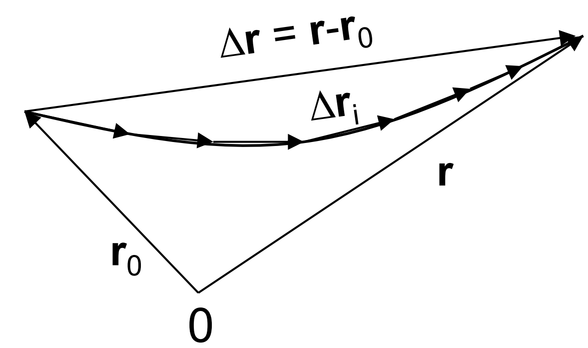
Electric potential

Electric potential is related to the electrostatic potential energy in the same way as the electric field is related to the electrostatic force. Electric potential V is defined as the potential energy E_p of a test charge q_0 divided by q_0: $V = E_p/q_0$. Unit: J/C = V (Volt).

From mechanics we know the definition of the potential energy of a physical body: it is the work of the external agent (say, your hand moving the body) on the way from a reference point $\mathbf{r}_0$ to the observation point $\mathbf{r}$. The motion is supposed to be slow and without acceleration, thus the total force is zero, that is, the force in the system $\mathbf{F}$ (electric force, gravity force, etc.) is just the opposite of the external agent force. It is more convenient to use the system force, thus:

$$E_p(\mathbf{r}) = -\sum_i \mathbf{F}_i \bullet \Delta\mathbf{r}_i + \text{const} \quad (\text{way from } \mathbf{r}_0 \text{ to } \mathbf{r})$$

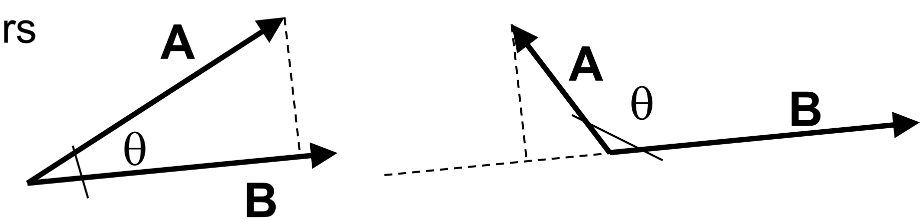


We split the whole way into small pieces that are practically straight and described by small displacements $\Delta\mathbf{r}_i$ and we sum the corresponding small works over *i*. The constant is the irrelevant reference potential energy $E_p(\mathbf{r}_0)$. The force $\mathbf{F}$ must be <u>conservative</u>.

Conservative forces are those forces for which the work is independent of the way from $\mathbf{r}_1$ to $\mathbf{r}_2$, and only in the case of conservative forces potential energy can be defined.

We use the dot product of two vectors

$$\mathbf{A} \bullet \mathbf{B} = AB\cos\theta$$

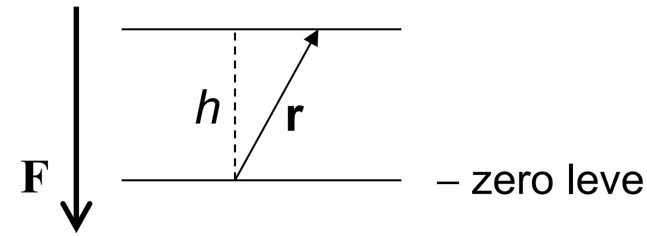


In the important particular case of constant force one obtains

$$E_p = -\mathbf{F}\cdot\sum_i \Delta\mathbf{r}_i + \text{const} = -\mathbf{F}\cdot(\mathbf{r}-\mathbf{r}_0) + \text{const} = -\mathbf{F}\cdot\mathbf{r} + \text{const}$$

Here the irrelevant constant has been redefined. It will be set to zero belw. One can see that $\Delta E_p > 0$ (that is, potential energy grows) if the displacement is directed against the system force ($\theta > 90^\circ$). For the gravity force that is a constant force directed vertically down, thus

$$E_p = -\mathbf{F}\cdot\ \mathbf{r} = Fh = mgh$$



This is a well-known formula in mechanics.

For electric charges, the electric potential V is defined by dividing the formula for the potential energy by q_0 and using $\mathbf{E} = \mathbf{F}/q_0$:

$$V = -\sum_i \mathbf{E}_i\cdot\Delta\mathbf{r}_i + \text{const}$$

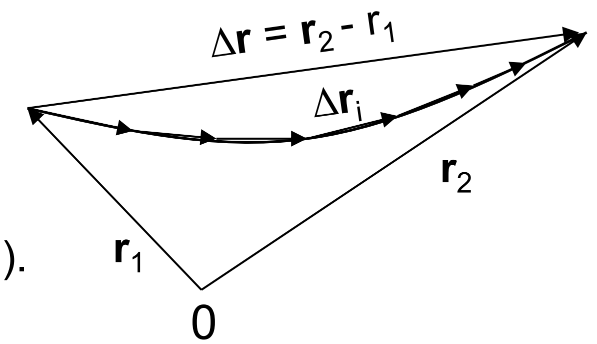
For the uniform electric field, this formula simplifies to

$$V = -\mathbf{E}\cdot\sum_i \Delta\mathbf{r}_i + \text{const} = -\mathbf{E}\cdot(\mathbf{r}-\mathbf{r}_0) + \text{const} = -\mathbf{E}\cdot\mathbf{r} + \text{const}$$

Electric potential increases when the observation point moves opposite to the electric-field vector (upstream), as in the case of gravity.

The difference of electric potentials ΔV between two points 1 and 2 (voltage) is defined as

$$\Delta V \equiv V_2 - V_1 = -\sum_i \mathbf{E}_i \cdot \Delta \mathbf{r}_i$$

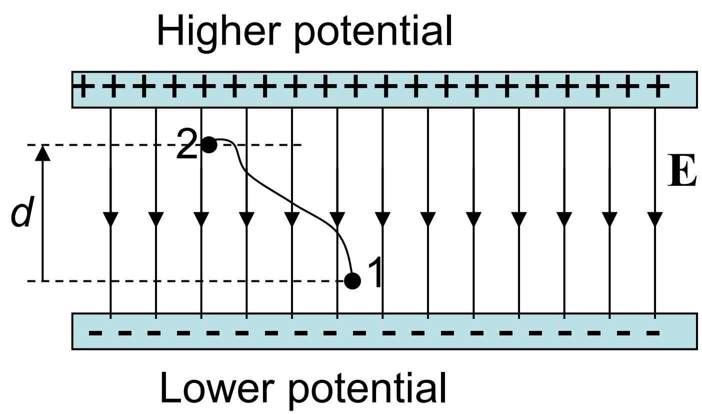


For the trajectory that goes from 1 to 2 (no constant in this formula).

Uniform electric field is an important particular case that is similar to the uniform gravitational field near the Earth´s surface. It is realized in the system of two large metal plates at a small distance from each other, charged with the opposite charges Q and -Q. The difference of the electric potential between the two points inside this system is

$$\Delta V \equiv V_2 - V_1 = Ed,$$

similarly to the case of gravity.



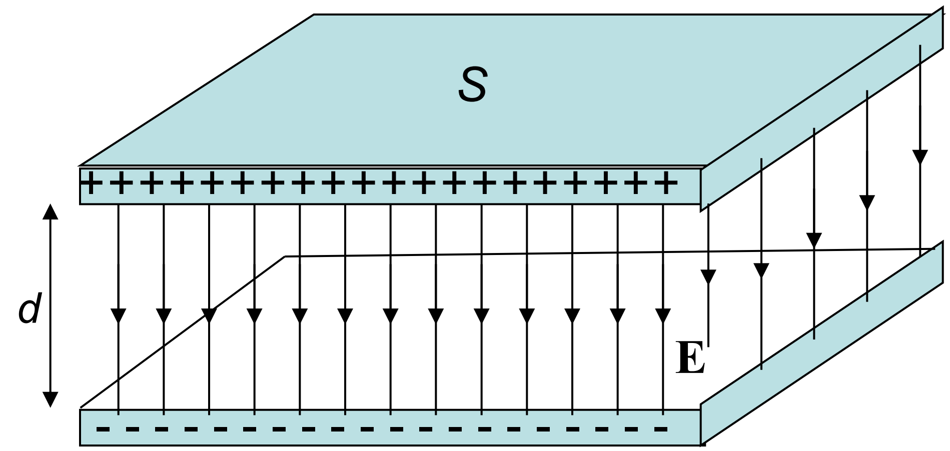
Electric field and electric potential in a capacitor

In most cases we will write V instead of ΔV to make formulas shorter.

Capacitors

A capacitor consists of two metal (or metal foil) plates of a big surface area S placed at a small distance *d* from each other. The two metal plates are charged by the opposite charges $\pm Q$ that are uniformly distributed over the surface with the surface charge density

$$\sigma = \frac{Q}{S}$$



The electric field in the capacitor is, according to the Gauss law,

$$E = \frac{\sigma}{\varepsilon_0}$$

Then the potential difference (voltage) on the capacitor is

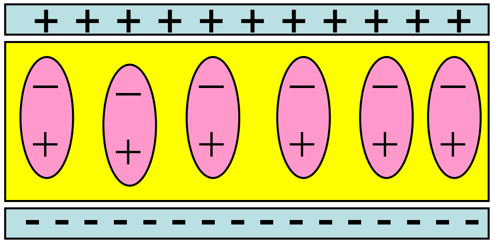
$$V = Ed = \frac{\sigma}{\varepsilon_0} d = \frac{Qd}{\varepsilon_0 S}$$

One can write the relation between the voltage V and the charge Q in the form

$$Q = CV, \qquad C = \frac{\varepsilon_0 S}{d}, \qquad \varepsilon_0 = 0.885 \times 10^{-11} \text{ in SI units}$$

where C is <u>capacitance</u>. The bigger the capacitance, the bigger charge is stored on the capacitor by the same voltage. To get a bigger capacitance, one has to take bigger surface S and smaller distance d. The unit of capacitance is <u>farade</u> (F).

Dielectrics



If the gap in a capacitor is filled by a dielectric, then the molecules of the dielectric are polarized by the electric field as shown above. This leads to the effective reduction of the charge of the plates by a factor of $\varepsilon > 1$ that depends on the dielectric material. Physically the charges induced in the dielectric partially screen the charges on the plates. As a result, the field in the capacitor is reduced by a factor of ε. For a condenser with dielectric one obtains

$$V = Ed = \frac{\sigma}{\varepsilon\varepsilon_0} d = \frac{Qd}{\varepsilon\varepsilon_0 S}$$

that is, the capacity of this capacitor increases because of the dielectric:

$$Q = CV, \qquad C = \frac{\varepsilon\varepsilon_0 S}{d}$$

For typical dielectrics ε is in the range from 2 to 10.

Energy of a charged capacitor

Process of charging a capacitor can be imagined as gradual transferring charge from one plate to the other. Then one plate becomes charged positively and the other negatively. The work to transfer a small charge ΔQ is given by

$$\Delta W = V\Delta Q$$

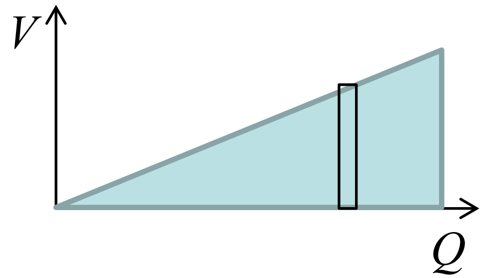
where we write V instead of ΔV for transparency and

$$V = \frac{Q}{C}$$

Thus

$$\Delta W = \frac{1}{C} Q\Delta Q$$

Total work = area under the line $V = Q/C$



- the work is bigger the bigger is the charge already transferred between the plates.
One can see that the process of charging a capacitor is similar to that of deforming a spring. The total work W done by charging the capacitor from zero (that is, by definition, the potential energy E_C of the capacitor) is thus the area of the triangle in the sketch:

$$E_C = \frac{1}{2} VQ = \frac{Q^2}{2C} = \frac{1}{2} CV^2$$

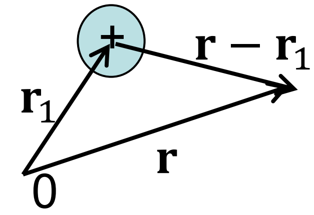
Electric potential of a point charge

To find the electric potential created by a point charge, one has to calculate the work by the coulomb force on the test charge and divide it by the test charge. This can be done with the help of calculus and the result is

$$V = k\frac{Q}{r}, \qquad k = \frac{1}{4\pi\varepsilon_0}$$

More general

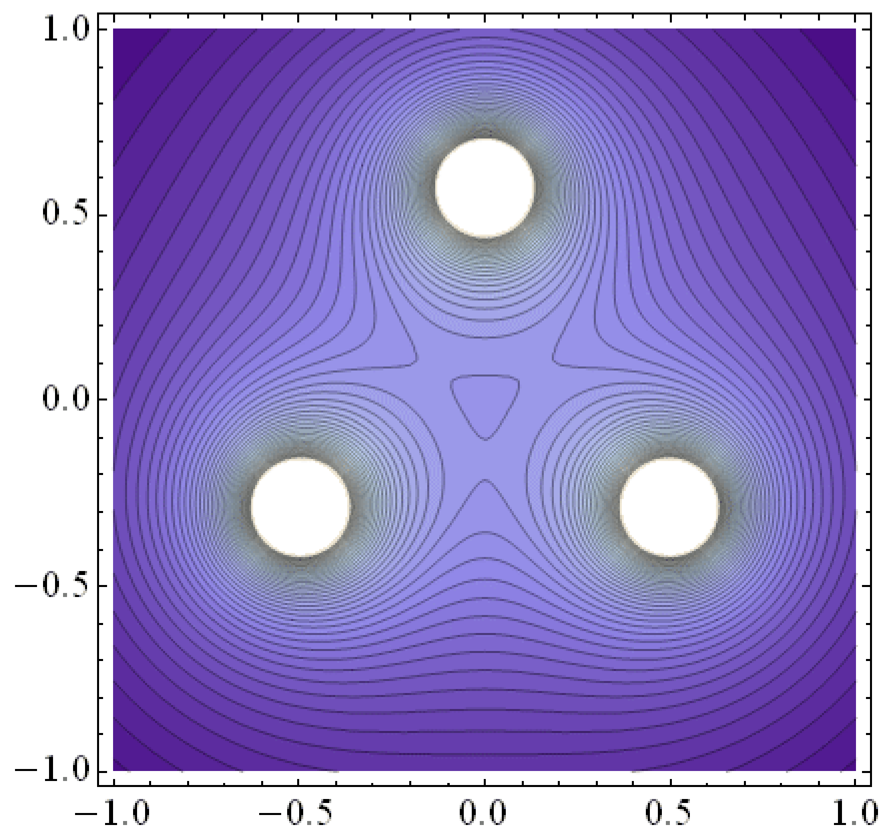
$$V = k\frac{Q}{|\mathbf{r} - \mathbf{r}_1|}$$



If the charge is positive, $Q > 0$, then V grows toward the point where the charge is placed.

The total electric potential created by several electric charges is the sum of the electric potentials created by each charge (the superposition law):

$$V(\mathbf{r}) = \sum_i V_i = \sum_i k\frac{Q_i}{|\mathbf{r} - \mathbf{r}_i|}$$



Equipotential lines of a system of three equal positive charges