7 – Impuls and Momentum; Conservation

Newton´s second law		Impuls - Momentum relation		Momentum conservation in isolated systems
$\mathbf{F} = m\mathbf{a}$	$\Rightarrow$ ($\mathbf{a} = \Delta\mathbf{v}/\Delta t$)	$\underbrace{\mathbf{F}\Delta t}_{\text{Impulse of the force}} = m\Delta\mathbf{v} \equiv \underbrace{\Delta\mathbf{P}}_{\text{Change of the momentum}}$	$\Rightarrow$ ($\mathbf{F}=\mathbf{0}$)	$\Delta\mathbf{P} = \mathbf{0} \Leftrightarrow \mathbf{P} = \text{const}$

Momentum

Definition of the (linear) momentum for one particle: $\boxed{\mathbf{P} = m\mathbf{v}}$

For a system of interacting objects, adding up Newton´s second laws for all objects:

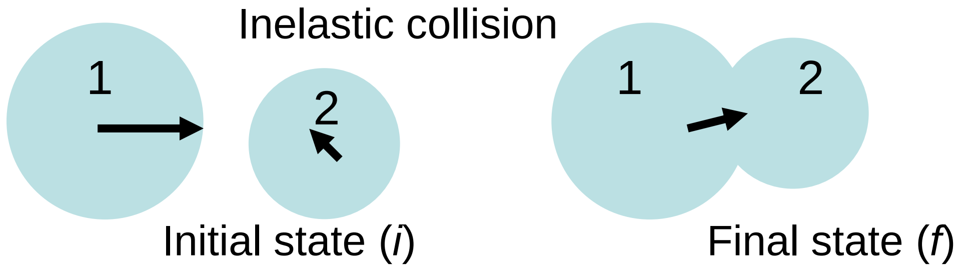
$$\boxed{\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \ldots} \Rightarrow \boxed{\mathbf{F}\Delta t = \Delta\mathbf{P}, \quad \text{where} \quad \mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \ldots}$$

(Sum only <u>*external*</u> forces.
<u>*Internal*</u> forces make no contribution according to Newton´s third law $\mathbf{F}_{21}=-\mathbf{F}_{12}$)

In isolated systems, ($\mathbf{F} = \mathbf{0}$) total momentum is conserved: $\boxed{\mathbf{P} = \text{const}}$

Application: <u>Collisions</u>. Collisions are very short, thus the impulse of the external forces on the system during a collision is negligible and the system can be considered as isolated. The interaction forces during the collision are very strong and they change the state of the system by changing the individual momenta, while the total momentum is conserved. We will consider collisions of two bodies. Collisions of many bodies, for instance, triple collisions, are rare.

Collisions and recoil



Two limiting cases of collisions:

- **Inelastic** (objects stick together after collision; Some part of the mechanical energy is lost)

$$\mathbf{P}_{1,i} + \mathbf{P}_{2,i} = \mathbf{P}_f \implies m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i} = (m_1 + m_2)\mathbf{v}_f$$

Solution: $\mathbf{v}_f \equiv \mathbf{u} = \frac{m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i}}{m_1 + m_2}$

- **Elastic** (Objects rebound; Mechanical energy is conserved)

$$\begin{aligned} \mathbf{P}_{1,i} + \mathbf{P}_{2,i} &= \mathbf{P}_{1,f} + \mathbf{P}_{2,f} \\ E_{1,i} + E_{2,i} &= E_{1,f} + E_{2,f} \end{aligned} \implies \begin{aligned} m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i} &= m_1\mathbf{v}_{1,f} + m_2\mathbf{v}_{2,f} \\ m_1v^2_{1,i} + m_2v^2_{2,i} &= m_1v^2_{1,f} + m_2v^2_{2,f} \end{aligned}$$

In 1D (head-on collision) one can remove vectors, then one can see that there are two equations with two unknowns that can be solved for $v_{1,f}$ and $v_{2,f}$. In 2D there are four unknowns (two components of each velocity) and only three equations – the system is underdetermined.

Recoil is a phenomenon inverse to the inelastic collision:

Forces acting between two objects cause them to move in different directions

Example: The gun and the bullet (the gun recoils back and hits the shoulder)

Whereas in the inelastic collision the energy is absorbed (mechanical energy is transformed into heat), in the recoil the energy is released (chemical energy is transformed into mechanical energy)

Energy lost in the inelastic collision (general case)

In the inelastic collision, the two masses stick together as the result of the collision, so that the conservation of the linear momentum has the form

$$m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i} = (m_1 + m_2)\mathbf{u},$$

where $\mathbf{u} \equiv \mathbf{v}_f$ is the velocity of the system in the final state. From here, one finds

$$\mathbf{u} = \frac{m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i}}{m_1 + m_2}.$$

Now, the energies in the initial and final states are given by

$$E_i = \frac{m_1\mathbf{v}_{1,i}^2}{2} + \frac{m_2\mathbf{v}_{2,i}^2}{2}, \qquad E_f = \frac{(m_1 + m_2)\mathbf{u}^2}{2}.$$

Here and elsewhere, the square of a vector is defined as the dot-product of the vector with itself, for instance:

$$\mathbf{u}^2 \equiv \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}||\mathbf{u}|\cos 0° = |\mathbf{u}|^2 = u^2.$$

The energy lost in the collision is defined as

$$E_{lost} \equiv E_i - E_f,$$

so that here

$$E_{lost} = \frac{m_1\mathbf{v}_{1,i}^2}{2} + \frac{m_2\mathbf{v}_{2,i}^2}{2} - \frac{(m_1 + m_2)\mathbf{u}^2}{2}.$$

Substituting the solution for $\mathbf{u}$, one obtains

$$E_{lost} = \frac{m_1 \mathbf{v}_{1,i}^2}{2} + \frac{m_2 \mathbf{v}_{2,i}^2}{2} - \frac{m_1 + m_2}{2} \left(\frac{m_1 \mathbf{v}_{1,i} + m_2 \mathbf{v}_{2,i}}{m_1 + m_2} \right)^2$$

$$= \frac{m_1 \mathbf{v}_{1,i}^2}{2} + \frac{m_2 \mathbf{v}_{2,i}^2}{2} - \frac{m_1^2 \mathbf{v}_{1,i}^2 + 2 m_1 m_2 \mathbf{v}_{1,i} \cdot \mathbf{v}_{2,i} + m_2^2 \mathbf{v}_{2,i}^2}{2(m_1 + m_2)}.$$

Bringing this expression into the form with the common denominator, one proceeds as follows

$$E_{lost} = \frac{(m_1 + m_2)\left(m_1 \mathbf{v}_{1,i}^2 + m_2 \mathbf{v}_{2,i}^2\right) - m_1^2 \mathbf{v}_{1,i}^2 - 2 m_1 m_2 \mathbf{v}_{1,i} \cdot \mathbf{v}_{2,i} - m_2^2 \mathbf{v}_{2,i}^2}{2(m_1 + m_2)}$$

$$= \frac{m_1 m_2 \mathbf{v}_{1,i}^2 + m_1 m_2 \mathbf{v}_{2,i}^2 - 2 m_1 m_2 \mathbf{v}_{1,i} \cdot \mathbf{v}_{2,i}}{2(m_1 + m_2)} = \frac{m_1 m_2 \left(\mathbf{v}_{1,i} - \mathbf{v}_{2,i}\right)^2}{2(m_1 + m_2)}.$$

In this calculation, the terms $m_1^2 \mathbf{v}_{1,i}^2$ and $m_2^2 \mathbf{v}_{2,i}^2$ cancel that leads to a great simplification. The formula obtained is very elegant and can be checked on particular cases. If one of the masses is zero or the initial velocities are equal to each other, there is actually no collision and the lost energy is zero.

See an application of this formula in the Problem Solving manual.

Recoil

In the recoil of an object into two parts with masses m_1 and m_2, the energy ΔE is released. Find the velocities of the parts 1 and 2.

Conservation laws for the linear momentum in this case has the form

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = \mathbf{0}.$$

As the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (the velocities in the final state) are proportional to each other, that is, directed along the same line (that has a random direction), one can choose the *x*-axis along this line. Then one can discard vectors and write

$$m_1 v_1 + m_2 v_2 = 0,$$

where v_1 and v_2 are projections of the velocity vectors onto the *x*-axis, that can be positive or negative. The energy balance in the process has the form

$$\Delta E = E_1 + E_2 = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2},$$

where ΔE is the energy released in the explosion. From the first equation, one obtains

$$\frac{v_2}{v_1} = -\frac{m_1}{m_2}.$$

This implies that the lighter part has a higher speed (think about the rifle and the bullet).

Expressing v_2 via v_1,

$$v_2 = -v_1 \frac{m_1}{m_2},$$

and substituting this into the energy equation, one obtains

$$\Delta E = \frac{m_1 v_1^2}{2} + \frac{m_2}{2}\left(-v_1 \frac{m_1}{m_2}\right)^2 = \frac{m_1 v_1^2}{2} + \frac{m_1^2 v_1^2}{2m_2} = \frac{m_1(m_1+m_2)v_1^2}{2m_2}.$$

From this one finds

$$v_1 = \sqrt{\frac{2m_2 \Delta E}{m_1(m_1+m_2)}} = \sqrt{\frac{2\Delta E}{m_1+m_2}}\sqrt{\frac{m_2}{m_1}}.$$

The last form of the result separates the parts symmetric and non-symmetric in 1 and 2. Now v_2 can be found using the formula for v_2 above:

$$v_2 = -v_1 \frac{m_1}{m_2} = -\sqrt{\frac{2\Delta E}{m_1+m_2}}\sqrt{\frac{m_2}{m_1}}\frac{m_1}{m_2} = -\sqrt{\frac{2\Delta E}{m_1+m_2}}\sqrt{\frac{m_1}{m_2}}.$$

In fact, this formula could be obtained immediately from the formula for v_1 by just exchanging $1 \rightleftharpoons 2$ and changing the sign.

Elastic collision in one dimension (head-on collision)

Transform
$$m_1\mathbf{v}_{1,i} + m_2\mathbf{v}_{2,i} = m_1\mathbf{v}_{1,f} + m_2\mathbf{v}_{2,f}$$
$$m_1 v^2_{1,i} + m_2 v^2_{2,i} = m_A v^2_{1,f} + m_2 v^2_{2,f}$$

– remove vectors in 1D –
two equations with two unknowns

as
$$m_1\left(v_{1,i} - v_{1,f}\right) = m_2\left(v_{2,f} - v_{2,i}\right) \quad (1)$$
$$m_1\left(v^2{}_{1,i} - v^2{}_{1,f}\right) = m_2\left(v^2{}_{2,f} - v^2{}_{1,i}\right)$$

– use
$a^2 - b^2 = (a+b)(a-b)$

and divide the second equation by the first, obtaining

$$v_{1,i} + v_{1,f} = v_{2,i} + v_{2,f} \quad (2)$$

Now we have a system of two linear equations, (1) and (2), with two unknowns. Solution:

$$\boxed{v_{1,f} = \frac{2m_2 v_{2,i} + (m_1 - m_2)v_{1,i}}{m_1 + m_2}, \qquad v_{2,f} = \frac{2m_1 v_{1,i} + (m_2 - m_1)v_{2,i}}{m_1 + m_2}}$$

Particular cases:

- For equal masses, $m_1 = m_2 = m$, one obtains

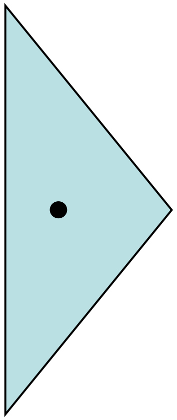
  $v_{1,f} = v_{2,i}, \qquad v_{2,f} = v_{1,i}$ (colliding bodies exchange velocities)

- One of the bodies is at rest (say, $v_{2,i} = 0$)

$$v_{1,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i}, \qquad v_{2,f} = \frac{2m_1}{m_1 + m_2} v_{1,i}$$

Direction depends on which mass is larger

Center of mass (CM) of an extended system



Definition of the position of the CM:

$$\mathbf{r}_{CM} \equiv \frac{1}{M}\sum_i m_i \mathbf{r}_i \,,$$

where $M \equiv \sum_i m_i$ is the total mass of the system.

For the bodies of symmetric shape (spheres, cubes, etc.) CM is in the center.

Displacement of the CM:

$$\Delta\mathbf{r}_{CM} = \frac{1}{M}\sum_i m_i \Delta\mathbf{r}_i \,.$$

Dividing it by Δt, one obtains the CM's velocity:

$$\mathbf{v}_{CM} \equiv \frac{1}{M}\sum_i m_i \mathbf{v}_i = \frac{\mathbf{P}}{M}.$$

For an isolated system the total momentum is conserved, $\mathbf{P} = const$, so that $\mathbf{v}_{CM} = const$.
If $\mathbf{P} = 0$, the CM is not moving, thus $\Delta\mathbf{r}_{CM} = 0$.

Acceleration of the CM:

$$\mathbf{a}_{CM} \equiv \frac{1}{M}\sum_i m_i \mathbf{a}_i = \frac{1}{M}\sum_i \mathbf{F}_i = \frac{\mathbf{F}}{M}$$

- this is Newton's second law for the CM – the acceleration is determined by the total external force and the total mass.

Applications, exercizes

Problem

A golf club exerts an average force of 500 N on a 0.1 kg golf ball, but the club is in contact with the ball for only a hundredth of a second. a) What is the magnitude of the impuls delivered by the club? b) What is the velocity acquired by the golf ball?

Solution

Given: $F=500$ N, $\Delta t=0.01$ s, $m=0.1$ kg

To find: a) $F\Delta t$ - ? b) v - ? (this is final velocity; initial velocity is zero: $v_1=0$, $v_2=v$)

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a) $F\Delta t = 500 \text{ N} \times 0.01 \text{ s} = 5 \text{ N s} = 5 \text{ kg m / s}$

b) $F\Delta t = m\Delta v = mv$; $v = F\Delta t/m = 5 \text{ N s} / 0.1 \text{ kg} = 50 \text{ m/s}$

Check units: $\text{N s / kg} = \text{kg m / s}^2 \times \text{s / kg} = \text{m/s}$, OK

Problem

Four railroad cars, all with the same mass of 20000 kg sit on a track. A fifth car of identical mass approaches them with a velocity of 15 m/s. This car collides with the other four cars. a) what is the initial momentum of the system? b) what is the velocity of the five coupled cars after the collision?

Solution

Given: m_1=4m, m_2=m, m=20000 kg, $v_{1,i}$=0, $v_{2,i}$=15 m/s

To find: a) P_i - ? b) $u = v_f$ - ?

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a) $P_i = m_2 v_{2,i}$ = 20000 kg 15 m/s = 300000 kg m / s

b) $m_1 v_{1,i} + m_2 v_{2,i} = (m_1 + m_2)u$, $\quad v_{1,i} = 0$

$$u = \frac{m_2 v_{2,i}}{m_1 + m_2} = \frac{m v_{2,i}}{5m} = \frac{1}{5} v_{2,i} = \frac{1}{5} 15 = 3\ m/s$$

Problem

Calculate the fraction of the energy lost in the railroad car collision of the problem above

Solution

$$E_{lost} \equiv -\Delta E = E_i - E_f = \frac{m_2 v^2{}_{2,i}}{2} - \frac{(m_1 + m_2)u^2}{2} = \frac{m v^2{}_{2,i}}{2} - \frac{5m\left(\frac{v_{2.i}}{5}\right)^2}{2}$$
$$= \frac{m_2 v^2{}_{2,i}}{2}\left(1 - \frac{1}{5}\right) = \frac{4}{5} E_i$$

Fraction of the energy lost:

$$\eta \equiv \frac{E_{lost}}{E_i} = \frac{4}{5}, \qquad \text{that is,} \qquad 80\%$$

Problem

Billiard ball 1 moving with a speed 3 m/s in the +x direction strikes an equal-mass ball 2 initially at rest. The two balls are observed to move off at 45° to the x axis, ball 1 above the x axis and ball 2 below. What are the speeds of the two ball after the collision?

Solution

Given: $m_1 = m_2 = m$, $v_{1,i} = 3$ m/s, $v_{2,i} = 0$, $\theta_{1,f} = 45°$, $\theta_{2\,f} = –45°$

To find: $v_{1,f}$ - ? $v_{2,f}$ - ? (magnitudes)

Momentum of the isolated system of two balls is conserved:

$$\mathbf{P} = \mathbf{P}_{1,i} + \mathbf{P}_{2,i} = \mathbf{P}_{1,f} + \mathbf{P}_{2,f}$$

Momentum conservation along the *y* axis:

$$0 = mv_{1,y,f} + mv_{2,y,f} = m\left(v_{1,f}\sin 45° + v_{2,f}\sin(-45°)\right) = m\frac{v_{1,f} - v_{2,f}}{\sqrt{2}}$$

$$\Rightarrow \quad v_{1,f} = v_{2,f}$$

Momentum conservation along the *x* axis:

$$P_x = mv_{1,i} = mv_{1,x,f} + mv_{2,x,f} = m\left(v_{1,f}\cos 45° + v_{2,f}\cos(-45°)\right)$$

$$= m\frac{v_{1,f} + v_{2,f}}{\sqrt{2}} = m\sqrt{2}v_{1,f} = mv_{1,i}$$

$$\Rightarrow \quad v_{1,f} = v_{2,f} = \frac{v_{1,i}}{\sqrt{2}} = 2.1\, m/s$$