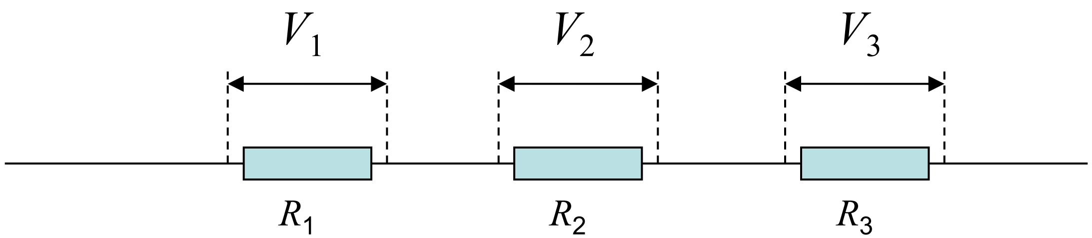
4 – Electric circuits

Serial and parallel resistors

Serial connection of resistors:



As the current I through each of serially connected resistors is the same, one can use Ohm's law and write

$$V = V_1 + V_2 + V_3 + \ldots = R_1 I + R_2 I + R_3 I + \ldots = (R_1 + R_2 + R_3) I$$

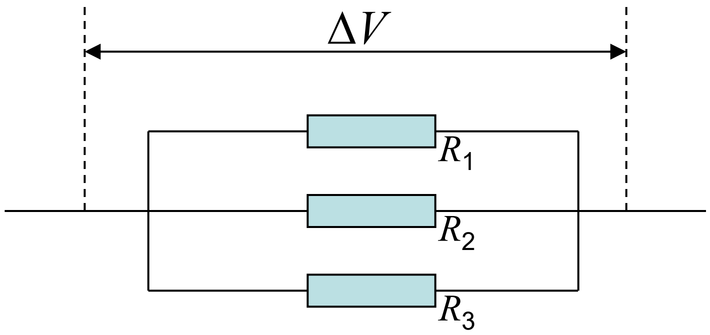
That is, one can consider serially connected resistors as one combined resistance:

$$V = RI$$

Thus

$$\boxed{R = R_1 + R_2 + R_3 + \ldots}$$

Parallel connection of resistors:



Here the potential difference V is common for all resistors, whereas total current is the sum of individual currents:

$$I = I_1 + I_2 + I_3 + \cdots = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} + \cdots$$

The total resistance is defined as

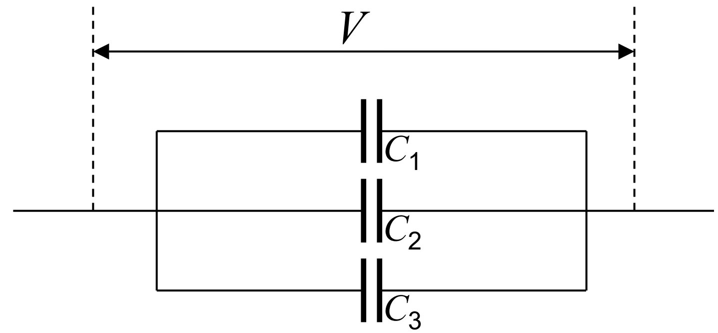
$$I = \frac{V}{R}$$

Thus

$$\boxed{\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots}$$

$1/R$ can be called <u>conductance</u>. In the case of parallel connection, conductances add up.

Serial and parallel capacitors



<u>Parallel</u> connection of capacitors:

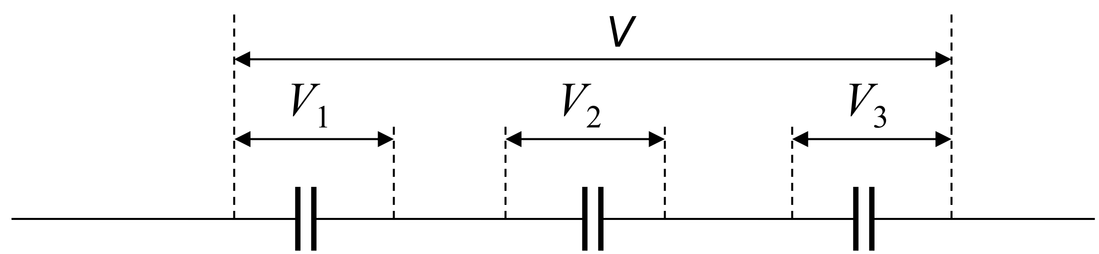
Several capacitors connected in parallel form an effective capacitor *C* whose charge is the sum of the charges on all capacitors, whereas the voltage is common:

$$Q = Q_1 + Q_2 + Q_3 + \ldots = C_1 V + C_2 V + C_3 V = (C_1 + C_2 + C_3)V = CV$$

Thus for the parallel connection of capacitors one obtains

$$\boxed{C = C_1 + C_2 + C_3}$$

<u>Serial</u> connection of capacitors:



Here voltages add up whereas the charge is common (on each of the capacitors the same)

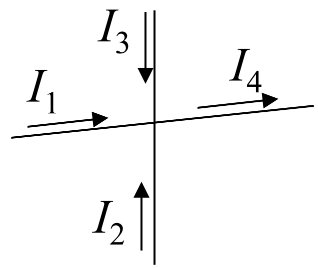
$$V = V_1 + V_2 + V_2 + \ldots = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3} + \ldots = Q\left(\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}\right) = \frac{Q}{C}$$

Thus for the serial connection of capacitors one obtains

$$\boxed{\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots}$$

Kirchhoff's laws

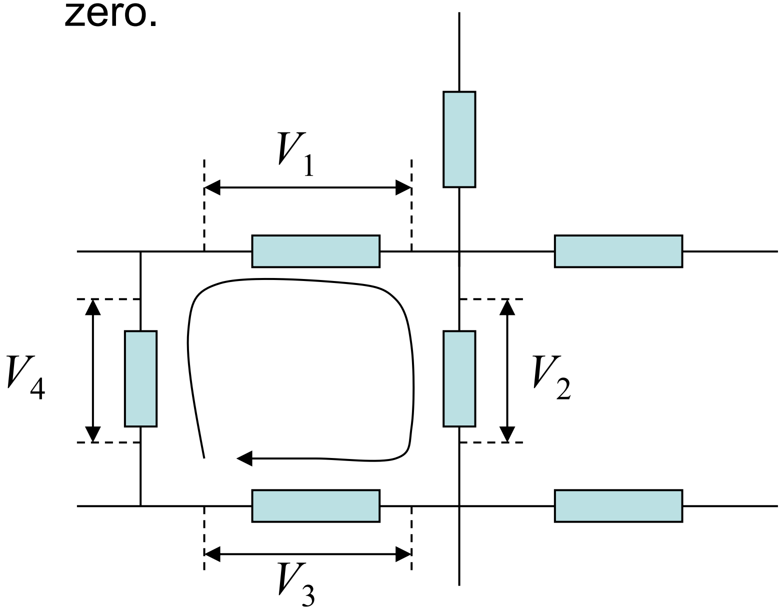
1. At any junction point, the sum of all currents entering the junction must be equal the sum of all currents leaving the junction



$$I_1 + I_2 + I_3 = I_4$$

(Electric charges are conserved and are not accumulating in the wire)

2. For any closed path in the circuit, the change of its electric potential around the path is zero.



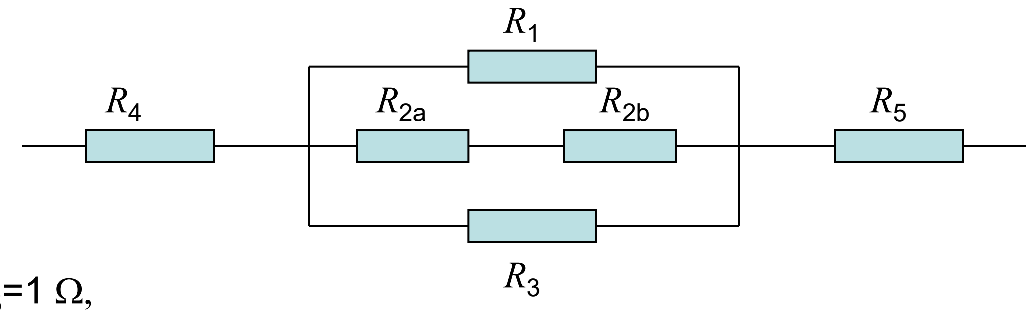
$$V_1 + V_2 + V_3 + V_4 = 0$$

(Electric field is conservative)

An equivalent formulation of the 2nd Kirchhoff's law: For any path between the two points in a circuit, the sum of all voltages on the path is equal to the voltage V between these two points:

$$\sum_i V_i = V$$

Problem



Calculate the effective resistance of this circuit. Obtain its numerical value for

R_1=1 Ω, R_{2a}=2 Ω, R_{2b}=1 Ω, R_3=3 Ω, R_4=1 Ω, R_5=1 Ω,

Solution: We at first replace the serially connected resistances $R_{2a}$and R_{2b} by the effective resistance $R_2 = R_{2a} + R_{2b}$. Then we replace the central group of parallel connected resistances by the effective resistance

$$\left(\frac{1}{R_1}+\frac{1}{R_2}+\frac{1}{R_3}\right)^{-1}$$

The last step is to replace the three serially connected resistances by the final effective resistance:

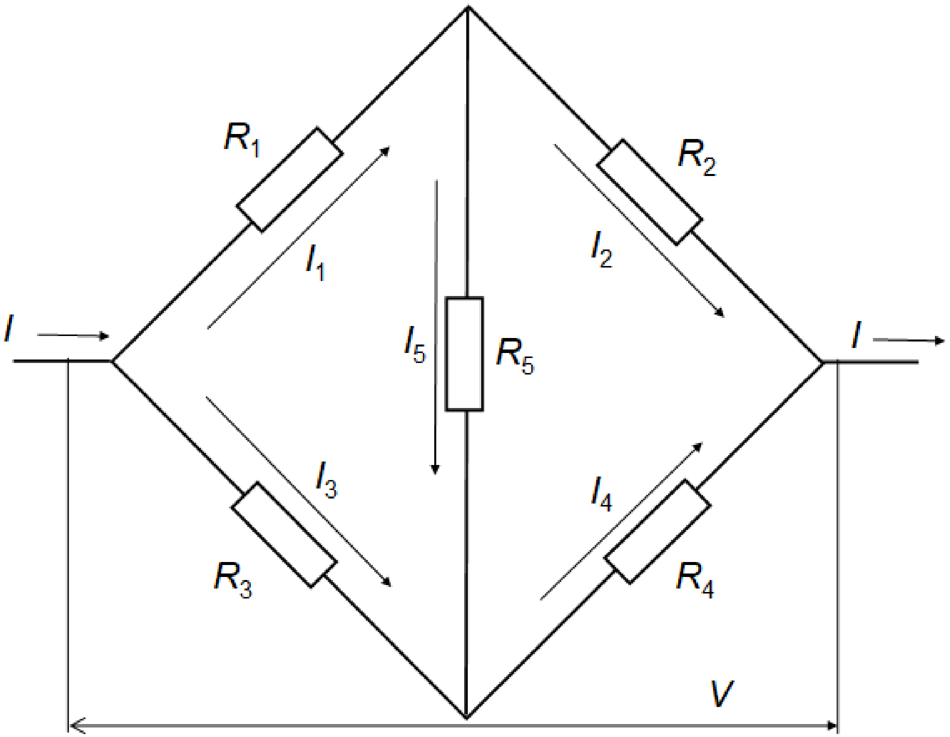
$$R = R_4 + \left(\frac{1}{R_1}+\frac{1}{R_{2a}+R_{2b}}+\frac{1}{R_3}\right)^{-1} + R_5$$

Plugging the numbers:

$$R = 1 + \left(\frac{1}{1}+\frac{1}{2+1}+\frac{1}{3}\right)^{-1} + 1 = 2.6\,\Omega$$

The Wheatstone bridge

Not all circuits can be calculated using the formulas for the serial and parallel connection of resistors. The simplest example is the so-called Wheatstone bridge. The Kirchhoff's equations for this circuit are the following.



1st Kirchhoff:

$$I = I_1 + I_3$$
$$I_1 = I_2 + I_5$$
$$I_3 + I_5 = I_4$$

2nd Kirchhoff + Ohm:

$$R_1 I_1 + R_2 I_2 = V$$
$$R_3 I_3 + R_4 I_4 = V$$
$$R_1 I_1 + R_5 I_5 + R_4 I_4 = V$$

This is a system of 6 linear equations for 6 unknowns – all currents. It can be solved by computer algebra. After finding all currents, one finds the effective resistance:

$$R = \frac{V}{I} = \frac{(R_1 + R_2)(R_3 + R_4)R_5 + (R_1 + R_3)R_2 R_4 + (R_2 + R_4)R_1 R_3}{(R_1 + R_2 + R_3 + R_4)R_5 + (R_1 + R_3)(R_2 + R_4)}$$

Limiting cases can be calculated easily and used to check the general formula above

$$R = \frac{R_1 R_3}{R_1 + R_3} + \frac{R_2 R_4}{R_2 + R_4}$$

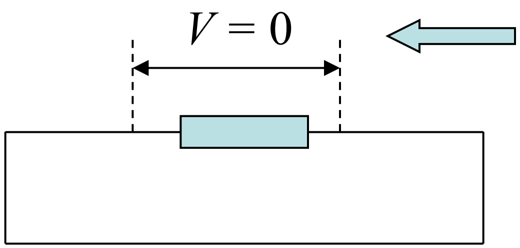
$$R_5 = 0$$

$$R = \frac{(R_1 + R_2)(R_3 + R_4)}{R_1 + R_2 + R_3 + R_4}$$

$$R_5 \Rightarrow \infty$$

Electromotive force

Consideration of closed electric circuits that consist of resistances only, like this one

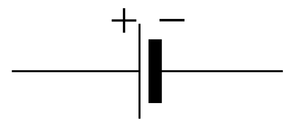


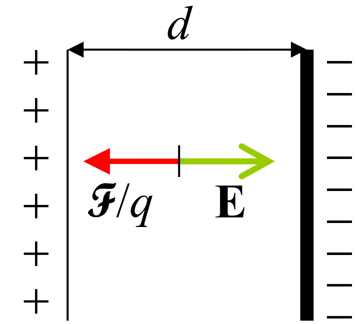
(According to the 2nd Kirchhoff's law)

shows that the electric current is zero. Indeed, there is no reason for the current to flow along the closed loops in the circuit because the total change of the potential across any loop is zero. What causes electric charges to flow are non-electric forces such as chemical forces in batteries. These forces are not potential forces because the work done by these forces along closed loops is nonzero. This is exactly the reason for the currents to flow in closed electric circuits.

Non-electric forces usually act in finite regions. In the case of batteries, they act only within batteries. The work done by non-electric forces on a test charge q that crosses the region of action of these forces, divided by q, is called Electromotive force and denoted by $\mathcal{E}$. Note that the term Electromotive force is misleading. First, it is of non-electric origin, although it moves electric charges. Second, it is not force but a quantity resembling the electric potential.

Batteries, EMF



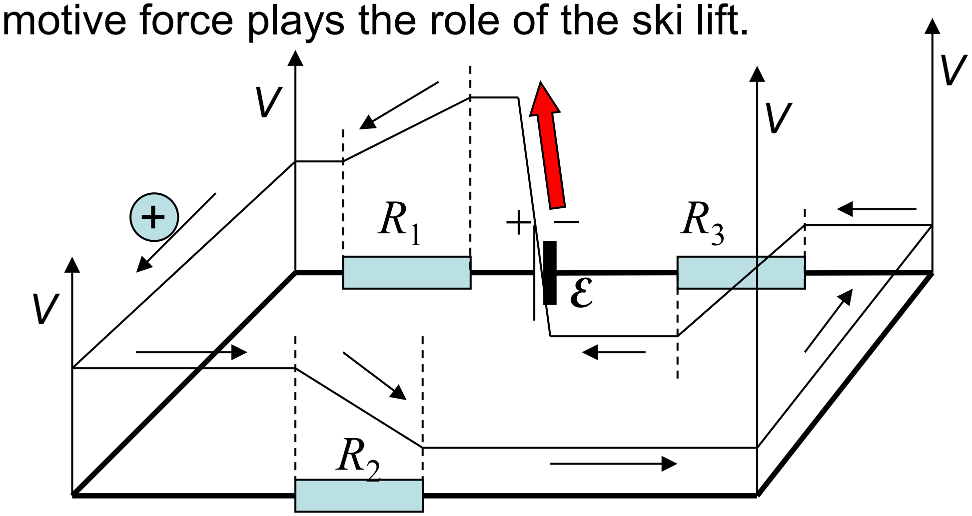


$$\mathcal{E} = \mathfrak{F}d/q$$

$$V = Ed$$

For an isolated (non-connected) battery the difference of the electric potential between the electrodes (the voltage) is equal to the electromotive force, $V = \mathcal{E}$. That is, electric and non-electric forces acting on the charges compensate each other everywhere inside the battery, so that the net force acting on a charge is zero and there is no electric current. The voltage on the battery arises because the chemical forces $\mathfrak{F}$ move electrons to the right (thus positive charges to the left) so that electrodes become charged positively and negatively, respectively.

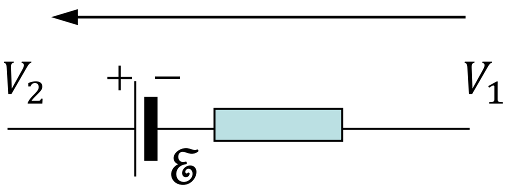
Change of the potential in a circuit with a battery. Potential increases across the battery and drops on the resistors, if we are moving in the shown direction of the current. Charges (positive) are moving outside the battery down the potential, like skiers in the mountains. Inside the battery they are moving up the potential under the influence of the electromotive force (an also electric force). The electromotive force plays the role of the ski lift.



Ohm's law with EMF

Consider a battery and a resistor connected serially. The resistor can stand for the internal resistance of the battery.

I - the chosen positive direction of the current



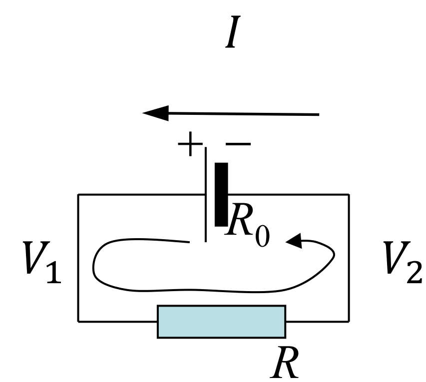
Voltage on the system

$$V = V_1 - V_2 = V_{\text{upstream}} - V_{\text{downstream}}$$

Ohm's law with EMF:

$$\boxed{V + \mathcal{E} = RI}$$

Consider the simplest closed circuit: battery and the load.



Voltage on the load resistor R is $V_1 - V_2 \equiv V$

Voltage on the battery is $V_2 - V_1 = -V$

Thus the Ohm's laws read

$$V = RI$$

$$-V + \mathcal{E} = R_0 I$$

Minus in front of V means that the voltage would move the current across the battery in the <u>negative</u> direction, in the absence of the electromotive force $\mathcal{E}$.

Adding these equations yields

$$\boxed{\mathcal{E} = (R + R_0)I} \Rightarrow I = \frac{\mathcal{E}}{R + R_0} \Rightarrow V = RI = \frac{R}{R + R_0}\mathcal{E}$$

2nd Kirchhoff's law with EMF

The second Kirchhoff's law states that for each closed loop in the circuit the sum of voltages is zero that reflects the fact that electric potential is defined unambiguously (and the work of the electric field over each closed trajectory is zero):

$$\sum_i V_i = 0.$$

To the Kirchhoff's laws, one has to add the Ohm's law

$$V_i = R_i I_i$$

for each resistor. On the top of it, there can be EMF's acting within resistors (batteries have their own internal resistance and thus can be considered as resistors) and pushing the current through them. With the EMF's, the Ohms law becomes

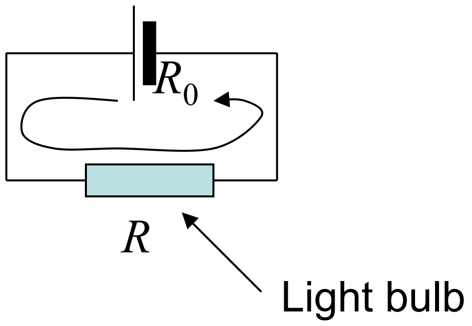
$$V_i + \mathcal{E}_i = R_i I_i.$$

Substituting $V_i = R_i I_i - \mathcal{E}_i$ into the second Kirchhoff's law, one obtains

$$\sum_i R_i I_i = \sum_i \mathcal{E}_i\,.$$

for each closed loop in a circuit.

Problem



A 1.5 V battery with the internal resistance 5 Ω is connected to a light bulb with a resistance of 20 Ω in a simple single-loop circuit. (a) What is the current flowing in the circuit? (b) What is the voltage difference across the light bulb?

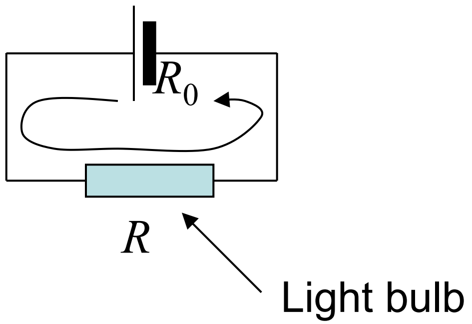
Solution: Use the results of the previous slide

$$I = \frac{\mathcal{E}}{R + R_0} = \frac{1.5}{5 + 20} = 0.06\ \mathrm{A}$$

$$V = IR = 0.06 \bullet 20 = 1.2\ \mathrm{V}$$

Problem

A 1.5 V battery with the internal resistance 5 Ω is connected to a light bulb with a resistance of 20 Ω in a simple single-loop circuit. (a) What is the power dissipated in the circuit? (b) What is the power dissipated in the light bulb and in the battery?



Solution: (a) The total dissipated power in the circuit is equal to the power of the electromotive force:

$$P = I\mathcal{E} = \frac{\mathcal{E}^2}{R + R_0} = \frac{1.5^2}{20 + 5} = 0.09 \text{ W}$$

(b) The power dissipated on the bulb is

$$P_{\text{bulb}} = I^2 R = \frac{\mathcal{E}^2 R}{(R + R_0)^2} = \frac{1.5^2 \bullet 20}{(20 + 5)^2} = 0.072 \text{ W}$$

The power dissipated inside the battery is

$$P_{\text{battery}} = I^2 R_0 = \frac{\mathcal{E}^2 R_0}{(R + R_0)^2} = \frac{1.5^2 \bullet 5}{(20 + 5)^2} = 0.018 \text{ W}$$

One can check that

$$P_{\text{bulb}} + P_{\text{battery}} = P$$