

$$\begin{aligned} F_1\left(q_1, \frac{\partial \mathcal{S}_0^{(1)}}{\partial q_1}\right) &= \alpha_1 \\ \mathcal{H}\left(q_{i\neq 1}, \frac{\partial \mathcal{S}_0^{(N-1)}}{\partial q_{i\neq 1}}, \alpha_1\right) &= E. \end{aligned} \tag{5.43}$$

A particular case of separation of variables is the case of a cyclic variable q_1 that does not enter the Hamiltonian. Then $F_1\left(q_1, \partial\mathcal{S}_0^{(1)}/\partial q_1\right)$ reduces to $\partial\mathcal{S}_0^{(1)}/\partial q_1$, so that the first equation (5.41) can be easily integrated and Eq. (5.39) becomes

$$\mathcal{S} = \mathcal{S}^{(N-1)}\left(q_{i\neq 1}, t\right) + \alpha_1 q_1. \tag{5.44}$$

Time also is a cyclic variable, and $-Et$ in Eq. (5.42) is similar to $\alpha_1 q_1$ in the above equation.

If there is another separating variable q_2, the second equation (5.41) can be further simplified in terms of ϕ_2, $\mathcal{S}_0^{(2)}$, and the remainder action $\mathcal{S}^{(N-2)}$. If all N variables separate, the procedure results in the complete integral of Hamilton-Jacobi equation of the completely additive form

$$\mathcal{S} = \mathcal{S}^{(0)}(t) + \sum_i^N \mathcal{S}_0^{(i)}\left(q_i, \{\alpha_i\}\right), \tag{5.45}$$

whereas $\mathcal{S}^{(0)}$ satisfies the equation

$$\frac{\partial \mathcal{S}^{(0)}}{\partial t}+\mathcal{H}\left(\alpha_1,\alpha_2,\ldots,\alpha_N,t\right)=0 \tag{5.46}$$

that also can be solved in quadratures. Each term of Eq. (5.45) depends on one or more constants α_i. For time-independent problems one obtains

$$\mathcal{S}^{(0)}(t)=-Et,\qquad E=\mathcal{H}\left(\alpha_1,\alpha_2,\ldots,\alpha_N\right). \tag{5.47}$$

The complete integral above can be used now to define system's dynamics as was explained in Sec. 5.1, see Eqs. (5.17) and (5.18). Let us write these equations again:

$$\beta_i=\frac{\partial \mathcal{S}}{\partial \alpha_i},\qquad \beta_i=\text{const.} \tag{5.48}$$

Note the equivalence $\alpha_i \Leftrightarrow P_i$ and $\beta_i \Leftrightarrow Q_i$. As, in general, $\mathcal{S}_0^{(i)}\left(q_i,\{\alpha_i\}\right)$ depends not only on its own constant α_i but on other constants as well, these equations for different i are coupled. For simpler problems such as three-dimensional harmonic oscillator, one has $\mathcal{S}_0^{(i)}\left(q_i,\{\alpha_i\}\right)=\mathcal{S}_0^{(i)}\left(q_i,\alpha_i\right)$, and different equations above are uncoupled.

5.4 Example: Particle's motion in spherical coordinates

As an example consider a generalization of Kepler problem in spherical coordinates having the Hamiltonian (1.18). As we have seen in Sec. 3.1, here the variables separate for

$$U(r,\theta,\phi)=a(r)+\frac{b(\theta)}{r^2}+\frac{c(\phi)}{r^2\sin^2\theta} \tag{5.49}$$

and integrals of motion can be found with the help of Poisson brackets, after which the problem can be integrated. Now we solve the same problem by Hamilton-Jacobi formalism. Hamilton-Jacobi equation has the form

$$\frac{\partial \mathcal{S}}{\partial t}+\frac{1}{2m}\left(\frac{\partial \mathcal{S}}{\partial r}\right)^2+a(r)+\frac{1}{2mr^2}\left\{\left(\frac{\partial \mathcal{S}}{\partial \theta}\right)^2+2mb(\theta)+\frac{1}{\sin^2\theta}\left[\left(\frac{\partial \mathcal{S}}{\partial \phi}\right)^2+2mc(\phi)\right]\right\}=0. \tag{5.50}$$

Searching for action $\mathcal{S}$ in the additive form

$$\mathcal{S}=-Et+\mathcal{S}_r(r)+\mathcal{S}_\theta(\theta)+\mathcal{S}_\phi(\phi), \tag{5.51}$$

for the parts of action one obtains equations

$$\begin{aligned}\left(\frac{\partial \mathcal{S}_\phi}{\partial \phi}\right)^2+2mc(\phi) &= \alpha_\phi=\text{const}\\ \left(\frac{\partial \mathcal{S}_\theta}{\partial \theta}\right)^2+2mb(\theta)+\frac{\alpha_\phi}{\sin^2\theta} &= \alpha_\theta=\text{const} \qquad (5.52)\\ \left(\frac{\partial \mathcal{S}_r}{\partial r}\right)^2+2ma(r)+\frac{\alpha_\theta}{r^2} &= 2mE=\text{const.} \qquad (5.53)\end{aligned}$$

Note that $\mathcal{S}_\theta$ depends on constants α_ϕ and α_θ, whereas $\mathcal{S}_r$ depends on α_θ and E. Integration of these equations yields

$$\mathcal{S}=-Et+\int dr\sqrt{2mE-2ma(r)-\frac{\alpha_\theta}{r^2}}+\int d\theta\sqrt{\alpha_\theta-2mb(\theta)-\frac{\alpha_\phi}{\sin^2\theta}}+\int d\phi\sqrt{\alpha_\phi-2mc(\phi)} \tag{5.54}$$

that depends on three constants E, α_ϕ, and α_θ. Differentiating $\mathcal{S}$ with respect to these constants and equating the results to other constants, one obtains equations of motion in the implicit form that include six constants, as it should be. These equations are the following:

$$\begin{aligned}
\beta_r &= \frac{\partial \mathcal{S}}{\partial E} = -t + \int \frac{m dr}{\sqrt{2mE - 2ma(r) - \alpha_\theta / r^2}} \\
\beta_\theta &= \frac{\partial \mathcal{S}}{\partial \alpha_\theta} = \int \frac{d\theta}{2\sqrt{\alpha_\theta - 2mb(\theta) - \alpha_\phi / \sin^2\theta}} - \int \frac{dr}{2r^2\sqrt{2m\left[E - a(r)\right] - \alpha_\theta / r^2}} \\
\beta_\phi &= \frac{\partial \mathcal{S}}{\partial \alpha_\phi} = \int \frac{d\phi}{2\sqrt{\alpha_\phi - 2mc(\phi)}} - \int \frac{d\theta}{2\sin^2\theta\sqrt{\alpha_\theta - 2mb(\theta) - \alpha_\phi / \sin^2\theta}}.
\end{aligned} \tag{5.55}$$

The first of these equations after integration and solving for r yields $r(t)$. Then the second equation after integration and solving for θ yields $\theta(t)$, expressed via $r(t)$. Finally, the third equation yields the solution $\phi(t)$, expressed via $\theta(t)$ that is in turn expressed via $r(t)$. The solution above is equivalent to the solution using Poisson brackets in Sec. 3.1 that is much less involved.

Practical usefullness of these solutions is questionable since, in general, the integrals cannot be done analytically. In this case direct numerical solution of the original problem is much simpler than numerical treatment of these integrals.

6 Angle-action variables and slow evolution of mechanical systems

6.1 Definition of angle-action variables

Consider a system with one degree of freedom with a time-independent Hamiltonian

$$\mathcal{H} = \frac{p^2}{2m} + U(q). \tag{6.1}$$

The Hamiltonian is conserved, $\mathcal{H} = E = \text{const}$. Assume that potential energy $U(q)$ is such that the system performs a periodic motion between turning points $q_\pm$. The short action

$$\mathcal{S}_0(q) = \int^q p dq' = \int^q \sqrt{2m\left[E - U(q')\right]} dq' \tag{6.2}$$

depends on energy E as a parameter. Let us introduce the so-called action variable

$$I = \frac{1}{2\pi} \oint p dq \tag{6.3}$$

that is proportional to the action over the period of motion, c.f. Eq. (5.37). One can see that I depends on E only. Thus one can consider $\mathcal{S}_0$ of Eq. (6.2) as parametrically dependent on I and use the function $\mathcal{S}_0(q, I)$ as a generating function of a time-independent canonical transformation, I being the new momentum and its canonically conjugate coordinate φ being the angle (see below). The relation between the old and new dynamic variables is given by Eq. (4.7) in the form

$$\varphi = \frac{\partial \mathcal{S}_0(q, I)}{\partial I}, \qquad p = \frac{\partial \mathcal{S}_0(q, I)}{\partial q}. \tag{6.4}$$

Since the Hamiltonian $\mathcal{H} = E(I)$ does not depend on φ, the latter is a cyclic variable. The Hamilton equations have the form

$$\begin{aligned}
\dot{I} &= -\frac{\partial \mathcal{H}}{\partial \varphi} = 0 \\
\dot{\varphi} &= \frac{\partial \mathcal{H}}{\partial I} = \frac{dE}{dI} = \omega,
\end{aligned} \tag{6.5}$$

where $\omega = 2\pi/T$ is the frequency of motion, $T = T(E)$ being the period of motion. The proof of the last relation is the following

$$\frac{dI}{dE} = \frac{d}{dE}\frac{1}{2\pi}\oint \sqrt{2m\left[E - U(q)\right]}dq = \frac{1}{2\pi}\oint \sqrt{\frac{m}{2\left[E - U(q)\right]}}dq = \frac{1}{2\pi}\oint \frac{dq}{\dot{q}} = \frac{1}{2\pi}\oint dt = \frac{T}{2\pi} = \frac{1}{\omega}. \qquad (6.6)$$

Note the difference between the relations $\omega = dE/dI$ and $\omega = E/I$ in Eq. (5.36). The latter is valid only for a harmonic oscillator. The choice of the action variable I above anticipates the nice result $\dot{\varphi} = \omega$ that allows to interpret φ as the phase or the *angle* of the periodic motion that linearly grows with time, $\varphi(t) = \omega t + \varphi_0$. Time dependence of q can be found from Eq. (6.4).

For the harmonic oscillator, $\mathcal{S}_0(q)$ and the relation between E and I are given by Eqs. (5.36) and (5.37). It can be obtained by integration of $dE/dI = \omega$ in the second equation (6.5) with $\omega =$ const. The relation between the (q, p) and (φ, I) variables that follows from Eq. (6.4) reads

$$q = \sqrt{\frac{2I}{m\omega}}\sin\varphi, \qquad p = \sqrt{2mI\omega}\cos\varphi. \qquad (6.7)$$

This is obviously a form of Eq. (4.28) or Eqs. (5.33) and (5.35). One can obtain a convenient presentation of $\mathcal{S}_0$ in terms of φ

$$\mathcal{S}_0 = \int^q p dq' = \int^\varphi p(\varphi')\frac{dq(\varphi')}{d\varphi'}d\varphi' = 2I\int^\varphi \cos^2\varphi' d\varphi' = I\left[\varphi + \frac{1}{2}\sin(2\varphi)\right]. \qquad (6.8)$$

Together with the first equation (6.7) this formula gives a parametric presentation of $\mathcal{S}_0(q)$ shown in Fig. 5.1.

Equation (6.3) can be written in the form

$$I = \frac{1}{2\pi}\int\int dpdq, \qquad (6.9)$$

where the double integral is the area circumscribed by the closed orbit in the phase plane (q, p). To the contrary, the trajectory of the system in the phase plane (φ, I) is just a straight line. One can say that transformation to the angle-action variables straightens the trajectory. Of course, the action-angle formalism is merely a variant of Hamilton-Jacobi method of solving mechanical problems. In the latter, the full action $\mathcal{S}$ is used as the generating function of a canonical transformation rather than the short action $\mathcal{S}_0$ here. Hamilton-Jacobi method is even more radical because after the canonical transformation trajectories reduce to points.

6.2 Integrable and non-integrable systems

If the motion of a system with N degrees of freedom can be determined by Hamilton-Jacobi method, e. g., in the case of separation of variables, one can also find a transformation to the angle-action variables $\{\varphi_i, I_i\}$, $i = 1, 2, \ldots, N$. Then transformed Hamiltonian of the system depends only on N constants I_i as $\mathcal{H} = \mathcal{H}(I_1, I_2, \ldots, I_N)$, whereas all angles linearly increase with time, $\varphi_i(t) = \omega_i t + \varphi_{0i}$. Thus, again, the trajectory of the system is a straight line in a $2N$-dimensional space. The corresponding motion in the real phase space $\{q_i, p_i\}$ is a multiperiodic motion with in general incommensurate periods. Topologically this is a motion on multidimensional *tori* that are known as *invariant tori*.

The systems that allow a complete transformation to angle-action variables are called *integrable systems*. Unfortunately, this definition does not provide an explicit integrability check for a particular system. Obviously systems with separating variables are integrable. However, one can transform a system with separating variables to a non-separating form by some wild canonical transformation. The resulting system will be integrable, too, although non-separable. It will be very difficult to integrate this system without knowing the canonical transformation. One may give up and erroneously conclude that this system is non-integrable.

Trajectories of integrable systems corresponding to different initial conditions are straight lines that do not cross and depend smoothly on the initial conditions. To the contrast, for nonintegrable systems (that are the majority of mechanical systems) angle-action variables cannot be found and trajectories cannot be straightened. This usually leads to an apparently irregular behavior known as *dynamical chaos*.

One apparent difference between integrable and non-integrable systems is that the former have many integrals of motion I_i, one for each separable degree of freedom i. These integrals of motion, expressed through the natural variables $\{q_i, p_i\}$, impose limitations on the regions in the phase space accessible to the system. This makes the motion of the system regular. To the contrary, non-integrable systems do not have integrals of motion depending on a small subset of dynamical variables. Thus much more phase space becomes accessible to them.

6.3 Time-dependent systems

Let us now consider a system with one degree of freedom and the Hamiltonian depending on time via a parameter $\lambda(t)$, i.e., $\mathcal{H} = \mathcal{H}(q, p, \lambda(t))$. If the time dependence of λ is slow, the energy E is a slow function of time as well. The formalism of angle-action variables allows one to separate this slow dynamics from the fast (or regular) orbiting dynamics of the system. One can define the function $\mathcal{S}_0$ by the same Eq. (6.2) formally considering λ as a parameter depending on the current time. This function $\mathcal{S}_0(q, I, \lambda(t))$ that can be called *adiabatic action* is not the short action since the latter would take into account the time dependence of λ in the integration over the time-dependent trajectory in Eq. (6.2). Similarly to Sec. (6.1), we use $\mathcal{S}_0(q, I, \lambda(t))$ as the generating function of a canonical transformation. Since this canonical transformation is *time dependent*, it changes the Hamiltonian,

$$\mathcal{H}' = \mathcal{H} + \frac{\partial \mathcal{S}_0}{\partial t} = E\left(I, \lambda\right) + \Lambda \dot{\lambda}, \tag{6.10}$$

where

$$\Lambda \equiv \left. \frac{\partial \mathcal{S}_0(q, I, \lambda)}{\partial \lambda} \right|_{q,I}. \tag{6.11}$$

In function Λ one has to eliminate q with the help of transformation formulas

$$\varphi = \frac{\partial \mathcal{S}_0(q, I, \lambda)}{\partial I}, \qquad p = \frac{\partial \mathcal{S}_0(q, I, \lambda)}{\partial q}, \tag{6.12}$$

after which $\Lambda = \Lambda\left(\varphi, I, \lambda\right)$. The Hamilton equations are modified by the term with $\dot{\lambda}$:

$$\begin{aligned} \dot{I} &= -\frac{\partial \mathcal{H}'}{\partial \varphi} = -\frac{\partial \Lambda}{\partial \varphi} \dot{\lambda} \\ \dot{\varphi} &= \frac{\partial \mathcal{H}'}{\partial I} = \omega\left(I, \lambda\right) + \frac{\partial \Lambda}{\partial I} \dot{\lambda}, \end{aligned} \tag{6.13}$$

where $\omega\left(I, \lambda\right) = \partial E\left(I, \lambda\right) / \partial I$. One can see that I and hence E are no longer integrals of motion because of the time dependence of λ.

Let us consider, as an illustration, a harmonic oscillator with time-dependent frequency, the Hamiltonian being given by Eq. (4.18) with $\omega = \omega(t)$. Short action is given by Eq. (5.36), where the dependence on ω enters via $\tilde{q}$ defined by

$$\tilde{q} \equiv \sqrt{\frac{m\omega}{2I}} q \tag{6.14}$$

that follows from Eqs. (5.31) and (5.36). Thus for Λ in Eq. (6.11) with the help of Eq. (5.32) one obtains

$$\Lambda = \left. \frac{\partial \mathcal{S}_0}{\partial \tilde{q}} \right|_I \left. \frac{\partial \tilde{q}}{\partial \omega} \right|_{q,I} = 2I\sqrt{1 - \tilde{q}^2} \frac{\tilde{q}}{2\omega} = \frac{I}{\omega} \sin\varphi \cos\varphi = \frac{I}{2\omega} \sin\left(2\varphi\right). \tag{6.15}$$

Now Eqs. (6.13) take the form

$$\begin{aligned} \dot{I} &= -I\cos(2\varphi)\,\frac{\dot{\omega}}{\omega} \\ \dot{\varphi} &= \omega + \frac{1}{2}\sin(2\varphi)\,\frac{\dot{\omega}}{\omega}. \end{aligned} \tag{6.16}$$

Note that the second of these equations is autonomic.

6.4 Adiabatic invariants

Function Λ of Eq. (6.11) is a periodic function of φ with zero average. Although short action $\mathcal{S}_0$ increases acquiring the increment $2\pi I$ every period of motion, Eq. (5.37), this increment does not depend on λ and makes no contribution into Λ. An example is Λ for the harmonic oscillator given by Eq. (6.15). Periodicity of Λ can be proven rigorously if one uses Legendre-transformed generating function

$$\mathcal{S}_0^*(q,\varphi,\lambda(t)) = \mathcal{S}_0(q,I,\lambda(t)) - I\varphi \tag{6.17}$$

instead of $\mathcal{S}_0(q,I,\lambda(t))$ for the canonical transformation. Then one obtains the same equations (6.10) and (6.13), whereas Λ is now given by

$$\Lambda \equiv \left.\frac{\partial \mathcal{S}_0^*(q,\varphi,\lambda)}{\partial\lambda}\right|_{q,\varphi} = \left.\frac{\partial \mathcal{S}_0(q,I,\lambda)}{\partial\lambda}\right|_{q,I}. \tag{6.18}$$

One can see that this Λ is the same as the above. However, $\mathcal{S}_0^*(q,\varphi,\lambda)$ does not grow on average with φ, because the increment $2\pi I$ of $\mathcal{S}_0^*(q,\varphi,\lambda)$ over the period of motion is exactly compensated for by the term $-I\times 2\pi$ in Eq. (6.17). Thus it is obvious that Λ is periodic with zero average.

In the first equation (6.13) the coefficient $\partial\Lambda/\partial\varphi$ is also periodic with zero average. If the change of λ over the period of oscillations T is small,

$$\left|\frac{\dot{\lambda}T}{\lambda}\right| \ll 1, \tag{6.19}$$

the change of I becomes very small upon integration on time as oscillations in the rhs average out. Thus I is the so-called *adiabatic invariant* of motion. To the contrary, the energy E is not an adiabatic invariant. For instance, if the frequency ω of a harmonic oscillator is slowly changing, one has

$$I = \frac{E}{\omega} = \text{const}, \qquad E \propto \omega. \tag{6.20}$$

Adiabatic invariants also emerge in the motion of a charged particle in a weakly non-uniform magnetic field and in quantum mechanics. Bohr-Sommerfeld quasiclassical quantization condition has the form

$$I = \frac{1}{2\pi}\oint p\,dq = n\hbar, \tag{6.21}$$

where n is an integer. If parameters of the system change slowly, I practically does not change, and so does not n. Were I not an adiabatic invariant, it would change continuously with the parameters of the system. However, n cannot change continuously. Thus we conclude that only adiabatic invariants are suitable to impose quantization while going from the classical to quantum mechanics.

Change of I becomes especially small if λ slowly changes over the time interval $(-\infty,\infty)$. In this case the integral

$$\Delta I = -\int_{-\infty}^{\infty} \frac{\partial\Lambda}{\partial\varphi}\dot{\lambda}\,dt \tag{6.22}$$

can be usually transformed by shifting the integration contour into the complex plane to suppress oscillations of the integrand. Then the dominant contribution to ΔI comes from the singularity of the integrand closest

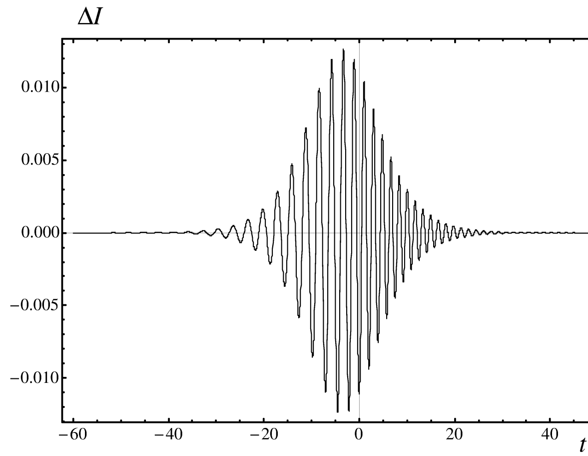

Figure 6.1: Time dependence of the change of the action variable I of a harmonic oscillator with slowly changing frequency. Since I is an adiabatic invariant, ΔI is small at all times. (See parameters for the numerical calculation in the text.)

to the real axis and ΔI becomes *exponentially* small. The smaller is $\dot{\lambda}$, the greater is the negative exponential in ΔI.

As a numerical example one can consider a harmonic oscillator with the energy function

$$E = \frac{m}{2}\left[\dot{x}^2 + \omega^2(t)x^2\right], \tag{6.23}$$

$\omega(t)$ changing from $\omega(-\infty)$ to $\omega(\infty)$ according to

$$\omega(t) = \omega_0 + \Delta\omega \frac{1 + \tanh(\alpha t)}{2} \tag{6.24}$$

with $\alpha > 0$. The change of $\omega(t)$ is slow if $\alpha \ll \omega_0$. Let us set $\alpha = \omega_0/10$. Also we set the oscillator mass $m = 2$. For $\omega_0 = \Delta\omega = 1$ one has $\omega(-\infty) = 1$ and $\omega(\infty) = 2$. For the initial state $x(-\infty) = 1$ and $\dot{x}(-\infty) = 0$ the initial oscillator energy is $E(-\infty) = 1$. Inital value of the action variable is $I(-\infty) = E(-\infty)/\omega(-\infty) = 1$. Numerical integration of the equation of motion $\ddot{x} + \omega^2(t)x = 0$ yields the results for $I(t)$ shown in Fig. 6.1. One can see that the change $\Delta I(t) \equiv I(t) - I(-\infty)$ is small at all times, that is, I is indeed an adiabatic invariant. Whereas the asymptotic change $\Delta I(\infty)$ is so small that it cannot be seen in the plot, $\Delta I(t)$ at t around zero, where the change of $\omega(t)$ occurs, is noticeable. Although $\Delta I(\infty)$ is exponentially small, $\Delta I(t)$ for a general t is not.

6.5 Parametric resonance via angle-action variables

The formalism of angle-action variables can be used for a more insightful solution of the parametric-resonance problem considered above. We have to rename the unperturbed frequency of the oscillator as $\omega \Rightarrow \omega_0$ since we need ω to denote the pumping frequency. The time-dependent frequency of the oscillator can be written as

$$\omega_0(t) = \omega_0[1 + \alpha\cos(\omega t)], \qquad \alpha \ll 1. \tag{6.25}$$

The second autonomic equation (6.16) has the explicit form

$$\dot{\varphi} = \omega_0[1 + \alpha\cos(\omega t)] - \frac{1}{2}\sin(2\varphi)\frac{\alpha\omega\cos(\omega t)}{1 + \alpha\cos(\omega t)}. \tag{6.26}$$

As we shall see, parametric resonance occurs if ω is close to $2\omega_0$. Thus we use

$$\omega = 2\omega_0 + \epsilon \tag{6.27}$$

with a small resonance detuning ϵ. Solution of the equation for φ can be searched for in the form

$$\varphi(t) = \left(\omega_0 + \frac{\epsilon}{2}\right) t + \frac{1}{2} f(t), \tag{6.28}$$

where $f(t)$ is a slow phase. $f(t)$ satisfies the equation

$$\dot{f}(t) = -\epsilon + 2\omega_0 \alpha \cos\left[(2\omega_0 + \epsilon)\, t\right] - \sin\left[(2\omega_0 + \epsilon)\, t + f(t)\right] \frac{\alpha\,(2\omega_0 + \epsilon) \sin\left[(2\omega_0 + \epsilon)\, t\right]}{1 + \alpha \cos\left[(2\omega_0 + \epsilon)\, t\right]}. \tag{6.29}$$

This is still an exact equation. Taking into account that $f(t)$ is a slow function of time and $\alpha \ll 1$, one can drop the term with α in the denominator and fast oscillating terms that average to zero. Using

$$2 \sin\left[(2\omega_0 + \epsilon)\, t + f(t)\right] \sin\left[(2\omega_0 + \epsilon)\, t\right] = \cos f(t) - \cos\left[2\,(2\omega_0 + \epsilon)\, t + f(t)\right] \tag{6.30}$$

and dropping the term $\sim \alpha\epsilon$, one obtains the slow equation

$$\dot{f}(t) = -\epsilon - \alpha\omega_0 \cos f(t). \tag{6.31}$$

This equation can be written in the "potential" form

$$\dot{f} = -\frac{dU_{\text{eff}}}{df}, \qquad U_{\text{eff}}(f) = \epsilon f + \alpha\omega_0 \sin f, \tag{6.32}$$

a tilted washboard potential. For small resonance detuning $|\epsilon| < \alpha\omega_0$, the potential U_{eff} has local maxima and minima, so that the phase $f(t)$ relaxes down to a constant value that satisfies

$$\cos f = -\frac{\epsilon}{\alpha\omega_0} \tag{6.33}$$

thus

$$\sin f = -\sqrt{1 - \left(\frac{\epsilon}{\alpha\omega_0}\right)^2}. \tag{6.34}$$

This result for $\sin f$ corresponds to the minima of $U_{\text{eff}}(f)$, whereas the solution with the sign $(+)$ in front of the square root corresponds to the maxima of $U_{\text{eff}}(f)$. The latter is unstable and it should be discarded. The result above means that the oscillator locks into the frequency $\omega/2$, as also follows from the solution in natural variables. In the case $|\epsilon| > \alpha\omega_0$ the potential $U_{\text{eff}}(f)$ is monotonic, and the $f(t)$ performs slow nonlinear motion with a *variable* rate without stopping anywhere.

To see the parametric instability that develops for $|\epsilon| < \alpha\omega_0$, let us now consider the first of the angle-action equations (6.16). This equation can be written as

$$\frac{d \ln I}{dt} \cong 2\omega_0 \alpha \cos\left[(2\omega_0 + \epsilon)\, t + f(t)\right] \sin\left[(2\omega_0 + \epsilon)\, t\right]. \tag{6.35}$$

Reducing the product of trigonometric functions as

$$2 \cos\left[(2\omega_0 + \epsilon)\, t + f(t)\right] \sin\left[(2\omega_0 + \epsilon)\, t\right] = -\sin f(t) + \sin\left[2\,(2\omega_0 + \epsilon)\, t + f(t)\right] \tag{6.36}$$

and dropping the fast oscillating term, one obtains

$$\frac{d \ln I}{dt} \cong -\omega_0 \alpha \sin f(t). \tag{6.37}$$

After $f(t)$ approaches a constant given by Eq. (6.34), this equation becomes

$$\frac{d\ln I}{dt} = 2\mu, \qquad \mu = \frac{1}{2}\sqrt{(\alpha\omega_0)^2 - \epsilon^2}, \tag{6.38}$$

where μ is the parametric resonance exponent. Solution of Eq. (6.38) has the form

$$I = I_0 e^{2\mu t}, \tag{6.39}$$

where I_0 is the initial value of I. One can see the exponential divergence of I and thus the oscillator's energy E in the region of parametric resonance. On the other hand, for large detuning, $|\epsilon| > \alpha\omega_0$, exponent μ is imaginary and I oscillates without growing.

As we have seen, analytical solution using angle-action variables is more elegant than the straightforward Newtonian solution and it provides more insight. Numerical solution can be done with both formalisms to the same effect. However, including damping and nonlinearities in the Newtonian formalism is straightforward, whereas in the angle-action formalism it requires a significant work.