Analytical Dynamics - Graduate Center CUNY - Fall 2007
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Small oscillations with many degrees of freedom

1 General formalism

Consider a dynamical system with N degrees of freedom near a minimum of the potential energy. The Lagrangian has a general form

$$\mathcal{L}=\frac{1}{2}\sum_{ij}\left(m_{ij}\dot{x}_i\dot{x}_j-k_{ij}x_ix_j\right), \tag{1}$$

where $x_i\equiv q_i-q_i^{(0)}$ are deviations from the minimum and the mass and stiffness coefficients are symmetric, $m_{ij}=m_{ji}$ and $k_{ij}=k_{ji}$ The Lagrangian can be written in the matrix form as

$$\mathcal{L}=\frac{1}{2}\left(\dot{\mathbf{X}}^T\cdot\mathbf{M}\cdot\dot{\mathbf{X}}-\mathbf{X}^T\cdot\mathbf{K}\cdot\mathbf{X}\right), \tag{2}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{K}$ is the stiffness matrix, $\mathbf{X}=(x_1,x_2,\ldots,x_N)$ (a column) and $\mathbf{X}^T$ its transposition (a row). The Lagrange equations read

$$\sum_j\left(m_{ij}\ddot{x}_j+k_{ij}x_j\right)=0,\qquad i=1\ldots N, \tag{3}$$

or, in the matrix form,

$$\mathbf{M}\cdot\ddot{\mathbf{X}}+\mathbf{K}\cdot\mathbf{X}=\mathbf{0}. \tag{4}$$

We search for the solution in the form

$$\mathbf{X}=\mathbf{a}\sin\left(\omega t\right) \tag{5}$$

[or the same with $\cos\left(\omega t\right)$] and obtain the system of linear algebraic equations

$$\left(-\omega^2\mathbf{M}+\mathbf{K}\right)\cdot\mathbf{a}=\mathbf{0}. \tag{6}$$

This system of equations has a nontrivial solution for $\mathbf{a}$ only if

$$\left|-\omega^2\mathbf{M}+\mathbf{K}\right|=0 \tag{7}$$

that defines N eigenfrequencies ω_α^2, $\alpha=1\ldots N$. One can prove that ω_α^2 are positive, if the potential energy is a positively defined bilinear form, as it is the case for the energy minimum. Then the vectors $\mathbf{a}_\alpha$ are real. We will assume the simplest case of ω_α^2 nondegenerate, $\omega_\alpha^2\neq\omega_\beta^2$ for $\alpha\neq\beta$. Technically, Eq. (6) is a generalized eigenvalue problem,

$$\mathbf{K}\cdot\mathbf{a}=\lambda\mathbf{M}\cdot\mathbf{a} \tag{8}$$

and $\mathbf{a}$ is eigenvector of $\mathbf{K}$ with respect to $\mathbf{M}$. Algorithm for solving this problem is implemented in Wolfram Mathematica.

The problem we are solving is resembling an eigenvalue problem. It is more complicated, however, since we have two matrices instead of one. Whereas in eigenvalue problems eigenvectors corresponding to different eigenvalues are orthogonal, here a generalized orthogonality relation takes place. To obtain it, write Eq. (6) with two different eigenvalues ω_α^2 and ω_β^2:

$$\begin{aligned}\omega_\alpha^2\mathbf{M}\cdot\mathbf{a}_\alpha &= \mathbf{K}\cdot\mathbf{a}_\alpha\\ \omega_\beta^2\mathbf{M}\cdot\mathbf{a}_\beta &= \mathbf{K}\cdot\mathbf{a}_\beta.\end{aligned} \tag{9}$$

Now multiply the first equation by $\mathbf{a}_\beta^T$ from the left, multiply the second equation by $\mathbf{a}_\alpha^T$ from the left, and subtract them from each other:

$$\omega_\alpha^2 \mathbf{a}_\beta^T \cdot \mathbf{M} \cdot \mathbf{a}_\alpha - \omega_\beta^2 \mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\beta = \mathbf{a}_\beta^T \cdot \mathbf{K} \cdot \mathbf{a}_\alpha - \mathbf{a}_\alpha^T \cdot \mathbf{K} \cdot \mathbf{a}_\beta. \tag{10}$$

Using symmetry of matrices $\mathbf{M}$ and $\mathbf{K}$, one can be easily show $\mathbf{a}_\beta^T \cdot \mathbf{M} \cdot \mathbf{a}_\alpha = \mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\beta$ and $\mathbf{a}_\beta^T \cdot \mathbf{K} \cdot \mathbf{a}_\alpha = \mathbf{a}_\alpha^T \cdot \mathbf{K} \cdot \mathbf{a}_\beta$. Thus the rhs vanishes and one obtains

$$\left(\omega_\alpha^2 - \omega_\beta^2\right) \mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\beta = 0. \tag{11}$$

This means that for $\alpha \neq \beta$ one has $\mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\beta = 0$. It is convenient also to require $\mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\alpha = 1$. This gives the generalized orthogonality condition

$$\mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\beta = \delta_{\alpha\beta}. \tag{12}$$

Now from the second of equations (9) one obtains

$$\omega_\beta^2 \mathbf{a}_\alpha^T \cdot \mathbf{M} \cdot \mathbf{a}_\beta = \omega_\beta^2 \delta_{\alpha\beta} = \mathbf{a}_\alpha^T \cdot \mathbf{K} \cdot \mathbf{a}_\beta \tag{13}$$

One can compose the $N \times N$ matrix of vectors $\mathbf{A}$ by stacking all $\mathbf{a}_\alpha$ together. In terms of $\mathbf{A}$ Eq. (12) takes the form

$$\mathbf{A}^T \cdot \mathbf{M} \cdot \mathbf{A} = \mathbf{I}, \tag{14}$$

where $\mathbf{I}$ is the unit matrix, whereas Eq. (13) becomes

$$\mathbf{A}^T \cdot \mathbf{K} \cdot \mathbf{A} = \mathbf{\Omega}^2, \tag{15}$$

where $\mathbf{\Omega} = \mathrm{diag}\left\{\omega_\alpha\right\}$.

Let us now introduce the normal-coordinate vector $\boldsymbol{\zeta}$ defined by

$$\mathbf{X} = \mathbf{A} \cdot \boldsymbol{\zeta}, \qquad \mathbf{X}^T = \left(\mathbf{A} \cdot \boldsymbol{\zeta}\right)^T = \boldsymbol{\zeta}^T \cdot \mathbf{A}^T. \tag{16}$$

These equations can be resolved for $\boldsymbol{\zeta}$ and ζ_α:

$$\boldsymbol{\zeta} = \mathbf{A}^{-1} \cdot \mathbf{X}, \qquad \zeta_\alpha = \left(\mathbf{A}^{-1}\right)_{\alpha i} x_i. \tag{17}$$

Note that, according to Eq. (14), $\mathbf{A}^{-1} \neq \mathbf{A}^T$ and $\left(\mathbf{A}^{-1}\right)_{\alpha i} \neq a_{\alpha i}$. Inserting Eq. (16) into the Lagrangian, Eq. (2), and using Eqs. (12) and (15), one obtains

$$\mathcal{L} = \frac{1}{2}\left(\dot{\boldsymbol{\zeta}}^T \cdot \mathbf{A}^T \cdot \mathbf{M} \cdot \mathbf{A} \cdot \dot{\boldsymbol{\zeta}} - \boldsymbol{\zeta}^T \cdot \mathbf{A}^T \cdot \mathbf{K} \cdot \mathbf{A} \cdot \boldsymbol{\zeta}\right) = \frac{1}{2}\left(\dot{\boldsymbol{\zeta}}^T \cdot \dot{\boldsymbol{\zeta}} - \boldsymbol{\zeta}^T \cdot \mathbf{\Omega}^2 \cdot \boldsymbol{\zeta}\right) \tag{18}$$

or

$$\mathcal{L} = \frac{1}{2}\sum_{\alpha=1}^{N}\left(\dot{\zeta}_\alpha^2 - \omega_\alpha^2 \zeta_\alpha^2\right), \tag{19}$$

where ζ_α are components of the normal-coordinate vector $\boldsymbol{\zeta}$ or normal coordinates. One can see that the Lagrangian is separable in terms of the normal coordinates, so that every normal coordinate ζ_α oscillates with its own frequency ω_α:

$$\zeta_\alpha(t) = C_{s\alpha} \sin\left(\omega_\alpha t\right) + C_{c\alpha} \cos\left(\omega_\alpha t\right). \tag{20}$$

From Eq. (16) one obtains

$$x_i(t) = \sum_{\alpha=1}^{N} a_{i\alpha} \zeta_\alpha(t) = \sum_{\alpha=1}^{N} a_{i\alpha}\left[C_{s\alpha} \sin\left(\omega_\alpha t\right) + C_{c\alpha} \cos\left(\omega_\alpha t\right)\right] \tag{21}$$

that can be compared with the tentative Eq. (5). To obtain the coefficients from the initial conditions, one writes

$$\dot{\zeta}_\alpha(t) = C_{s\alpha}\omega_\alpha \cos(\omega_\alpha t) - C_{c\alpha}\omega_\alpha \sin(\omega_\alpha t) \tag{22}$$

and at $t = 0$ from Eqs. (17), (20), and (22) obtains

$$C_{c\alpha} = \left(\mathbf{A}^{-1}\right)_{\alpha i} x_i(0), \qquad C_{s\alpha} = \left(\mathbf{A}^{-1}\right)_{\alpha i} \frac{\dot{x}_i(0)}{\omega_\alpha}. \tag{23}$$

2 Example: Double pendulum

Consider a double pendulum that consists of the pendulum 1 attached in a standard way and the pendulum 2 attached to the end of pendulum 1. For simplicity we consider the case $m_1 = m_2 = m$ and $l_1 = l_2 = l$. Describing the pendula with the angles θ_1 and θ_2, one writes

$$\begin{aligned} x_1 &= l\sin\theta_1, \qquad x_2 = l\sin\theta_1 + l\sin\theta_2 \\ y_1 &= -l\cos\theta_1, \qquad y_2 = -l\cos\theta_1 - l\cos\theta_2. \end{aligned}$$

The potential energy reads

$$U = mg\left(y_1 + y_2\right) = -mgl\left(2\cos\theta_1 + \cos\theta_2\right)$$

and the kinetic energy is given by

$$\begin{aligned} T &= \frac{m}{2}\left(\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2\right) = \frac{ml^2}{2}\left[\dot{\theta}_1^2 + \left(-\cos\theta_1\dot{\theta}_1 - \cos\theta_2\dot{\theta}_2\right)^2 + \left(\sin\theta_1\dot{\theta}_1 + \sin\theta_2\dot{\theta}_2\right)^2\right] \\ &= \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right)\dot{\theta}_1\dot{\theta}_2\right]. \end{aligned}$$

Near the ground state $\theta_1 = \theta_2 = 0$ one leaves only quadratic terms that results in

$$\mathcal{L} = T - U \cong \frac{ml^2}{2}\left(2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{\theta}_1\dot{\theta}_2\right) - \frac{mgl}{2}\left(2\theta_1^2 + \theta_2^2\right).$$

Identifying this with Eq. (1) yields

$$\mathbf{M} = ml^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathbf{K} = mgl\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

The generalized eigenvalue problem, Eq. (6), has the form

$$\begin{pmatrix} 2\left(-\omega^2 + \omega_0^2\right) & -\omega^2 \\ -\omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0,$$

where we have introduced

$$\omega_0^2 = \frac{g}{l}.$$

The secular equation is

$$\begin{aligned} 0 &= 2\left(-\omega^2 + \omega_0^2\right)^2 - \omega^4 = \left[\sqrt{2}\left(\omega^2 - \omega_0^2\right) + \omega^2\right]\left[\sqrt{2}\left(\omega^2 - \omega_0^2\right) - \omega^2\right] \\ &= \left[\left(\sqrt{2} + 1\right)\omega^2 - \sqrt{2}\omega_0^2\right]\left[\left(\sqrt{2} - 1\right)\omega^2 - \sqrt{2}\omega_0^2\right]. \end{aligned}$$

Thus the eigenfrequencies are given by

$$\omega_1^2 = \frac{\sqrt{2}\omega_0^2}{\sqrt{2} + 1} = \left(2 - \sqrt{2}\right)\omega_0^2, \qquad \omega_2^2 = \frac{\sqrt{2}\omega_0^2}{\sqrt{2} - 1} = \left(2 + \sqrt{2}\right)\omega_0^2.$$

Let us find now generalized eigenvectors and normal modes of the system. Eq. (**??**) becomes

$$\begin{aligned} 2\left(\omega_\alpha^2-\omega_0^2\right)a_{1\alpha}+\omega_\alpha^2 a_{2\alpha} &= 0 \\ \omega_\alpha^2 a_{1\alpha}+\left(\omega_\alpha^2-\omega_0^2\right)a_{2\alpha} &= 0. \end{aligned}$$

For instance, from the second equation one obtains

$$\mathbf{a}_\alpha=\mu_\alpha\begin{pmatrix} -\omega_\alpha^2+\omega_0^2 \\ \omega_\alpha^2 \end{pmatrix},$$

where μ_α are normalization coefficients. In particular,

$$\begin{aligned} \mathbf{a}_1 &= \mu_1\begin{pmatrix} -\omega_1^2+\omega_0^2 \\ \omega_1^2 \end{pmatrix}=\mu_1\omega_0^2\begin{pmatrix} \sqrt{2}-1 \\ 2-\sqrt{2} \end{pmatrix}=\mu_1\omega_0^2\left(\sqrt{2}-1\right)\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\ \mathbf{a}_2 &= \mu_2\begin{pmatrix} -\omega_2^2+\omega_0^2 \\ \omega_2^2 \end{pmatrix}=\mu_2\omega_0^2\begin{pmatrix} -\sqrt{2}-1 \\ 2+\sqrt{2} \end{pmatrix}=\mu_2\omega_0^2\left(\sqrt{2}+1\right)\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}. \end{aligned}$$

Let us check the orthogonality of these eigenvectors, Eq. (12):

$$\begin{aligned} \mathbf{a}_1^T\cdot\mathbf{M}\cdot\mathbf{a}_2 &= \mu_1\mu_2 ml^2\omega_0^4\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}=\mu_1\mu_2 ml^2\omega_0^4\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} -2+\sqrt{2} \\ -1+\sqrt{2} \end{pmatrix} \\ &= \mu_1\mu_2 ml^2\omega_0^4\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}\left(\sqrt{2}-1\right)=0, \end{aligned}$$

as expected. Now calculate normalization factors:

$$\begin{aligned} 1 &= \mathbf{a}_1^T\cdot\mathbf{M}\cdot\mathbf{a}_1=\mu_1^2 ml^2\omega_0^4\left(\sqrt{2}-1\right)^2\begin{pmatrix} 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\ &= \mu_1^2 ml^2\omega_0^4\left(\sqrt{2}-1\right)^2 2\sqrt{2}\left(\sqrt{2}+1\right)=\mu_1^2 ml^2\omega_0^4\left(\sqrt{2}-1\right)2\sqrt{2} \end{aligned}$$

and

$$\begin{aligned} 1 &= \mathbf{a}_2^T\cdot\mathbf{M}\cdot\mathbf{a}_2=\mu_2^2 ml^2\omega_0^4\left(\sqrt{2}+1\right)^2\begin{pmatrix} -1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix} \\ &= \mu_2^2 ml^2\omega_0^4\left(\sqrt{2}+1\right)^2 2\sqrt{2}\left(\sqrt{2}-1\right)=\mu_2^2 ml^2\omega_0^4\left(\sqrt{2}+1\right)2\sqrt{2}. \end{aligned}$$

Thus

$$\begin{aligned} \mu_1\omega_0^2 &= \frac{1}{\sqrt{ml^2}}\left(\frac{1}{\left(\sqrt{2}-1\right)2\sqrt{2}}\right)^{1/2}=\frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2} \\ \mu_2\omega_0^2 &= \frac{1}{\sqrt{ml^2}}\left(\frac{1}{\left(\sqrt{2}+1\right)2\sqrt{2}}\right)^{1/2}=\frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2} \end{aligned}$$

and, finally,

$$\begin{aligned} \mathbf{a}_1 &= \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \\ \mathbf{a}_2 &= \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\begin{pmatrix} -1 \\ \sqrt{2} \end{pmatrix}. \end{aligned}$$

Now, the angles expressed through the eigenmodes are given by Eqs. (16) or (21):

$$\begin{aligned}
\theta_1 &= a_{11}\zeta_1 + a_{12}\zeta_2 = \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\zeta_1 - \frac{1}{\sqrt{ml^2}}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\zeta_2 \\
\theta_2 &= a_{21}\zeta_1 + a_{22}\zeta_2 = \frac{1}{\sqrt{ml^2}}\sqrt{2}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\zeta_1 + \frac{1}{\sqrt{ml^2}}\sqrt{2}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\zeta_2.
\end{aligned}$$

The inverse transformation has the form

$$\begin{aligned}
\zeta_1 &= \frac{\det_1}{\det} = \frac{\theta_1 a_{22} - \theta_2 a_{12}}{a_{11}a_{22} - a_{21}a_{12}} = (\theta_1 a_{22} - \theta_2 a_{12})\, ml^2 = \sqrt{ml^2}\sqrt{2}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\theta_1 + \sqrt{ml^2}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\theta_2 \\
&= \sqrt{ml^2}\left(\frac{\sqrt{2}+1}{2\sqrt{2}}\right)^{1/2}\left(\sqrt{2}\theta_1 + \theta_2\right) \\
\zeta_2 &= \frac{\det_2}{\det} = (a_{11}\theta_2 - a_{21}\theta_1)\, ml^2 = \sqrt{ml^2}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\theta_2 - \sqrt{ml^2}\sqrt{2}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\theta_1 \\
&= \sqrt{ml^2}\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)^{1/2}\left(-\sqrt{2}\theta_1 + \theta_2\right).
\end{aligned}$$

One can see that in the normal mode 1 both pendula are swinging in-phase and in the mode 2 they are swinging antiphase. The frequency of the mode 1 is lower.

3 Example: Two coupled oscillators

Another model that illustrate general principles and has more interesting physical content is the model of two coupled oscillators with the Lagrangian

$$\mathcal{L} = \frac{m}{2}\left(\dot{x}_1^2 - \omega_1^2 x_1^2 + \dot{x}_2^2 - \omega_2^2 x_2^2 - \Delta^2 x_1 x_2\right). \tag{24}$$

We will show that if $|\omega_1 - \omega_2| \gg \Delta$, the oscillators behave nearly independently from each other and oscillate with their own frequencies $\omega_{1,2}$, whereas in the opposite limit they strongly hybridize and both normal modes involve both coordinates. Here the matrices $\mathbf{M}$ and $\mathbf{K}$ are given by

$$\mathbf{M} = m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{K} = m\begin{pmatrix} \omega_1^2 & \Delta^2/2 \\ \Delta^2/2 & \omega_2^2 \end{pmatrix}$$

and the eigenvalue problem, Eq. (6), is

$$\begin{pmatrix} -\omega^2 + \omega_1^2 & \Delta^2/2 \\ \Delta^2/2 & -\omega^2 + \omega_2^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0.$$

Note that this is a standard eigenvalue problem, not generalized. The secular equation has the form

$$0 = \left(\omega^2 - \omega_1^2\right)\left(\omega^2 - \omega_2^2\right) - \Delta^4/4 = \omega^4 - \omega^2\left(\omega_1^2 + \omega_2^2\right) + \omega_1^2\omega_2^2 - \Delta^4/4$$

and the eigenfrequencies are

$$\omega_\pm^2 = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 \pm \sqrt{\left(\omega_1^2 + \omega_2^2\right)^2 - 4\omega_1^2\omega_2^2 + \Delta^4}\right) = \frac{1}{2}\left(\omega_1^2 + \omega_2^2 \pm \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}\right). \tag{25}$$

One can see that, indeed, for $|\omega_1 - \omega_2| \gg \Delta$ the eigenfrequencies $\omega_\pm$ coincide with $\omega_{1,2}$. On the other hand, for $\omega_1 = \omega_2 = \omega_0$ one obtains

$$\omega_\pm^2 = \omega_0^2 \pm \frac{1}{2}\Delta^2. \tag{26}$$

Let us now find the eigenvectors and normal modes. From the equation

$$\left(-\omega_\pm^2 + \omega_1^2\right) a_{1\pm} + \left(\Delta^2/2\right) a_{2\pm} = 0$$

one obtains

$$\mathbf{a}_\pm = \mu_\pm \begin{pmatrix} \Delta^2/2 \\ \omega_\pm^2 - \omega_1^2 \end{pmatrix},$$

where μ_α is a normalization coefficient. This form is nonsymmetric and it is convenient to symmetrize it. To this end, consider the case $\omega_1 < \omega_2$. One can see from Eq. (25) that the frequencies are ordered as

$$\omega_- < \omega_1 < \omega_2 < \omega_+.$$

Now $\mathbf{a}_+$ can be symmetrized as

$$\mathbf{a}_+ = \mu_+ \begin{pmatrix} \sqrt{\frac{\Delta^2}{2\left(\omega_+^2 - \omega_1^2\right)}} \\ \sqrt{\frac{2\left(\omega_+^2 - \omega_1^2\right)}{\Delta^2}} \end{pmatrix} \tag{27}$$

with another μ_+. Here

$$\frac{2\left(\omega_+^2 - \omega_1^2\right)}{\Delta^2} = \frac{-\omega_1^2 + \omega_2^2 + \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}}{\Delta^2}$$

and

$$\begin{aligned} \frac{\Delta^2}{2\left(\omega_+^2 - \omega_1^2\right)} &= \frac{\Delta^2}{-\omega_1^2 + \omega_2^2 + \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}} = \frac{-\omega_1^2 + \omega_2^2 - \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}}{-\Delta^2} \\ &= \frac{\omega_1^2 - \omega_2^2 + \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}}{\Delta^2}. \end{aligned}$$

To the contrary, $\mathbf{a}_-$ can be symmetrized as

$$\mathbf{a}_- = \mu_- \begin{pmatrix} \sqrt{\frac{\Delta^2}{2\left(\omega_1^2 - \omega_-^2\right)}} \\ -\sqrt{\frac{2\left(\omega_1^2 - \omega_-^2\right)}{\Delta^2}} \end{pmatrix}. \tag{28}$$

Here

$$\frac{2\left(\omega_1^2 - \omega_-^2\right)}{\Delta^2} = \frac{\omega_1^2 - \omega_2^2 + \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}}{\Delta^2}$$

and

$$\begin{aligned} \frac{\Delta^2}{2\left(\omega_1^2 - \omega_-^2\right)} &= \frac{\Delta^2}{\omega_1^2 - \omega_2^2 + \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}} = \frac{\omega_1^2 - \omega_2^2 - \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}}{-\Delta^2} \\ &= \frac{-\omega_1^2 + \omega_2^2 + \sqrt{\left(\omega_1^2 - \omega_2^2\right)^2 + \Delta^4}}{\Delta^2}. \end{aligned}$$

Obviously eigenvectors $\mathbf{a}_+$ and $\mathbf{a}_-$ are orthogonal. Let us now calculate the normalization factors.

$$\begin{aligned}
1 &= \mathbf{a}_+^T \cdot \mathbf{a}_+ = \frac{\mu_+^2}{\Delta^2} \left(\sqrt{-\omega_1^2+\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}} \quad \sqrt{\omega_1^2-\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}} \right) \\
&\qquad \cdot \left(\begin{array}{c} \sqrt{-\omega_1^2+\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}} \\ \sqrt{\omega_1^2-\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}} \end{array} \right) \\
&= \frac{\mu_+^2}{\Delta^2} \left[-\omega_1^2+\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4} + \omega_1^2-\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4} \right] = \frac{\mu_+^2}{\Delta^2} 2\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}.
\end{aligned}$$

The equation for μ_- is the same. Thus one obtains

$$\mu_\pm = \sqrt{\frac{\Delta^2}{2\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}}.$$

After that Eqs. (27) and (28) become

$$\mathbf{a}_+ = \sqrt{\frac{\Delta^2}{2\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \left(\begin{array}{c} \sqrt{\frac{\omega_1^2-\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}{\Delta^2}} \\ \sqrt{\frac{-\omega_1^2+\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}{\Delta^2}} \end{array} \right) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \sqrt{1+\frac{\omega_1^2-\omega_2^2}{\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \\ \sqrt{1-\frac{\omega_1^2-\omega_2^2}{\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \end{array} \right)$$

and

$$\mathbf{a}_- = \sqrt{\frac{\Delta^2}{2\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \left(\begin{array}{c} \sqrt{\frac{-\omega_1^2+\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}{\Delta^2}} \\ -\sqrt{\frac{\omega_1^2-\omega_2^2+\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}{\Delta^2}} \end{array} \right) = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \sqrt{1-\frac{\omega_1^2-\omega_2^2}{\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \\ -\sqrt{1+\frac{\omega_1^2-\omega_2^2}{\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \end{array} \right).$$

Both eigenvectors can be written as

$$\mathbf{a}_\pm = \frac{1}{\sqrt{2}} \left(\begin{array}{c} \sqrt{1\pm\frac{\omega_1^2-\omega_2^2}{\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \\ \pm\sqrt{1\mp\frac{\omega_1^2-\omega_2^2}{\sqrt{\left(\omega_1^2-\omega_2^2\right)^2+\Delta^4}}} \end{array} \right).$$

Actually this result is valid for an arbitrary relation between ω_1 and ω_2. For $\omega_2 - \omega_1 \gg \Delta$ one has

$$\mathbf{a}_+ \cong \left(\begin{array}{c} 0 \\ 1 \end{array} \right), \qquad \mathbf{a}_- \cong \left(\begin{array}{c} 1 \\ 0 \end{array} \right)$$

that means that $\omega_+ \cong \omega_2$ corresponds to the second oscillator and $\omega_- \cong \omega_1$ corresponds to the first oscillator. In the resonance case $\omega_1^2 = \omega_2^2$ one obtains

$$\mathbf{a}_\pm = \frac{1}{\sqrt{2}} \left(\begin{array}{c} 1 \\ \pm 1 \end{array} \right)$$

that means that the modes are strongly mixed.

The original coordinates expressed through the normal modes are given by

$$x_i = \sum_{\alpha=\pm} a_{i\alpha}\zeta_\alpha,$$

i.e.,

$$x_1 = a_{1+}\zeta_+ + a_{1-}\zeta_-, \qquad x_2 = a_{2+}\zeta_+ + a_{2-}\zeta_-.$$

The normal modes behave as independent harmonic oscillators with frequencies $\omega_\pm$. Thus the time dependence of both x_1 and x_2 is a superposition of these two oscillations.

Now consider, as an illustration, the resonance case $\omega_1 = \omega_2 = \omega_0$ with a small coupling $\Delta \ll \omega_0$. Here

$$x_1 = \frac{1}{\sqrt{2}}\left(\zeta_+ + \zeta_-\right), \qquad x_2 = \frac{1}{\sqrt{2}}\left(\zeta_+ - \zeta_-\right) \tag{29}$$

and, inversely,

$$\zeta_+ = \frac{1}{\sqrt{2}}\left(x_1 + x_2\right), \qquad \zeta_- = \frac{1}{\sqrt{2}}\left(x_1 - x_2\right).$$

Suppose at $t = 0$ oscillator 1 was in a general state while oscillator 2 was in its ground state. Then the initial condition for the normal modes is

$$\begin{aligned} \zeta_+(0) &= \zeta_-(0) = \frac{1}{\sqrt{2}}x_1(0) \\ \dot{\zeta}_+(0) &= \dot{\zeta}_-(0) = \frac{1}{\sqrt{2}}\dot{x}_1(0). \end{aligned}$$

Thus the time dependence of the normal-mode coordinates is given by

$$\begin{aligned} \zeta_+(t) &= \zeta_+(0)\cos(\omega_+ t) + \dot{\zeta}_+(0)\frac{1}{\omega_+}\sin(\omega_+ t) \\ &= \frac{x_1(0)}{\sqrt{2}}\cos(\omega_+ t) + \frac{\dot{x}_1(0)}{\sqrt{2}}\frac{1}{\omega_+}\sin(\omega_+ t) \\ \zeta_-(t) &= \zeta_-(0)\cos(\omega_- t) + \frac{\dot{\zeta}_-(0)}{\omega_-}\sin(\omega_- t) \\ &= \frac{x_1(0)}{\sqrt{2}}\cos(\omega_- t) + \frac{\dot{x}_1(0)}{\sqrt{2}}\frac{1}{\omega_-}\sin(\omega_- t). \end{aligned}$$

Inserting this into Eq. (29) one obtains

$$\begin{aligned} x_1(t) &= \frac{1}{\sqrt{2}}\left[\zeta_+(t) + \zeta_-(t)\right] = \frac{1}{2}\left\{x_1(0)\left[\cos(\omega_+ t) + \cos(\omega_- t)\right] + \dot{x}_1(0)\left[\frac{1}{\omega_+}\sin(\omega_+ t) + \frac{1}{\omega_-}\sin(\omega_- t)\right]\right\} \\ x_2(t) &= \frac{1}{\sqrt{2}}\left[\zeta_+(t) - \zeta_-(t)\right] = \frac{1}{2}\left\{x_1(0)\left[\cos(\omega_+ t) - \cos(\omega_- t)\right] + \dot{x}_1(0)\left[\frac{1}{\omega_+}\sin(\omega_+ t) - \frac{1}{\omega_-}\sin(\omega_- t)\right]\right\}. \end{aligned}$$

In the small-coupling case the eigenfrequencies are close to each other, see Eq. (26), one can approximate

$$\frac{1}{\omega_+} \cong \frac{1}{\omega_-} \cong \frac{1}{\omega_0}.$$

On the other hand, one cannot make this approximation in the arguments of sin and cos since it will breakdown at large times. From Eq. (26) in the weak-coupling case follows

$$\omega_\pm = \omega_0\sqrt{1 \pm \frac{\Delta^2}{2\omega_0^2}} \cong \omega_0 \pm \frac{\Delta^2}{4\omega_0}.$$

Usung this and trigonometric relations, one obtains

$$\begin{aligned} x_1(t) &= x_1(0)\cos\left(\frac{\omega_+ + \omega_-}{2}t\right)\cos\left(\frac{\omega_+ - \omega_-}{2}t\right) + \frac{\dot{x}_1(0)}{\omega_0}\sin\left(\frac{\omega_+ + \omega_-}{2}t\right)\cos\left(\frac{\omega_+ - \omega_-}{2}t\right) \\ &\cong \left[x_1(0)\cos(\omega_0 t) + \frac{\dot{x}_1(0)}{\omega_0}\sin(\omega_0 t)\right]\cos\left(\frac{\Delta^2}{4\omega_0}t\right). \end{aligned}$$

This is a standard time dependence for a harmonic oscillator (square brackets) multiplied by a slowly oscillating function of time. Using

$$\dot{x}_1(t) \cong [-x_1(0)\omega_0 \sin(\omega_0 t) + \dot{x}_1(0)\cos(\omega_0 t)] \cos\left(\frac{\Delta^2}{4\omega_0}t\right)$$

(we do not differentiate the slow function), one can compute the time dependence of the energy of the first oscillator as

$$\begin{aligned}
e_1(t) &= \frac{m}{2}\left(\dot{x}_1^2(t) + \omega_0^2 x_1^2(t)\right) \\
&= \frac{m}{2}\left\{[-x_1(0)\omega_0 \sin(\omega_0 t) + \dot{x}_1(0)\cos(\omega_0 t)]^2 + \omega_0^2\left[x_1(0)\cos(\omega_0 t) + \frac{\dot{x}_1(0)}{\omega_0}\sin(\omega_0 t)\right]^2\right\}\cos^2\left(\frac{\Delta^2}{4\omega_0}t\right) \\
&= \frac{m}{2}\left\{\dot{x}_1^2(0) + \omega_0^2 x_1^2(0)\right\}\cos^2\left(\frac{\Delta^2}{4\omega_0}t\right) = e_1(0)\cos^2\left(\frac{\Delta^2}{4\omega_0}t\right).
\end{aligned}$$

The same can be done for the second oscillator. For instance, its energy behaves as

$$e_2(t) = e_1(0)\left[1 - \cos^2\left(\frac{\Delta^2}{4\omega_0}t\right)\right],$$

so that the total energy $e_1(t) + e_2(t) = e_1(0)$ is conserved, whereas the energy is slowly migrating between the two oscillators.