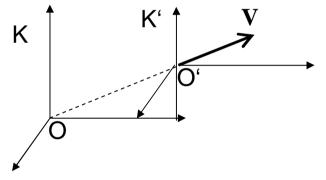# Transformations and conservation laws

Galilean transformation

We mainly use <u>inertial</u> frames in which a free body (no forces applied) moves with a constant velocity. A frame moving with a constant velocity with respect to an inertial frame is inertial, too. Thus there is an infinite number of inertial frames. The equations of motion are invariant with respect to transformations from one inertial frame to another, and the transformed Lagrange function can differ from the initial one only by an irrelevant full derivative. This is the principle of the <u>Galilean invariance</u>, i.e., invariance with resperct to <u>Galilean transfromations</u>, that is valid in the classical mechanics.

Frame K' moves with a velocity $\mathbf{V} = \mathrm{const}$ with respect to frame K

$$\mathbf{v}' = \mathbf{v} \cdot \mathbf{V}, \qquad \mathbf{r}' = \mathbf{r} - \mathbf{V}t$$

$$t' = t$$

Transformation of the Lagrange function of a free particle:

$$\mathcal{L}' = \frac{m}{2}\mathbf{v}'^2 = \frac{m}{2}(\mathbf{v} - \mathbf{V})^2 = \frac{m}{2}\mathbf{v}^2 - m\mathbf{v} \bullet \mathbf{V} + \frac{m}{2}\mathbf{V}^2$$

$$= \mathcal{L} + \underbrace{\frac{d}{dt}\left(-m\mathbf{r} \bullet \mathbf{V} + \frac{m}{2}\mathbf{V}^2 t\right)}_{\text{Irrelevant full derivative}}$$

- Thus the Lagrange equations are the same in both frames
(Prove the same for systems with interaction)

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The proof of the Galilean invariance on the previous page (Landau and Lifshitz) is not fully satisfactory since the Largange function in the K' frame is expressed as the function of the variables in the K frame. The true check of the Galilean invariance should be that of the <u>covariance</u> of the Lagrange equations, that is,

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = 0 \quad \Rightarrow \quad \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \mathbf{v}'} = 0$$

under the Galilean transformation with the same functional forms of the Lagrange function. The proof is simple. With

$$\mathcal{L} = \frac{m}{2}\mathbf{v}^2$$

one obtains

$$0 = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \mathbf{v}} = \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}(m(\mathbf{v}' + \mathbf{V})) = \frac{d}{dt}(m\mathbf{v}') = \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \mathbf{v}'} = 0$$

where

$$\mathcal{L}' = \frac{m}{2}\mathbf{v}'^2$$

has the same functional form, as required. Note that the latter was just assumed in the LL reasoning.

The proof of the Galilean invariance can be easily extended for the systems of particles with pair interaction

$$\mathcal{L} = \sum_i \frac{m}{2}\mathbf{v}_i^2 - \frac{1}{2}\sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|)$$

since the Galilean transformation does not change the distances between particles,

$$|\mathbf{r}_i - \mathbf{r}_j| = |\mathbf{r}_i' - \mathbf{r}_j'|$$

Energy E : Conserved if the Lagrange function does not explicitly depend on time (the problem is invariant with respect to time translations)

$$L = L(\dot{q}, q)$$

Consider the quantity

$$E = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Its time derivative with the help of the Lagrange equations $\dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}_i} - \dfrac{\partial L}{\partial q_i} = 0$

can be shown to be zero:

$$\dot{E} = \sum_i \ddot{q}_i \frac{\partial L}{\partial \dot{q}_i} + \sum_i \dot{q}_i \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \sum_i \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i - \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i = \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} - \sum_i \frac{\partial L}{\partial q_i}\dot{q}_i = 0$$

Thus E is conserved. As the Lagrange function is a bilinear function of the velocities, one has

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T \quad \Rightarrow \quad E = 2T - (T - U) = T + U$$

That is, E is indeed the full energy of the system.

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Generalized momentum p_i : Conserved if the Lagrange function does not depend on the particular q_i (q_i is the cyclic coordinate)

Indeed, from the Lagrange equation follows

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \quad \Rightarrow \quad p_i \equiv \frac{\partial L}{\partial \dot{q}_i} = \text{const}$$

Example: Consider a constant force $\mathbf{F} = \textbf{const}$ of a particular form $\quad \mathbf{F} = F\mathbf{e}_z$

The corresponding potential energy is given by $\quad U = -\mathbf{F}\bullet\mathbf{r} = -Fz$

Thus the coordinate components x and y are cyclic so that

$$p_x = m\dot{x} = mv_x = \text{const}$$
$$p_y = m\dot{y} = mv_y = \text{const}$$

whereas

$$p_z \neq \text{const}$$

Total momentum **P** of a system: Conserved if the system is isolated

The Lagrangian of an isolated system (no external forces acting) is invariant under translations

$$\mathbf{r}_i \Rightarrow \mathbf{r}_i + \boldsymbol{\varepsilon}$$

Considering $\boldsymbol{\varepsilon}$ as infinitesimal, one obtains

$$0 = \delta L = \boldsymbol{\varepsilon} \bullet \sum_i \frac{\partial L}{\partial \mathbf{r}_i} \quad \Rightarrow \quad \sum_i \frac{\partial L}{\partial \mathbf{r}_i} = 0$$

As $\quad \dfrac{\partial L}{\partial \mathbf{r}_i} = -\dfrac{\partial U}{\partial \mathbf{r}_i} = \mathbf{F}_i \quad$ one further obtains $\quad \sum_i \mathbf{F}_i = 0 \quad$ (Newton's third law)

On the other hand, from the Lagrange equations follows

$$\frac{d}{dt} \sum_i \frac{\partial L}{\partial \mathbf{v}_i} = 0 \quad \Rightarrow \quad \sum_i \frac{\partial L}{\partial \mathbf{v}_i} = \sum_i \mathbf{p}_i \equiv \mathbf{P} = const$$

Alternatively, all relations above could be obtained (in a slightly less general form) using the potential energy

$$U = \frac{1}{2} \sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|)$$

(that is explicitly translationally invariant) to calculate the sum of all forces etc. (Home exercise)

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Angular momentum **L**: Conserved if the system is rotationally invariant (in particular, for any isolated system)

An infinitesimal rotation $\quad \delta \mathbf{r}_i = [\delta \boldsymbol{\varphi} \times \mathbf{r}_i], \quad \delta \mathbf{v}_i = [\delta \boldsymbol{\varphi} \times \mathbf{v}_i]$

leaves the Lagrangian unchanged:

$$0 = \delta \mathcal{L} = \sum_i \left(\frac{\partial \mathcal{L}}{\partial \mathbf{r}_i} \bullet \delta \mathbf{r}_i + \frac{\partial \mathcal{L}}{\partial \mathbf{v}_i} \bullet \delta \mathbf{v}_i \right) = \sum_i \left(\dot{\mathbf{p}}_i \bullet [\delta \boldsymbol{\varphi} \times \mathbf{r}_i]_i + \mathbf{p}_i \bullet [\delta \boldsymbol{\varphi} \times \mathbf{v}_i] \right)$$

$$= \delta \boldsymbol{\varphi} \bullet \sum_i \left([\mathbf{r}_i \times \dot{\mathbf{p}}_i]_i + [\mathbf{v}_i \times \mathbf{p}_i] \right) = \delta \boldsymbol{\varphi} \bullet \frac{d}{dt} \sum_i [\mathbf{r}_i \times \mathbf{p}_i]$$

It follows then $\qquad \mathbf{L} \equiv \sum_i [\mathbf{r}_i \times \mathbf{p}_i] = const$

In particular, rotational invariance of the Lagrangian $\qquad \mathcal{L} = \sum_i \frac{m}{2} \mathbf{v}_i^2 - \frac{1}{2} \sum_{ij} V(|\mathbf{r}_i - \mathbf{r}_j|)$

is obvious since rotations do not change the distances between the particles and the squares of the velocities. Rotational invariance can be incomplete. If, for instance, the Lagrangian is invariant with respect to rotations only around the z axis, then only L_z is conserved (the proof is similar).

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In the Lagrangian formalism the angular momentum is just a generalized momentum.

For the components of $\mathbf{L}$ one can write

$$L_\alpha = \frac{\partial L}{\partial \dot{\varphi}_\alpha}, \quad \alpha = x, y, z$$

where $\dot{\varphi}_\alpha$ are components of the angular velocity (to be considered later in the course)

Example:

$$L_z = m[\mathbf{r} \times \mathbf{v}]_z = m(x\dot{y} - \dot{x}y)$$

In the cylindrical coordinate system $x = \rho \cos\varphi, \quad y = \rho \sin\varphi, \quad z = z$

one has $\dot{x} = \dot{\rho}\cos\varphi - \rho\sin\varphi\,\dot{\varphi}, \quad \dot{y} = \dot{\rho}\sin\varphi + \rho\cos\varphi\,\dot{\varphi}$

that finally yields

$$L_z = m\rho^2\dot{\varphi}$$

On the other hand, the kinetic energy can be written as $T = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\varphi}^2 + \dot{z}^2)$

Thus the formula at the top yields

$$L_z = \frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial T}{\partial \dot{\varphi}} = m\rho^2\dot{\varphi}$$

in accordance with the direct calculation.

Center of mass (CM)

The position of the CM is defined by

$$\mathbf{R} \equiv \frac{1}{M} \sum_i m_i \mathbf{r}_i, \quad M \equiv \sum_i m_i$$

For the velocity of the CM one obtains

$$\mathbf{V} = \dot{\mathbf{R}} = \frac{1}{M} \sum_i m_i \mathbf{v}_i = \frac{\mathbf{P}}{M}$$

that is,

$$\mathbf{P} = M\mathbf{V}$$

and one can consider the motion of a system as the whole in terms of the total mass M and the velocity of the CM V.

(see page 1)

Momentum:

$$\mathbf{P}' = \sum_i m_i \mathbf{v}_i' = \sum_i m_i (\mathbf{v}_i - \mathbf{V}) = \mathbf{P} - M\mathbf{V}$$

or

$$\mathbf{P} = \mathbf{P}' + M\mathbf{V}$$

If the frame K' moves with the velocity of the CM, then

$$\mathbf{P} = M\mathbf{V}$$

and

$$\mathbf{P}' = 0$$

(the „internal" momentum of the system is zero)

Energy:

$$E' = \frac{1}{2}\sum_i m_i(\mathbf{v}_i')^2 + U = \frac{1}{2}\sum_i m_i(\mathbf{v}_i - \mathbf{V})^2 + U = \frac{1}{2}\sum_i m_i(\mathbf{v}_i^2 - 2\mathbf{v}_i \bullet \mathbf{V} + \mathbf{V}^2) + U = E - \mathbf{P} \bullet \mathbf{V} + \frac{1}{2}MV^2$$

or

$$E = E' + \mathbf{P} \bullet \mathbf{V} - \frac{1}{2}MV^2$$

If the frame K' moves with the velocity of the CM, then

$$\mathbf{P} = M\mathbf{V}$$

and

$$E = E' + \frac{1}{2}MV^2$$

(the sum of the „internal" energy and the kinetic energy of the CM)

Angular momentum:

Here not only the velocity $\mathbf{V}$ of the frame K' with respect to the frame K but also the position of O' with respect to K is important. Thus we use the Galilean transformation in the form

$$\mathbf{v}' = \mathbf{v} \cdot \mathbf{V}, \qquad \mathbf{r}' = \mathbf{r} - \mathbf{a}, \qquad (\dot{\mathbf{a}} = \mathbf{V})$$

and transform

$$\mathbf{L}' = \sum_i [\mathbf{r}'_i \times \mathbf{p}'_i] = \sum_i m_i [(\mathbf{r}_i - \mathbf{a}) \times (\mathbf{v}_i - \mathbf{V})] = \mathbf{L} - [\mathbf{a} \times \mathbf{P}] - M[\mathbf{R} \times \mathbf{V}] + M[\mathbf{a} \times \mathbf{V}]$$

or

$$\mathbf{L} = \mathbf{L}' + [\mathbf{a} \times \mathbf{P}] + M[\mathbf{R} \times \mathbf{V}] - M[\mathbf{a} \times \mathbf{V}]$$

One can see that in the case $\mathbf{V} = 0$ but $\mathbf{a} \neq 0$ (statically shifted frames) the angular momenta in the two frames are different, if the total momentum $\mathbf{P} = \mathbf{P}'$ is nonzero and not collinear to $\mathbf{a}$.

If the origin O' of the frame K' is located at the CM, then

$$\mathbf{a} = \mathbf{R}$$

and the transformation of the angular momentum simplifies to

$$\mathbf{L} = \mathbf{L}' + [\mathbf{R} \times \mathbf{P}]$$

(the sum of the „internal" angular momentum and angular momentum of the CM). Note that Landau and Lifshitz obtain this formula setting $\mathbf{a} = 0$ (at a given moment of time!) and using $\mathbf{P} = M\mathbf{V}$
Their derivation can be valid at one moment of time only.