1 – Vectors, coordinates, motion

Most of the physical quantities are either <u>scalars</u> or <u>vectors</u>. These mathematical objects are described in more details in the math summary „Refreshing..“

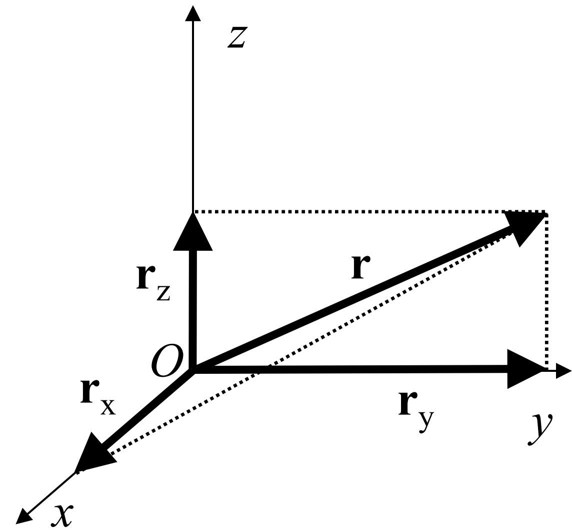
Scalars are simple numbers that, in general, can be positive and negative (Examples: Time *t*, distance *d*, mass *m*, Temperature *T*, etc).

Vectors can be imagined as arrows that have length and direction. (Examples: position in space **r**, velocity **v**, force **F**, etc.). Vectors are denoted by boldface characters (**F**) in printed texts and by overhead arrows ($\vec{F}$) in handwriting.

Magnitude (length) of a vector $\mathbf{A}$ is denoted as $|\mathbf{A}|$ or simply as A. Vectors of <u>unit length</u>, $|\mathbf{A}| = 1$, describe <u>directions</u> only. Each vector can be represented in the form $\mathbf{A} = A\mathbf{n}$, where $A > 0$ is the magnitude (length) of the vector $\mathbf{A}$ and $\mathbf{n}$ is a unit vector directed along $\mathbf{A}$.

We will also need an extended definition according to which $A < 0$ is allowed. In this case, the direction of $\mathbf{A}$ is opposite to that of $\mathbf{n}$. This will be used in the next slide.

Space and coordinate systems



To define position of a point object in space, one needs a coordinate system. In our three-dimensional world, a typical coordinate system consists of an origin *O* and three mutually perpendicular axes *x,y,z*.

Given the coordinate system, a position of an object is represented by a position vector **r** (or **R**) that goes from the origin *O* to the point in space, where the object is located (see Figure).

The position vector **r** can be represented as the sum of the three vectors, each directed along one of the coordinate axes:

$$\mathbf{r} = \mathbf{r}_x + \mathbf{r}_y + \mathbf{r}_z$$

This can be done by drawing perpendiculars to all three axes from the tip of the vector **r**, that is, by projecting vector **r** onto the three coordinate axes.

It is convenient to introduce unit vectors $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$, ($|\mathbf{e}_x| = 1$ etc.) that are directed along the three axes. Then one can write $\mathbf{r}_x = r_x\mathbf{e}_x$, etc., where the scalars r_x, r_y, r_z, are called coordinates of our point object. Coordinates can be positive and negative, depending on the orientations of $\mathbf{r}_x$ and $\mathbf{e}_x$ (collinear or opposite, see the preceding slide). Then one can represent the vector **r** through its coordinates as

$$\mathbf{r} = r_x\mathbf{e}_x + r_y\mathbf{e}_y + r_z\mathbf{e}_z$$

While the three **e**-vectors belong to the coordinate system, the three coordinates (r_x, r_y, r_z) completely specify vector **r** in three-dimensional space, so that one can use the short notation

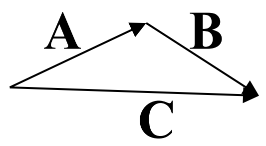
$$\mathbf{r} = (r_x, r_y, r_z)$$

Operations on vectors

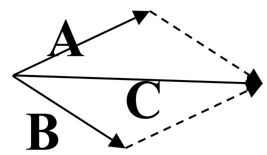
Vectors can be added and subtracted. Vectors can be multiplicated and divided by scalars. There are two kinds of multiplication of two vectors that will be considered later. Division by a vector does not exist.

Addition of vectors can be done graphically with the help of either the tail-to-tip rule or the parallelogramm rule

$$\mathbf{A}+\mathbf{B}=\mathbf{C}$$



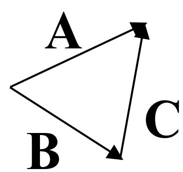
(tail-to-tip)



(parallelogramm)

Subtraction of vectors:

$$\mathbf{A}-\mathbf{B}=\mathbf{C}$$



because $\mathbf{A}=\mathbf{B}+\mathbf{C}$

Geometrical definition above is inconvenient practically. It is much easier to represent vectors in numerical form via their components and make operations on them numerically. Addition of vectors can be done as

$$\mathbf{A}+\mathbf{B}=A_x\mathbf{e}_x+A_y\mathbf{e}_y+A_z\mathbf{e}_z+B_x\mathbf{e}_x+B_y\mathbf{e}_y+B_z\mathbf{e}_z=(A_x+B_x)\mathbf{e}_x+(A_y+B_y)\mathbf{e}_y+(A_z+B_z)\mathbf{e}_z$$

$$\mathbf{A}+\mathbf{B}=\mathbf{C}=C_x\mathbf{e}_x+C_y\mathbf{e}_y+C_z\mathbf{e}_z \quad \Rightarrow \quad \boxed{C_x=A_x+B_x, \quad C_y=A_y+B_y, \quad C_z=A_z+B_z}$$

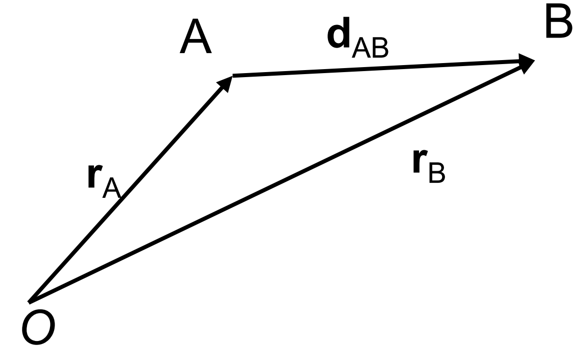
In the short notation to be used practically,

$$\mathbf{A}+\mathbf{B}=\left(A_x,A_y,A_z\right)+\left(B_x,B_y,B_z\right)=(A_x+B_x,\ A_y+B_y,\ A_z+B_z)$$

That is, to add vectors, one has just to add their components. In the case of subtraction, one just has to subtract components from each other. Multiplication and division of a vector by a scalar is also done on components. Geometrically it changes vector length and (for the negative scalar) direction.

Distance and displacement

If a point object moves from A (initial state) to B (final state) that are specified by position vectors $\mathbf{r}_A$ and $\mathbf{r}_B$, then displacement $\mathbf{d}_{AB}$ is defined as a vector that goes from A to B:



According to the rules of addition and subtraction of vectors,

$$\mathbf{d}_{AB} = \mathbf{r}_B - \mathbf{r}_A$$

so that

$$\mathbf{r}_A + \mathbf{d}_{AB} = \mathbf{r}_B$$

The distance d_{AB} between A and B is a non-negative scalar that is equal to the length of the displacement:

$$d_{AB} = |\mathbf{d}_{AB}| = |\mathbf{r}_B - \mathbf{r}_A|$$

The length of any vector **A** is given by the Pythagoras theorem as

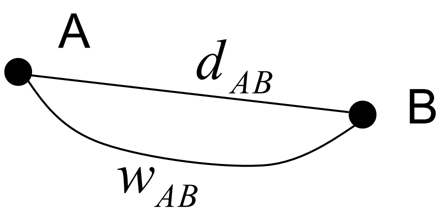
$$|\mathbf{A}| \equiv A = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

so that in our case

$$|\mathbf{d}_{AB}| \equiv d_{AB} = \sqrt{(r_{A,x} - r_{B,x})^2 + (r_{A,y} - r_{B,y})^2 + (r_{A,z} - r_{B,z})^2}$$

The way from A to B is not necessarily straight. The length of this way w_{AB} is usually greater than the distance between A and B:

$$w_{AB} \geq d_{AB}$$



Average velocity and average speed

If an object travels from A to B within time interval Δt, one can define the average velocity of this object. It is convenient to write the displacement $\mathbf{d}_{AB}$ as $\Delta\mathbf{r}$, then the average velocity is defined as the vector

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t}$$

The absolute value of the average velocity, v, is a scalar, the distance covered divided by the time elapsed:

$$v = |\mathbf{v}| = \frac{|\Delta\mathbf{r}|}{\Delta t} = \frac{d}{\Delta t}$$

The average speed *s* is defined as the ratio of the length of the way to the time elapsed:

$$s = \frac{w}{\Delta t}$$

Since the way cannot be shorter than the distance (see previous page) the average speed is not less than the absolute value of the average velocity:

$$s \geq v$$

Instantaneous velocity and speed

We know from our everyday life that the velocity and speed of a moving object can change. In particular, a moving car can accelerate and decelerate. It then makes sense to introduce the instantaneous velocity and speed by splitting the trajectory up into many small pieces of vanishingly small lengths. The time required to travel these small pieces of the trajectory will be vanishingly small, too. The definitions of the instantaneous velocity and speed are

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t}, \qquad v = \lim_{\Delta t \to 0} \frac{d}{\Delta t}, \qquad s = \lim_{\Delta t \to 0} \frac{w}{\Delta t}$$

where $\lim_{\Delta t \to 0}$ means limiting transition to vanishingly small Δt.

Note that a very small piece of a trajectory is practically straight, thus $d = w$ and, as a result, the instantaneous speed is equal to the absolute value of the instantaneous velocity, $s = v$. In the following, we will be using v rather than s for the instantaneous speed.

If position vector is defined as a function of time, $\mathbf{r} = \mathbf{r}(t)$, then instantaneous velocity $\mathbf{v}(t)$ is just time derivative of $\mathbf{r}(t)$. Calculating derivatives of functions is a subject of calculus and belongs to a calculus-based course of physics (PHY168).

Acceleration

If the velocity of an object changes by $\Delta\mathbf{v}$ during the time Δt, one can define the average acceleration by

$$\mathbf{a} = \frac{\Delta\mathbf{v}}{\Delta t}$$

The instantaneous acceleration is defined as the limit $\Delta t \to 0$ of the average acceleration:

$$\mathbf{a} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{v}}{\Delta t}$$

Note that acceleration is a vector. Acceleration is due to both the change of the absolute value of the velocity and the change of its direction. For instance, if a car is travelling a curved trajectory with a constant speed, its velocity changes its direction. Thus there is an acceleration in spite of the speed being constant.

Centripetal acceleration

Acceleration of an object moving along a circular trajectory with a constant speed is directed to the center of the circle. The value of this so-called <u>centripetal</u> acceleration is given by

$$a_c = \frac{v^2}{R}$$

where *R* is the radius of the circle and *v* is absolute value of the instantaneous velocity, that is, of the speed. As said above, the centripetal acceleration is directed toward the center of the circle