8 – Rotational motion of solid objects

Kinematics of rotations

In this Lecture we call „solid objects“ such extended objects that are rigid (nondeformable) and thus retain their shape. In contrast to point objects that are characterized by their position in the space, extended solid objects are characterized by both position and orientation. These objects can move as the whole and change their orientation (that is, rotate) at the same time.

The simplest case of rotational motion is rotation around a fixed axis, like rotation of a door around a hinge. In this case each point of the object performs circular motion studied in Lecture 5. If the object does not have a fixed axis of rotation (like an arbitrary object thrown into the air) its rotation can be very complicated, with the rotation axis permanently changing its direction with time. It can be shown that very small (infinitesimal) rotations can be considered as rotations around some fixed axis.

We will mostly speak about rotations around fixed axis here, leaving general rotations for more advanced physics courses.

Rotational velocity and acceleration

Rotational velocity (angular velocity):

$$\omega = \frac{\Delta\theta}{\Delta t}$$

Rotational acceleration (angular acceleration):

$$\alpha = \frac{\Delta\omega}{\Delta t}$$

(Δt is the time elapsed. For finite Δt this is the average angular velocity, for very small Δt this is the instantaneous angular velocity, as usual)

Motion with constant angular acceleration

$$\omega = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

(Similar to the translational motion with constant acceleration)

Similarities between translational and rotational motion (kinematics)

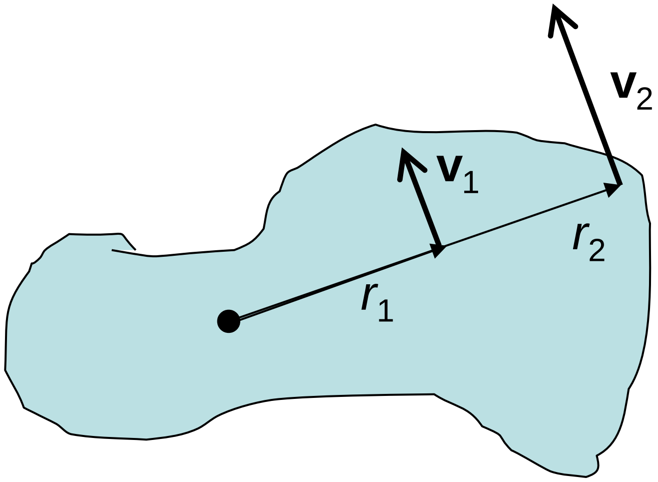
Position: x	Orientation: θ
Displacement: Δx	Rotational displacement: $\Delta\theta$
Velocity: v	Angular velocity: ω
Acceleration: a	Angular acceleration: α

Relation between the linear and angular velocities

For objects rotating around an axis, the linear velocity v increases with the distance from the axis:

$$v = \omega r$$

That is, the object cannot be characterized by a unique linear velocity but it can be characterized by a unique angular velocity



Similarly the relation between the linear and rotational accelerations has the form

$$a = \alpha r$$

Statics of rotations

In the mechanics of translational motion, two forces balance each other if their sum is zero

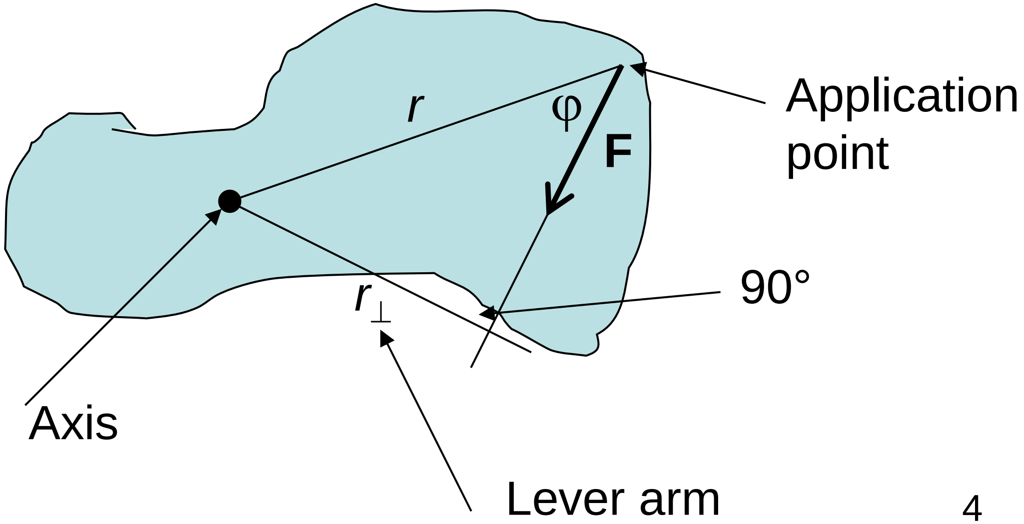
In the mechanics of rotational motion, two forces balance each other if the sum of their torques is zero, in other words, if the two torques balance each other.

Definition of torque (from the textbook): Product of the force and the lever arm

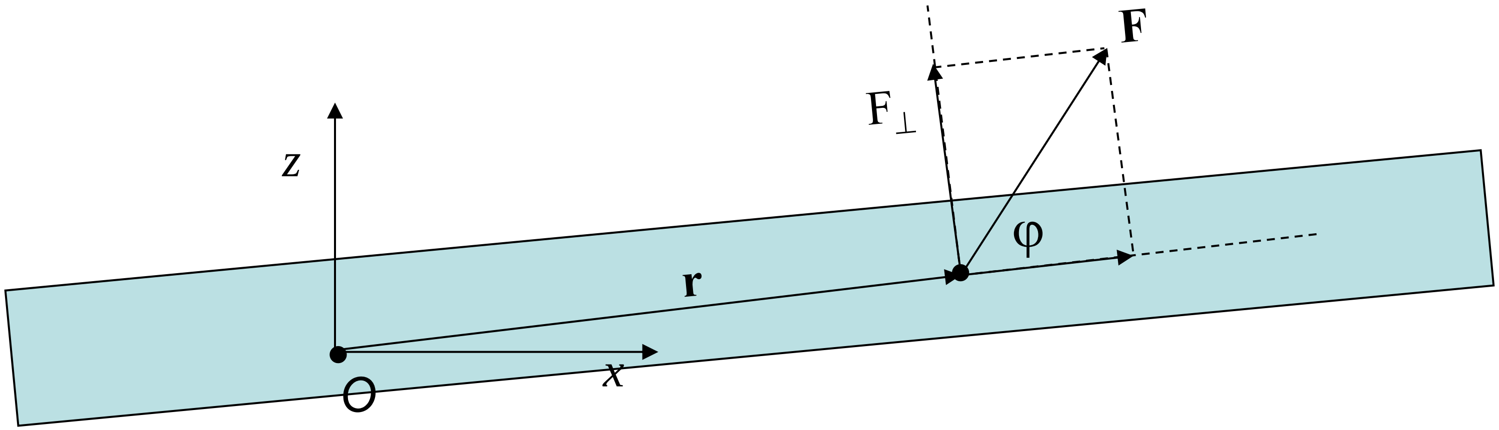
$$\boxed{\tau = F r_{\perp} = F r \sin\varphi}$$

- Force with a greater lever arm has a stronger effect!

One should assign a sign to the torque: Torques rotating counterclockwise are positive and torques rotating clockwise are negative. Having different signs, torques can balance each other



More convenient definition of the torque



$$\tau = F_{\perp} r = Fr\ \sin\varphi$$

This result is the same as on the preceding page.
Here we call r lever arm.

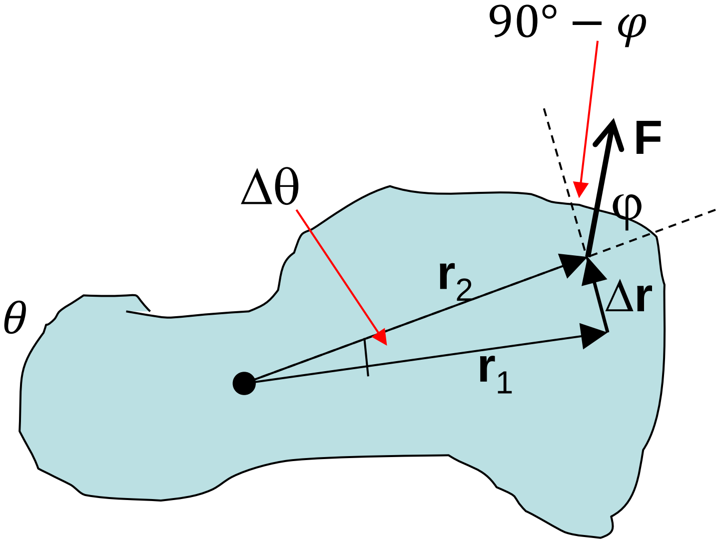
The unit of torque: N m (but not J).

Work-energy principle for the rotational motion and the condition of rotational equilibrium

The concept of the torque follows from the work-energy principle: The solid object is balanced if its displacements and rotations result in a zero work of applied forces. Indeed, if the work is nonzero, then displacements or rotations accompanied by the work will result in the change of its kinetic energy, that is, the object will accelerate or decelerate and thus it is unbalanced.

For translational displacement it obviously follows from this principle that the sum of all forces should be zero. For rotations it follows that the sum of all **torques** should be zero. The proof is the following:

For a small rotation $\Delta\theta$, the application point of the force makes a linear displacement $\Delta r = r\Delta\theta$ that is perpendicular to **r**, so that the work done is



$$\Delta W = F\Delta r\cos(90° - \varphi) = F\Delta r\sin\varphi = Fr\sin\varphi\,\Delta\theta = \tau\Delta\theta$$

If several forces are applied, then the work is

$$\Delta W = (\tau_1 + \tau_2 + \cdots)\Delta\theta$$

$$r_1 = r_2 = r$$
$$\Delta r \cong r\Delta\theta$$

since $\Delta\theta$ is the same for all forces. The work will be zero if $\boxed{\tau_1 + \tau_2 + \ldots = 0}$ - the rotational equilibrium condition

Center of mass and stability

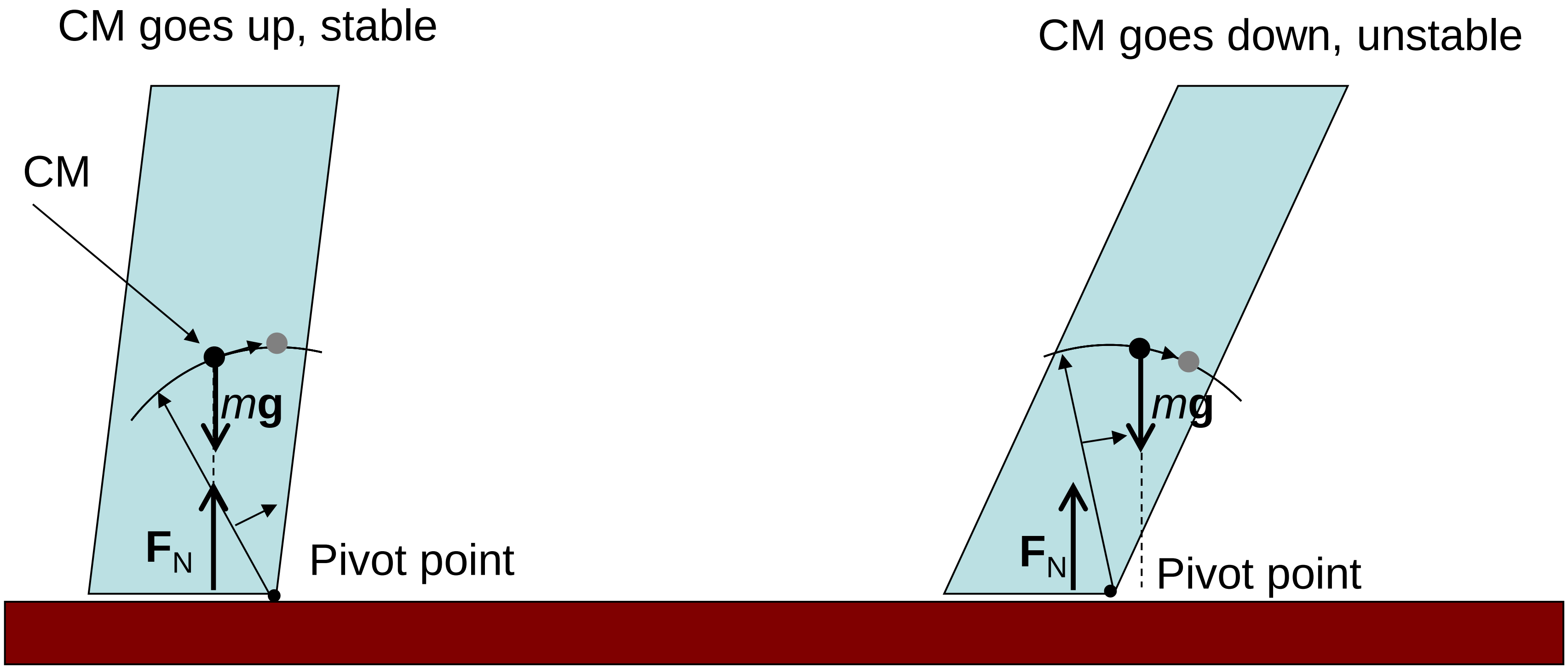
Center of mass (CM), also called center of gravity, is a point about which gravitational forces applied to different parts of the object produce no torque. That is, if we choose an axis going through or a pivot point in CM, the object will be balanced. One can consider gravitational force as applied to CM, so that its lever arm is zero and the torque vanishes. The object does not rotationally accelerate around ist own CM under the influence of gravitational force and it only can have linear (translational) acceleration. If we now choose another pivot point that does not coincide with CM, then the torque will be nonzero, the lever arm being defined by the distance between CM and the pivot point. The position of CM is given by

$$\mathbf{r} = \frac{1}{M}\sum_i m_i \mathbf{r}_i \qquad \text{where} \quad M = \sum_i m_i \quad \text{total mass of a system}$$

For symmetric objects such as a cube, CM is in the geometric center.

CM is important, in particular, for the analysis of the stability of solid objects with respect to tipping. If, as the result of a small rotation away from the initial position, CM goes up, the object is stable. Indeed, such a rotation leads to the increase of the potential energy and thus it is impossible without violation of the energy conservation law. On the contrary, if a small rotation leads to CM going down, then the object is unstable. In this case the decrease of the potential energy is compensated for by the increase of the kinetic energy, and the object will rotationally accelerate away from the initial position. See illustration on the next page.

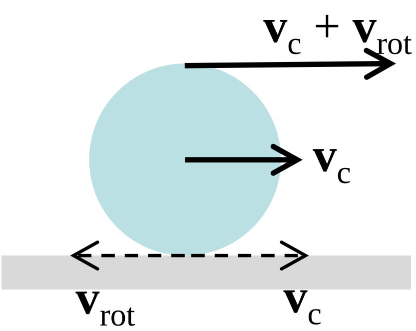
„Pisa tower“



Torque of the gravity force is <u>positive</u>. It is compensated for by the negative torque of the normal force.

Torque of the gravity force <u>negative</u>. It is <u>not</u> compensated for by negative torque of the normal force: The two torques have the same sign!

Rolling without slipping



If a round object (a sphere, a cylinder) is rolling on a plane without slipping, the velocity of its lowest point that is in contact with the plane is zero. The velocity of any point of the body is the sum of two velocities:

1) the velocity of the center $\mathbf{v}_c$ due to its translational motion and

2) the velocity $\mathbf{v}_{rot}$ due to the rotation of the body around its center.

The magnitude of the latter is $v_{rot} = \omega R$ at the distance R from its center, while the directions are different.

At the contact point, the velocities are opposite and compensate each other, so that the total velocity is zero (the no-slipping condition). For their magnitudes one obtains

$$v_c = v_{rot} = \omega R.$$

At the top of the rolling body, the two velocities are parallel and thus adding up:

$$v_{top} = v_c + v_{rot} = 2v_c = 2\omega R.$$

Example

68. A large spool of rope rolls on the ground with the end of the rope lying on the top edge of the spool. A person grabs the end of the rope and walks a distance L, holding onto it, Fig. 8–50. The spool rolls behind the person without slipping. What length of rope unwinds from the spool? How far does the spool's center of mass move?

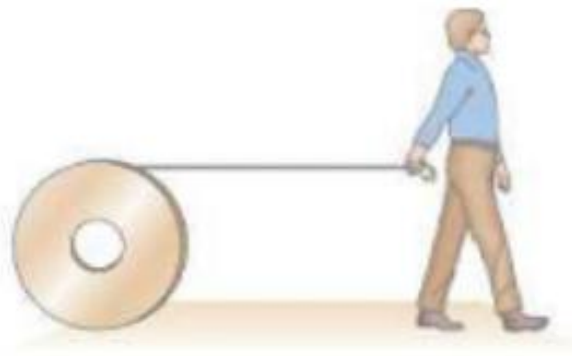


FIGURE 8–50
Problem 68.

The displacement of the person is

$$L = v_{top}\Delta t = 2v_c\Delta t.$$

The displacement of the center of the spool is

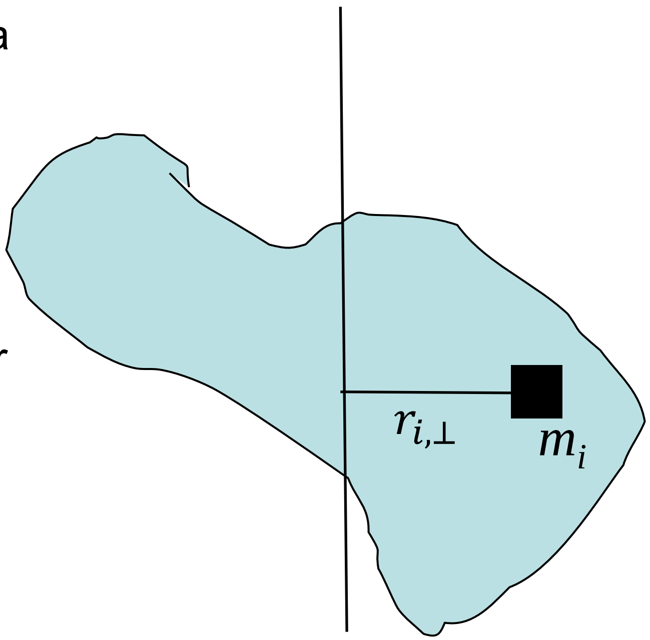
$$d_c = v_c\Delta t = L/2.$$

The length of the unwound rope is just the difference $L - d_c = L - L/2 = L/2$.

Rotational dynamics

If the torques acting on a body are unbalanced, that is, the total torque is nonzero, the body will rotationally accelerate. This is governed by the second Newton´s law for rotations.

Newton‘s second law for the rotational motion



To obtain Newton’s second law for the solid body rotating around a fixed axis, we split the body into small pieces of masses m_i and introduce the distances $r_{i,\perp}$ between each elementary mass and the axis of rotation, drawing perpendiculars from each elementary mass to the axis.

For each elementary mass, we write Newton’s second law for their tangential accelerations:

$$F_i = m_i a_i.$$

One can express all tangential accelerations via the angular acceleration α that is the same for the whole solid body: $a_i = \alpha r_{i,\perp}$. Next, we multiply Newton’s second law by $r_{i,\perp}$ to express it in terms of the torques:

$$\tau_i = F_i r_{i,\perp} = m_i {r_{i,\perp}}^2 \alpha.$$

Now, we sum over all elementary masses to obtain Newton’s second law for the rotational motion:

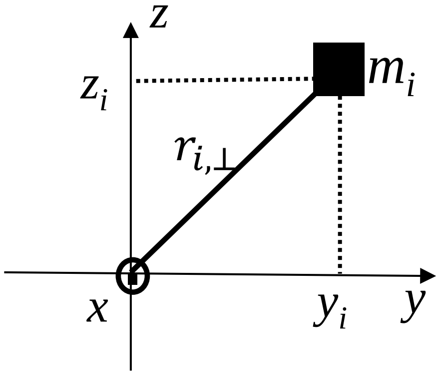
$$\tau = \sum_i \tau_i = I\alpha, \qquad I \equiv \sum_i m_i {r_{i,\perp}}^2 .$$

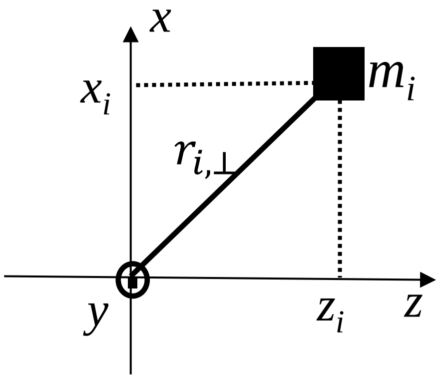
Here τ is the total torque exerted on the system from <u>external</u> forces and I is the moment of inertia of the body with respect to the given rotation axis.

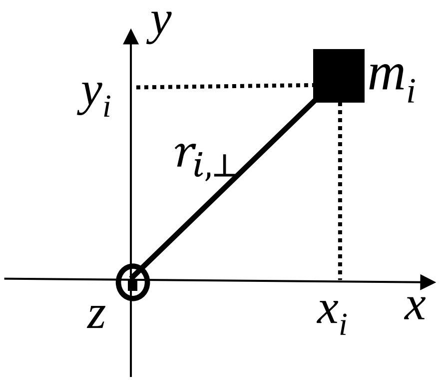
Practical formulas for calculating moments of inertia with respect to different axes

With respect to the x, y, and z axes :

$$I_x = \sum_i m_i({y_i}^2+{z_i}^2), \qquad I_y = \sum_i m_i({z_i}^2+{x_i}^2), \qquad I_z = \sum_i m_i({x_i}^2+{y_i}^2),$$







For a flat body in the xy plane one has $z_i = 0$, thus

$$I_x = \sum_i m_i {y_i}^2\,, \qquad I_y = \sum_i m_i {x_i}^2\,, \qquad I_z = \sum_i m_i({x_i}^2+{y_i}^2) = I_x + I_y,$$

Expressions for I for typical symmetric bodies can be obtained with calculus:

Rod of length l with axis through its end: $I = \frac{1}{3}Ml^2$

Rod of length l with axis through center: $I = \frac{1}{12}Ml^2$

Rectangle with sides a and b around the ⊥ axis z: $I_z = \frac{1}{12}M(a^2 + b^2)$

Ring or hollow cylinder of radius R around the symmetry axis z: $I_z = MR^2$

Ring around in-plane axes x,y: $I_x = I_y = \frac{1}{2}MR^2$

Disc or solid cylinder around the symmetry axis z: $I_z = \frac{1}{2}MR^2$

Disc around in-plane axes x,y: $I_x = I_y = \frac{1}{4}MR^2$

Hollow sphere: $I_x = I_y = I_z = \frac{2}{3}MR^2$

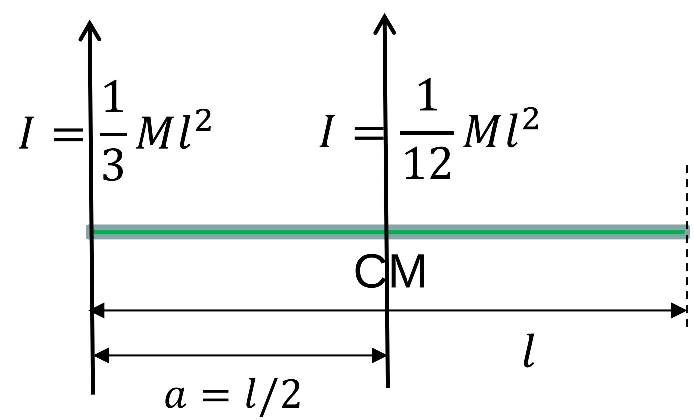
Solid sphere: $I_x = I_y = I_z = \frac{2}{5}MR^2$

Steiner theorem: the moment of inertia I_{CM} with respect to the axis going through the CM is minimal; the moment of inertia with respect to any other axis parallel to the former is given by

$$I = I_{CM} + Ma^2,$$

where a is the distance between the axis and the CM. Example: for the rod $I_{CM} = Ml^2/12$. With respect to the rod's end, with $a = l/2$

$$I = I_{CM} + M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2.$$



Angular momentum and its conservation

Newton's second law for the rotational motion can be written as

$$\tau = I\frac{\Delta\omega}{\Delta t} \Rightarrow \frac{\Delta(I\omega)}{\Delta t} = \frac{\Delta L}{\Delta t}, \qquad\qquad L \equiv I\omega.$$

Here L is the angular momentum of the body. In the absence of the total torque, $\tau = 0$, the angular momentum is conserved, $L = const$.

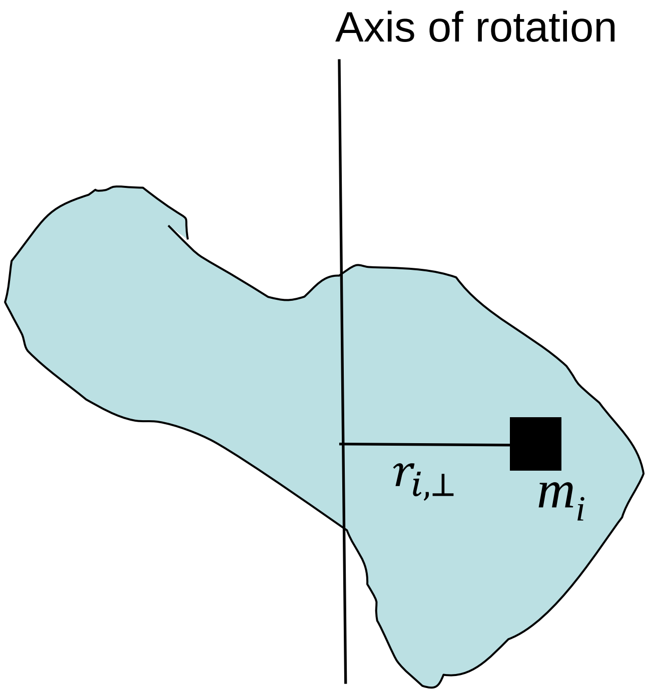
For instance, the gravitational force acting upon the planets from the sun has a zero lever arm, so that the torque on the planets is zero and their angular momentum is conserved. This leads to Kepler´s second law.

An interesting difference between translational and rotational motions is that, whereas the mass is constant, the moment of inertia I can be changed by changing the distances of masses from the axis. If the torque is zero and L is conserved, changing I results in the change of the angular velocity ω. However, strictly speaking, here it was not proven that the transformation above shown by $\Rightarrow$ is valid when I changes. This can be easily proven within the calculus-based physics course.

Rotational kinetic energy

The kinetic energy of the body rotating around a fixed axis can be written as the sum of kinetic energies of elementary masses and then transformed using the relation between the linear and angular velocity $v = \omega r$:

$$E_{rot} = \sum_i \frac{m_i {v_i}^2}{2} = \frac{1}{2} \sum_i m_i (\omega r_{i,\perp})^2 = \frac{I\omega^2}{2}.$$



In general, the total kinetic energy is the sum of the translational kinetic energy of the center of mass and the kinetic energy due to the rotation around the CM

$$E_k = E_{tr} + E_{rot} = \frac{M {v_{CM}}^2}{2} + \frac{I_{CM}\omega^2}{2}.$$

Corresponding concepts of translational (linear) and rotational (angular) motion in dynamics

Concept	Translational motion	Rotational motion
Inertia	m, M	I
Newton´s second law	$F = ma$	$\tau = I\alpha$
Momentum	$\mathbf{P} = m\mathbf{v}$	$L = I\omega$
Kinetic energy	$E_k = \frac{1}{2}mv^2$	$E_k = \frac{1}{2}I\omega^2$