3 – Motion with constant acceleration:

Linear and projectile motion

In the preceding Lecture we have considered motion with constant acceleration along the *x* axis:

$$\boxed{\begin{aligned} &v_x = v_{x0} + a_x t, \qquad a_x = \mathrm{const} \\ &x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2 \end{aligned}}$$

Note that x_0, v_{x0}, a_x can be positive and negative that leads to a variety of behaviors. Clearly, similar equations can be written for the motion along other axes:

$$v_y = v_{y0} + a_y t, \qquad a_y = \mathrm{const}$$
$$y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2$$

$$v_z = v_{z0} + a_z t, \qquad a_z = \mathrm{const}$$
$$z = z_0 + v_{z0} t + \frac{1}{2} a_z t^2$$

All these motions can happen at the same time independently from each other. We can write our equations in a more general form with <u>shifted</u> time

$$v_x = v_{x0} + a_x (t - t_0)$$
$$x = x_0 + v_{x0} (t - t_0) + \frac{1}{2} a_x (t - t_0)^2$$

etc. This means that at $t = t_0$ the velocity is/was v_{x0} and the *x* coordinate is/was x_0.

Below we will consider different problems as illustrations

Linear motion with constant acceleration

An example is a free fall of an object under the influence of gravity if the air resistance is neglected. We use the z axis directed <u>upward</u>, then the acceleration along the z axis is

$$a_z = -g, \qquad g = 9.8\, m/s^2$$

So that the equations of motion become

$$\boxed{v_z = v_{z0} - gt, \qquad z = z_0 + v_{z0}t - \frac{1}{2}gt^2}$$

Do not use any other form!

Problem: A stone is dropped from the level (a) z_0 = h = 10 m, (b) z_0 = 100 m above the ground with zero initial velocity. What is the fall time?

<u>Solution</u>: Use the instance of the general formula for the coordinate z corresponding to the stone hitting the ground.

$$z = z_0 - \frac{1}{2}gt_{\text{fall}}^2 = 0$$

and find

$$\boxed{t_{\text{fall}} = \sqrt{\frac{2z_0}{g}}}$$

Analytical result

(Frame your final analytical result!)

Now plug numbers into your final result:

(Symbolic answers are superior as they can be used with any input numbers)

$$(a): \quad t_{\text{fall}} = \sqrt{\frac{2z_0}{g}} = \sqrt{\frac{2\times 10}{9.8}} = 1.42\,\text{s}$$

$$(b): \quad t_{\text{fall}} = \sqrt{\frac{2\times 100}{9.8}} = 4.52\,\text{s}$$

Numerical results

Problem:

A stone is thrown upward from the ground ($z_0 = 0$) with the initial velocity (a) $v_{z0} = 5$ m/s; (b) $v_{z0} = 10$ m/s. What is the maximal height that the stone reaches? What time does it take for the stone to fall on the ground?

<u>Solution</u>: The maximal height corresponds to the zero velocity. Thus we use

$$v_z = v_{z0} - g t_{\text{z-max}} = 0$$

This yields

$$t_{\text{z-max}} = \frac{v_{z0}}{g}$$

Now the maximal height can be found as

$$z_{\max} = v_{z0} t_{z-\max} - \frac{1}{2} g t_{z-\max}^2$$

$$= v_{z0} \frac{v_{z0}}{g} - \frac{1}{2} g \left(\frac{v_{z0}}{g} \right)^2 = \boxed{\frac{v_{z0}^2}{2g} = z_{\max}}$$

(Frame your final result!)

Now plug numbers into the final result:

$$(a): \quad z_{\max} = \frac{v_{z0}^2}{2g} = \frac{5^2}{2 \times 9.8} = 1.28\,\text{m}$$

$$(b): \quad z_{\max} = \frac{v_{z0}^2}{2g} = \frac{10^2}{2 \times 9.8} = 5.1\,\text{m}$$

To find the fall time, use

$$0 = z = v_{z0} t_{\text{fall}} - \frac{1}{2} g t_{\text{fall}}^2 = t_{\text{fall}} \left(v_{z0} - \frac{1}{2} g t_{\text{fall}} \right)$$

The two solutions of this equation are t_{fall}=0 and

$$\boxed{t_{\text{fall}} = \frac{2 v_{z0}}{g}} = 2 t_{z-\max}$$

of which the first is trivial and second is relevant.

Plug numbers…

Problem:

Two stones are dropped from the same point with zero initial velocity, one at t = 0 and another at t = t_0 > 0. How the distance between the two stones depends on time? What do you expect before you start solving the problem?

Solution: Use subscript 1 for the first stone and 2 for the second stone. The hights of the two stones at time t are given by

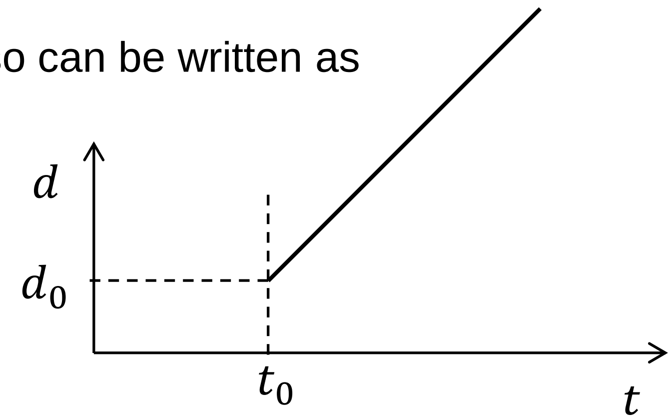
$$z_1 = z_0 - \frac{1}{2} g t^2, \qquad z_2 = z_0 - \frac{1}{2} g (t - t_0)^2 \quad (t > t_0)$$

The distance between the two stones is thus

$$d \equiv z_2 - z_1 = -\frac{1}{2} g (t - t_0)^2 + \frac{1}{2} g t^2 = \frac{1}{2} g [t^2 - (t - t_0)^2]$$
$$= \frac{1}{2} g [t^2 - t^2 + 2 t t_0 - t_0{}^2] = g t t_0 - \frac{1}{2} g t_0{}^2.$$

The distance linearly increases with time! This also can be written as

$$d = d_0 + g t_0 (t - t_0), \quad d_0 \equiv \frac{1}{2} g t_0{}^2$$



Moral: Always make a good start..

Projectile motion

Projectile motion is the motion of objects in the gravitational field of the earth if they do not go far away from the Earth's surface and the air resistance is neglected. This is the motion with constant acceleration directed downward and having the value *g*. There is no acceleration in the directions parallel to the Earth's surface, thus projections of the velocity on these directions remains constant.

Formulas for the projectile motion can be put into the form

$$v_z = v_{z0} - gt \qquad v_x = v_{x0}$$

$$z = z_0 + v_{z0}t - \frac{1}{2}gt^2 \qquad x = x_0 + v_{x0}t$$

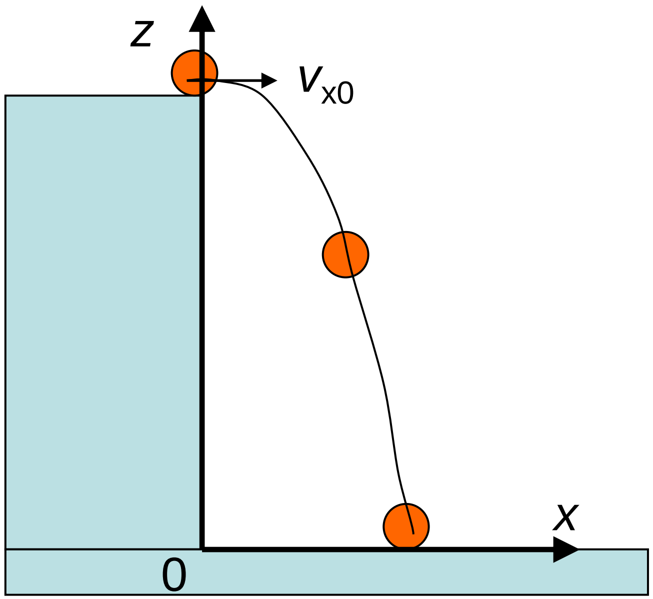
with the appropriate choice of the axes (so that $y = v_y = 0$). The trajectory is a parabola, if $v_{x0} \neq 0$. Let us show it in the particular case $x_0 = z_0 = 0$. In this case $t = x/v_{x0}$ and substituting into the formula for *z* yields

$$z = \frac{v_{z0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2$$

that is, *z* is a quadratic function of *x*.

Problem:

A ball rolls off a shelf with a horizontal velocity of 1 m/s. At what horizontal distance from the shelf does the ball land if the shelf is 2 m above the floor?



Solution: Use equations

$$z = z_0 - \frac{1}{2}gt^2, \qquad x = v_{x0}t$$

with $z_0 = h = 2$ m and $v_{x0} = 1$ m/s. From the first equation follows the fall time

$$t_{\text{fall}} = \sqrt{\frac{2z_0}{g}}$$

Plugging it into the second equation gives

$$\boxed{d = v_{x0}t_{\text{fall}} = v_{x0}\sqrt{\frac{2z_0}{g}}}$$

This is our final analytical result. Now plug numbers to find numerical result:

$$d = v_{x0}\sqrt{\frac{2z_0}{g}} = 1\sqrt{\frac{2\times 2}{9.8}} = 0.64\,\text{m}$$

Problem:

A missile was launched from the surface level with the initial velocity v_0 at the angle θ with the horizon. What horizontal distance d will it travel before hitting the ground? For which value of θ is this distance maximal?

Solution: The components of the initial velocity $\mathbf{v}_0$ are given by $v_{x0} = v_0 \cos\theta$, $v_{z0} = v_0 \sin\theta$. The horizontal distance is given by

$$d = v_{0x}\, t_{\text{fall}},$$

where t_{fall} is the fall time, the time from the shot until reaching the target. Here it is important that $v_x = v_{0x}$ is a constant. The fall time can be found considering motion along z:

$$z = v_{0z}\, t - (1/2)gt^2 = t(v_{0z} - gt/2).$$

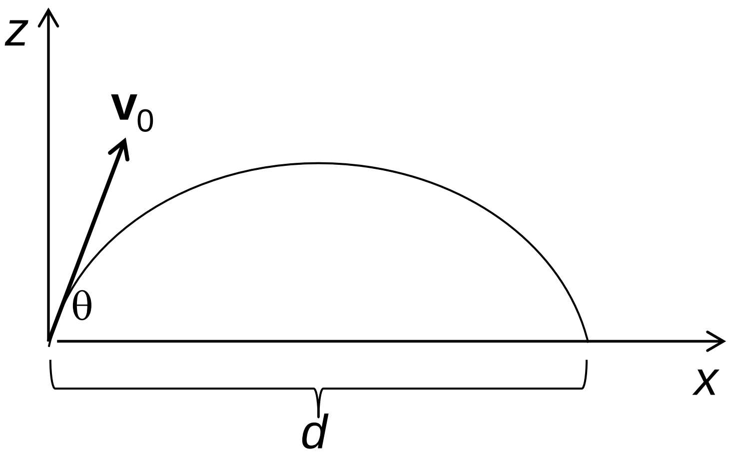
The condition $z = 0$ results in the fall time

$$t_{\text{fall}} = 2v_{0z}/g.$$

This yields

$$d = 2v_{0x}v_{0z}/g = 2v_0^2 \sin\theta\cos\theta/g = v_0^2 \sin(2\theta)/g.$$

The distance has a maximum at $\theta = 45^\circ$.



Relative velocity, addition of velocities

In some cases the body (for instance, a person) is moving with respect to a larger body or media (for instance a train or a river) that is, in turn, moving with respect to the main frame of reference such as the Earth. We call the main frame of reference *laboratory frame* and another frame (train, river) a *moving frame*. If

v' - velocity of the body with respect to the moving frame (relative velocity)
u - velocity of the moving frame with respect to the laboratory frame

Then the velocity of the body with respect to the laboratory frame (absolute velocity) is

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}$$

ihis simply means that velocities add up. The latter holds in the non-relativistic mechanics where all speeds are much smaller than the speed of light $c = 300000$ km/s $= 3\times10^8$ m/s.

Example: A person walks in a train moving at $u = 200$ km/h. The speed of the walker is $v' = 5$ km/h. This is the relative velocity (speed). The value of the absolute velocity is $v = \pm v' + u$, dependent on the direction of the walker, that is, 205 or 195 km/h.

Problem:

A swimmer swims with a speed 5 km/h across a river flowing with a speed of 3 km/h. At what angle with respect the straight crossing line should the swimmer swim to cross the river perpendicularly? What will be a speed with which the swimmer is crossing the river?

Mathematical formulation: v' = 5 km/h, u = 3 km/h; $\theta = ?$ v_y = ?

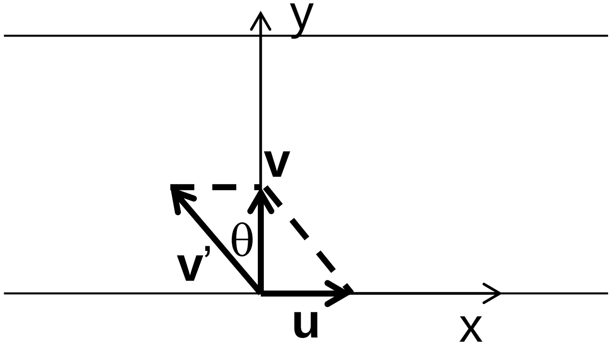
Solution: Use velocity-adding formula **v** = **v'** + **u.** Choose coordinate system as shown, in which $u_x = u$ and $u_y = 0$. **v** must be perpendicular to **u**, that is, $v_x = 0$.

Project onto x axis to find the angle:

$$v_x = v'_x + u_x \quad \Rightarrow \quad 0 = -v' \sin\theta + u \Rightarrow \sin\theta = \frac{u}{v'} \Rightarrow \theta = \arcsin\frac{u}{v'} = 36.7°$$

Project onto y axis to find v_y:

$$v_y = v'_y + u_y \quad \Rightarrow \quad v_y = v' \cos\theta = v'\sqrt{1 - \sin^2\theta}$$
$$= v'\sqrt{1 - (u/v')^2} = \sqrt{v'^2 - u^2} = 4 \text{ km/h}$$



Analyze solution: Solution exists only for v' > u, as expected.