12 – Oscillations and waves

Oscillations or vibrations are periodic motions in physical systems (such as mass on a spring) under the influence of restoring forces. Waves are motions of distributed systems (such as string) that are periodic in both time and space.

Oscillations

Example: mass on a spring

Restoring force:

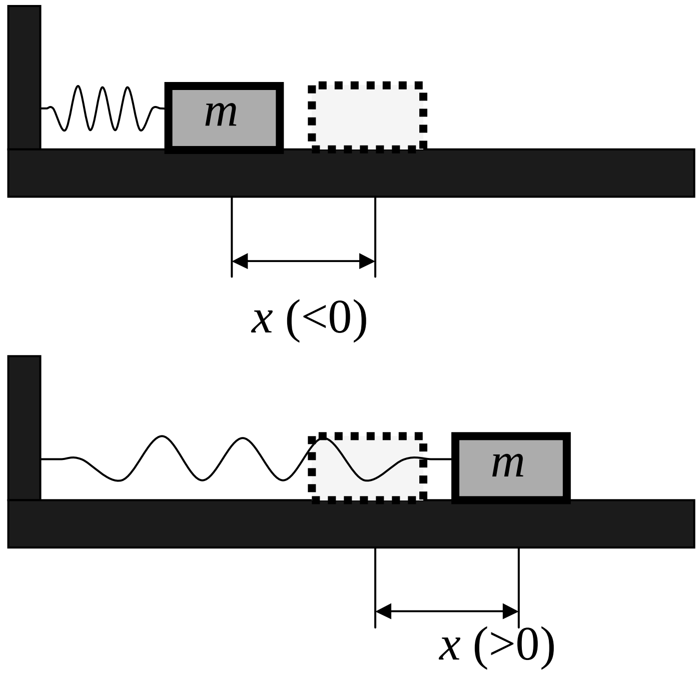
$$F = -kx, \quad k > 0 \ \text{(stiffness constant)}$$

Kinetic energy:

$$E_k = \frac{mv^2}{2}$$

Potential energy:

$$E_p = \frac{kx^2}{2}$$



Conservation of energy:

$$E_k + E_p = \frac{mv^2}{2} + \frac{kx^2}{2} = \text{const}$$

Newton's second law: $F = ma \quad \Rightarrow \quad a = \frac{F}{m} = -\frac{k}{m}x$

$$a + \omega^2 x = 0$$

– General equation for all kinds of oscillating systems (oscillators)

In our case $\omega = \sqrt{\frac{k}{m}}$

ω is the angular frequency of oscillations

Note that Newton's second law does not result in the case of oscillations to a motion with constant acceleration because the force is not constant and depends on the displacement. Solutions of Newton's second law for oscillations in the general form

$$\boxed{a + \omega^2 x = 0}$$

can be easily obtained with the calculus. It has a sinusoidal form $\boxed{x = A\cos(\omega t + \varphi_0)}$

where A is the amplitude of oscillations, ω is the anguilar velocity and φ_0 is a phase that depends on the initial conditions. Once $x(t)$ is known, one obtains the acceleration:

$$a = -\omega^2 x = -\omega^2 A\cos(\omega t + \varphi_0)$$

the velocity v can be obtained with the calculus or, alternatively, from the energy conservation law. At the turning points of the motion where $x = \pm A$ the velocity is zero and the whole energy is potential energy $kA^2/2$. Thus the energy conservation can be written in the form

$$\frac{mv^2}{2} + \frac{kx^2}{2} = \frac{kA^2}{2} \quad \Rightarrow \quad \left(\frac{v}{\omega}\right)^2 + x^2 = A^2$$

It follows then

$$v = \omega\sqrt{A^2 - x^2} = \omega A\sqrt{1 - \cos^2(\omega t + \varphi_0)} = \omega A\sqrt{\sin^2(\omega t + \varphi_0)} = \pm\omega A\sin(\omega t + \varphi_0) \quad \Rightarrow \quad -\omega A\sin(\omega t + \varphi_0)$$

Thus, all together,

$$\boxed{\begin{aligned} x &= A\cos(\omega t + \varphi_0) \\ v &= -\omega A\sin(\omega t + \varphi_0) \\ a &= -\omega^2 A\cos(\omega t + \varphi_0) \end{aligned}}$$

– for all oscillators!

(Correct sign can be easily obtained with calculus)

Problem

Find the phase φ_0 and the dependence $x(t)$ for the oscillatory motion that starts at $t = 0$ in the state where (a) velocity is zero and the displacement is maximal; (b) velocity is zero and the displacement is minimal; (c) displacement is zero and velocity is positive; (d) displacement is zero and velocity is positive.

Solution: (a) take the general solution $x = A\cos(\omega t + \varphi_0)$ and plug $t = 0$ and $x = A$:

$$A = A\cos(\varphi_0) \quad \Rightarrow \quad 1 = \cos(\varphi_0) \quad \Rightarrow \quad \varphi_0 = 0$$

$$x = A\cos(\omega t)$$

(b) take $x = A\cos(\omega t + \varphi_0)$ and plug $t = 0$ and $x = -A$:

$$-A = A\cos(\varphi_0) \quad \Rightarrow \quad -1 = \cos(\varphi_0) \quad \Rightarrow \quad \varphi_0 = \pi$$

$$x = A\cos(\omega t + \pi) = -A\cos(\omega t)$$

(c) take $x = A\cos(\omega t + \varphi_0)$ and plug $t = 0$ and $x = 0$:

$$0 = A\cos(\varphi_0) \quad \Rightarrow \quad 0 = \cos(\varphi_0) \quad \Rightarrow \quad \varphi_0 = \pm\frac{\pi}{2}$$

We see that there are two solutions. To find the proper one consider the velocity at $t = 0$:

$$v(t=0) = -\omega A\sin(\varphi_0) = -\omega A\sin\left(\pm\frac{\pi}{2}\right) = \mu\omega A$$

Positive velocity corresponds to the lower sign, thus take the lower sign to obtain

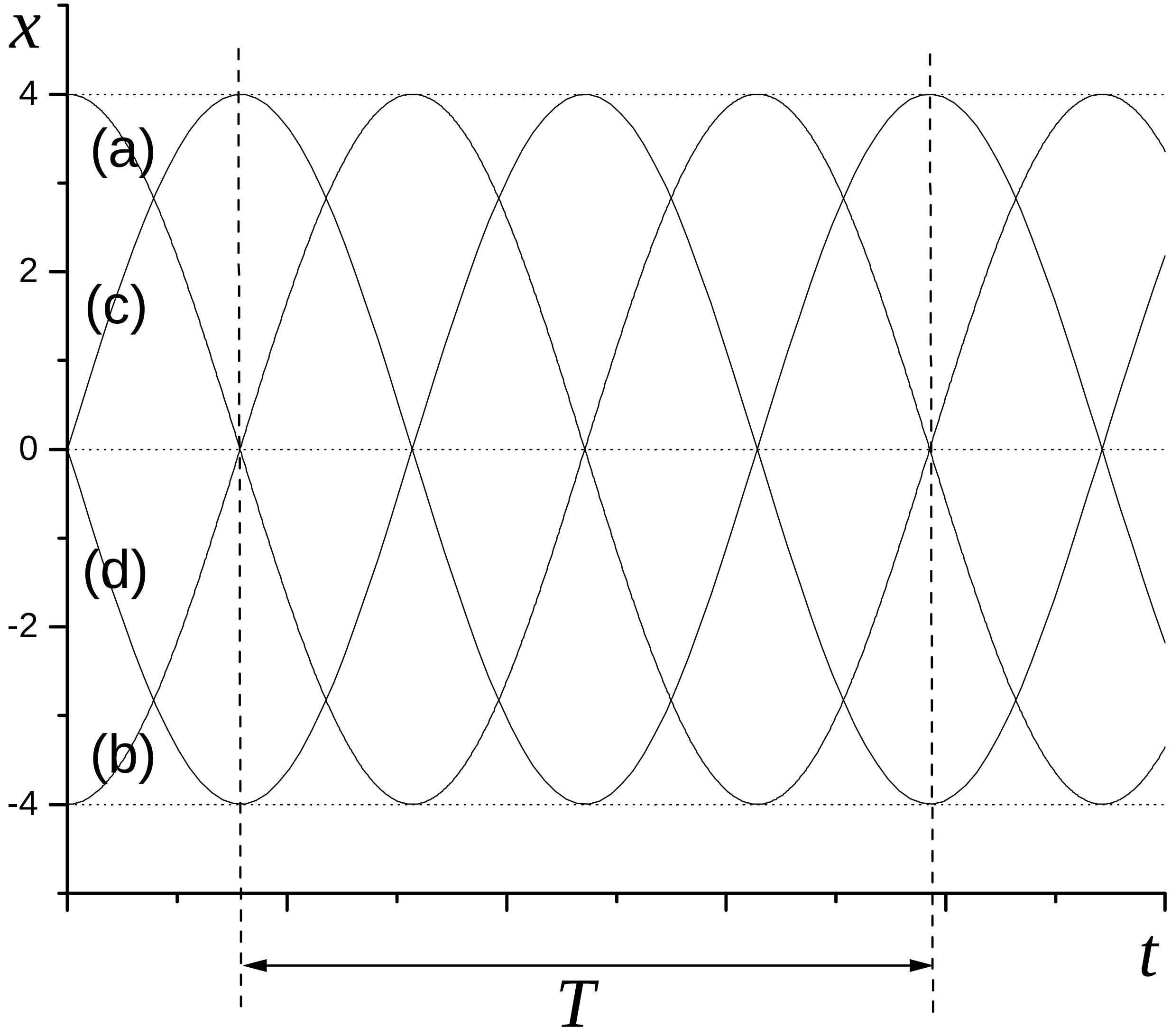
$$x = A\cos\left(\omega t - \frac{\pi}{2}\right) = A\sin(\omega t)$$

Here we take the upper sign in (c) and obtain

$$x = A\cos\left(\omega t + \frac{\pi}{2}\right) = -A\sin(\omega t)$$

Plots for (a-d)

$A = 4$



Frequency and period of oscillations

Frequency: $f = \dfrac{\omega}{2\pi}$

Period: $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

Problem

Mass m = 0.5 kg is attached to a spring with the stiffness constant k = 20 N/m on a horizontal frictionless table. The mass is pushed with the velocity v = 2 m/s in the positive direction out of the equilibrium position. What is (a) the amplitude of harmonic oscillations; (b) the maximal acceleration; (c) full time dependence $x(t)$? (d) Time to achieve maximal displacement for the first time?

Solution: (a) Use the energy conservation in the form $\frac{mv^2}{2}+\frac{kx^2}{2}=\frac{kA^2}{2}$

In the initial state $x = 0$ and v is known, thus the amplitude A is

$$A^2=\frac{m}{k}v^2 \quad \Rightarrow \quad A=\sqrt{\frac{m}{k}}v=\sqrt{\frac{0.5}{20}}2=0.316\,\mathrm{m}$$

(b) From Newton's second $F=ma$ follows $a=\frac{F}{m}=-\frac{k}{m}x$

The maximal acceleration corresponds to the maximal x, that is, to $x=A$. One obtains

$$a_{\max}=\frac{k}{m}A=\frac{k}{m}\sqrt{\frac{m}{k}}v=\sqrt{\frac{k}{m}}v=\sqrt{\frac{20}{0.5}}2=12.6\,\mathrm{m/s^2}$$

(c) Obviously $x=A\sin(\omega t),\qquad \omega=\sqrt{\frac{k}{m}}=6.32\,\mathrm{s^{-1}}$ is the only solution that satisfies $x = 0$ and $v>0$

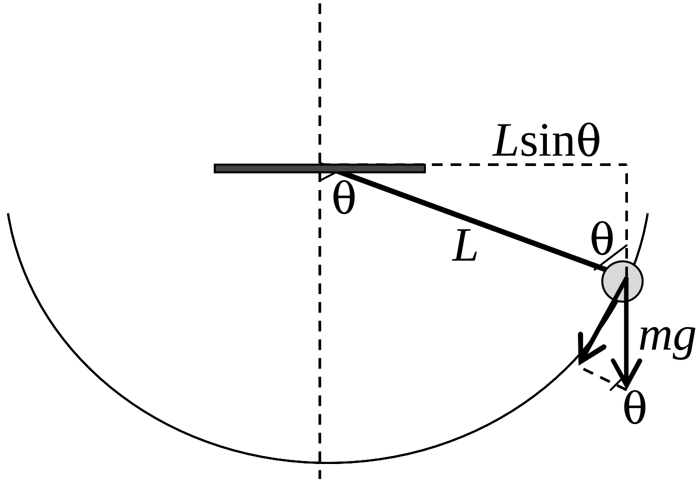
(d) The required time satisfies $\omega t = \pi/2$, so that the sine attains its maximum. Thus

$$t=\frac{\pi}{2\omega}=\frac{\pi}{2\times 6.32}=0.248\,\mathrm{s}$$

Oscillations, pendulum

Pendulum performs a rotational motion, thus we have to write down the rotational Newton's second law

$$\tau = I\alpha$$



The torque and the moment of inertia for a point-mass pendulum are given by

$$\tau = -mgL\sin\theta, \qquad I = mL^2$$

The sign (–) in the torque shows that the torque is <u>restoring</u>. The Newton's second law above can be rewritten as

$$\alpha = \frac{\tau}{I} = -\frac{mgL\sin\theta}{mL^2} = -\frac{g}{L}\sin\theta$$

or

$$\alpha + \omega^2 \sin\theta = 0, \qquad \omega = \sqrt{\frac{g}{L}}$$

For small-amplitude oscillations $\theta \ll 1$ one can use $\sin\theta \cong \theta$ that leads to the equation

$$\alpha + \omega^2 \theta = 0, \qquad \omega = \sqrt{\frac{g}{L}}$$

This equation is similar to that for a mass on a string above, with the replacements $a \Rightarrow a$ and $x \Rightarrow \theta$. Thus the solution for a pendulum is a sinusoidal (harmonic) motion with frequency ω.

Waves

Oscillations occur in localized systems such as mass on a spring or pendulum and they are periodic in time. Wave occur in distributed (non-localized) systems such as guitar string or water in the sea or the air, and they are periodic both in time and space. Dependence of the deviation variable X in a plane harmonic wave has the form

$$X = A\cos(\omega t - \mathbf{k} \bullet \mathbf{r} + \varphi_0)$$

where A is the amplitude, ω is the frequency, and $\mathbf{k}$ is the wave vector that shows the direction of motion of the wave. Solution for a wave on one dimension (that is, along the x axis) can be written as

$$X = A\cos(\omega t - kx + \varphi_0)$$

where $k > 0$ corresponds to a wave that goes to the right (in the positive direction along the x axis) and $k < 0$ corresponds to a wave that goes to the left. The period T and wave length λ are given by

$$T = \frac{2\pi}{\omega}, \qquad \lambda = \frac{2\pi}{|k|}$$

because the periods in time and space T and λ are defined by

$$\omega T = 2\pi, \qquad |k|\lambda = 2\pi$$

the velocity of the wave is given by $$v = \frac{\omega}{k}$$

and the wave equation can be rewritten as

$$X = A\cos\left(k(vt - x) + \varphi_0\right) = A\cos\left(k(x - vt) - \varphi_0\right)$$

In a particular media (such as air) waves with different values of ω and k are possible. However the speed of the wave v is a constant for a particular material.

In fluids sound can be only in the form of longitudinal waves. In solids that resist shear deformations, there is both longitudinal and transverse sound waves. In the longitudinal waves, displacement of the media is along the wave vector whereas in transverse waves it is perpendicular to the wave vector. In longitudinal waves pressure oscillates around the equilibrium level, that is why they are sometimes called pressure waves. Note that the velocities of the media's particles in the wave have no relation to the speed of sound. The former depends on the amplitude of the wave whereas the latter does not.

Material	Speed of sound
Rubber	60 m/s
Air	355 m/s
Water	1400 m/s
Glass	4540 m/s
Stone	5971 m/s
Lead	1210 m/s
Copper	3100 m/s