Introductory Physics: Problems solving

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Introduction

Solving problems is an inherent part of the physics course that requires a more active approach than just reading the theory or listening to lectures. Making only the latter, the student can have an illusion of having understood the material but it is not the case until s/he becomes able to apply one's knowledge to solving problems that is, working actively with the material.

The main purpose of our Introductory Physics course, for the majority of our students, is to acquire a conceptual understanding of physics, to develop a scientific way of thinking. The latter means relying on the scientific definitions, simple logic, and the common sense, opposed to making wild assumptions at every step that leads to wrong results and loss of points.

PHY166 and PHY167 courses are algebra based, while PHY168 and PHY169 are calculus based. Both types of courses require that problems are solved algebraically and an algebraic result, that is, a formula is obtained. Only after that the numbers are plugged in the resulting formula and the numerical result is obtained. One should understand that physics is mainly about formulas, not about numbers, thus the main result of problem solving is the algebraic result, while the numerical result is secondary.

Unfortunately, most of the students taking part in our physics courses reject algebra and try to work out the solution numerically from the very beginning. Probably, bad teachers at the high school taught the students that problem solving consists in finding the "right" formula and plugging the numbers into it. This is fundamentally wrong.

There are several arguments for why the algebraic approach to problem solving is better than the numeric approach.

1. Algebraic manipulations leading to the solution are no more difficult than the corresponding operations with numbers. In fact, they are easier as a single symbol, such as a, stands for a number that usually requires much more efforts to write without mistakes.
2. Numerical calculations are for computers, while algebraic calculations are for humans. Computers do not understand what they are computing, and they are proceeding blindly along prescribed routes. The same does a human trying to operate with numbers. However, the human forgets what do these numbers stand for and loses the clue very soon. If a human operates with algebraic symbols, s/he is not losing the clue as the symbols speak for themselves. For instance, a usually is an acceleration or a distance, m usually is a mass, etc.
3. The value of a formula is much higher than that of the numerical answer because the formula can be used with another set of input values while the numerical result cannot. In all more or less intelligent devices formulas are implemented that work as "black boxed": one supplies the input values and collects the output values.
4. Formulas allow analysis of their dependence on the input values or parameters. This is important for understanding the formula and for checking its validity on simple

particular cases in which one can obtain the result in a simpler way. This is impossible to do with numerical answers. Actually, one can hardly understand them.

Probably, the reasons given above are sufficient to abandon attempts to ignore the algebraic approach, especially as the absence of the algebraic result does not give a full score, even if the numerical answer is correct.

In this collection, the reader will find some exemplary solutions of Introductory Physics problems that show the efficient methods and approaches. It is recommended to read my collection of math used in our course, “REFRESHING High-School Mathematics”.

This collection of physics problems solutions does not intend to cover the whole Introductory Physics course. Its purpose is to show the right way to solve physics problems.

Here some useful tips.

1. Always try to find out what a problem is about, which part of the physics course is in question
2. Drawings are very helpful in most cases. They help to understand the problem and its solution
3. Write down basic formulas that will be used in the solution
4. Write comments in a good scientific language. It will make the solution more readable and will help you to understand it. Solution that consists only of formulas and numbers is not good.
5. Frame your resulting formulas. This shows to the grader that you really understand where your results are.

Physics part I

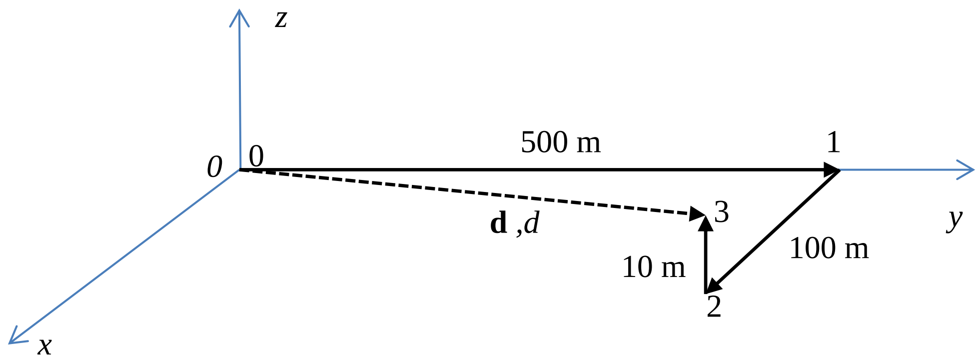
Kinematics

Vectors, coordinates, displacement, distance, velocity, speed, acceleration, projectile motion, etc.

1. Professor's way to work

A professor going to work first walks 500 m along the campus wall, then enters the campus and goes 100 m perpendicularly to the wall towards his building, after that takes an elevator and mounts 10 m up to his office. The trip takes 10 minutes.

Calculate the displacement, the distance between the initial and final points, the average velocity and the average speed.



Solution: The total trajectory can be represented by three vectors going from 0 to 1, then from 1 to 2, then from 2 to 3. The displacement is the vector sum of the three displacement vectors:

$$\mathbf{d} = \boldsymbol{r}_{01} + \boldsymbol{r}_{12} + \boldsymbol{r}_{23.}$$

It is convenient to choose the coordinate axes xyz that coincide with these three mutually orthogonal vectors, as shown in the figure. Then, using, for any vector

$$\mathbf{r} = \left(r_x, r_y, r_z\right),$$

one writes

$$\boldsymbol{r}_{01} = (0{,}500{,}0)\text{ m}, \quad \boldsymbol{r}_{12} = (100{,}0{,}0)\text{ m}, \quad \boldsymbol{r}_{23} = (0{,}0{,}10)\text{ m}.$$

The addition of these vectors is performed as follows:

$$\mathbf{d} = (0 + 100 + 0,\, 500 + 0 + 0,\, 0 + 0 + 10) = (100{,}500{,}10)\text{ m}.$$

The distance d between the initial and final points is the magnitude of the displacement $\mathbf{d}$:

$$d = |\mathbf{d}| = \sqrt{d_x^2 + d_y^2 + d_z^2} = \sqrt{100^2 + 500^2 + 10^2}$$

$$= \sqrt{10000 + 250000 + 100} = \sqrt{260100} = 510\text{ m}.$$

The trajectory length (the way length) is given by

$$w = r_{01} + r_{12} + r_{23} = 500 + 100 + 10 = 610 \text{ m}$$

and it is longer than the distance. Now, the average velocity is

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{d}}{\Delta t} = \frac{(100{,}500{,}10)}{10 \times 60} = (0.167, 0.833, 0.017) \text{ m/s}.$$

The magnitude of the average velocity is

$$v = |\mathbf{v}| = \frac{d}{\Delta t} = \frac{510}{10 \times 60} = 0.85 \text{ m/s}.$$

The average speed is

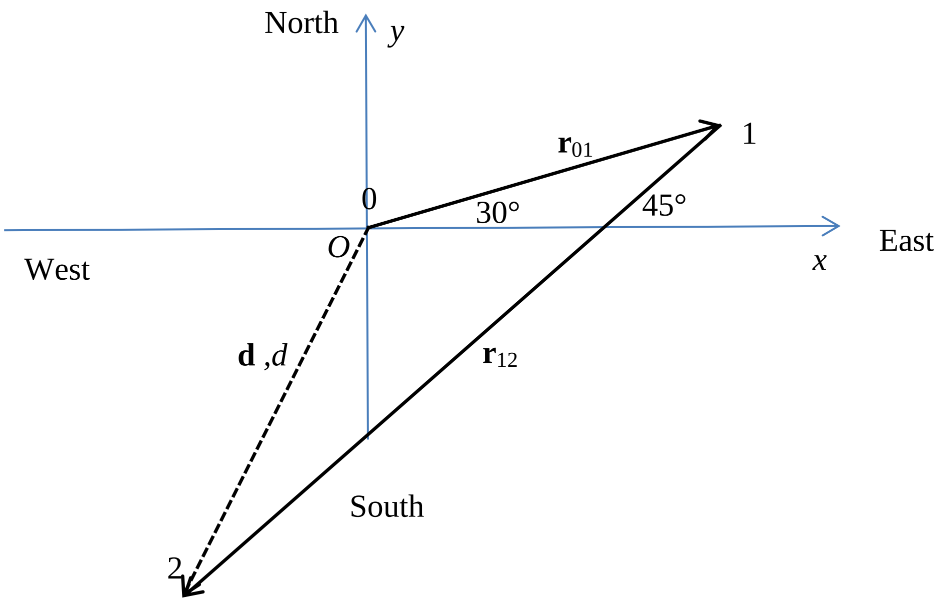
$$s = \frac{w}{\Delta t} = \frac{610}{10 \times 60} = 1.02 \text{ m/s}.$$

One can see that $s \geq v$, as it should be.

2. A 2D walker

A walker goes 1000 m the direction 30 degrees North of East, then 2000 m in the South-West direction. The trip takes 30 minutes.

Find the displacement, way length, average velocity and average speed.



Solution: The displacement is given by

$$\mathbf{d} = \boldsymbol{r}_{01} + \boldsymbol{r}_{12},$$

where

$$\boldsymbol{r}_{01} = \left(r_{01,x}, r_{01,y}\right) = (r_{01}\cos 30°, r_{01}\sin 30°) = \left(1000\frac{\sqrt{3}}{2},\ 1000\frac{1}{2}\right) = \left(500\sqrt{3},\ 500\right) \text{ m}$$

and

$$\boldsymbol{r}_{12} = (r_{12,x}, r_{12,y}) = (-r_{12}\cos 45°, -r_{12}\sin 45°) = \left(-2000\frac{\sqrt{2}}{2}, -2000\frac{\sqrt{2}}{2}\right)$$
$$= (-1000\sqrt{2}, -1000\sqrt{2},)\ \mathrm{m}.$$

Better is to write

$$\boldsymbol{r}_{12} = (r_{12,x}, r_{12,y}) = (r_{12}\cos 125°, r_{12}\sin 125°) = \left(2000\left(-\frac{\sqrt{2}}{2}\right), 2000\left(-\frac{\sqrt{2}}{2}\right)\right)$$
$$= (-1000\sqrt{2}, -1000\sqrt{2},)\ \mathrm{m}$$

that gives the same result. Now,

$$\mathbf{d} = (r_{01,x} + r_{12,x}, r_{01,y} + r_{12,y}) = (500\sqrt{3} - 1000\sqrt{2},\ 500 - 1000\sqrt{2})$$
$$\approx (-548.2, -914.2)\ \mathrm{m}$$

The distance is given by

$$d = |\mathbf{d}| = \sqrt{d_x^2 + d_y^2} = \sqrt{(-548.2)^2 + (-914.2)^2} \approx 1066\ \mathrm{m}$$

The length of the trajectory is

$$w = r_{01} + r_{12} = 1000 + 2000 = 3000\ \mathrm{m}.$$

The velocity:

$$\mathbf{v} = \frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{d}}{\Delta t} = \frac{(-548.2, -914.2)}{30 \times 60} = \left(\frac{-548.2}{30 \times 60}, \frac{-914.2}{30 \times 60}\right) = (\ldots, \ldots)\ \mathrm{m/s}.$$

The magnitude of the average velocity:

$$v = |\mathbf{v}| = \frac{d}{\Delta t} = \frac{1066}{30 \times 60} = 0.59\ \mathrm{m/s}.$$

The average speed:

$$s = \frac{w}{\Delta t} = \frac{3000}{30 \times 60} = 1.67\ \mathrm{m/s} > v.$$

3. Motion with constant acceleration

A car started moving from rest with a constant acceleration. At some moment of time, it covered the distance x and reached the speed v. Find the acceleration and the time.

Solution. The formulas for the motion with constant acceleration read

$$v = at, \quad x = \frac{1}{2}at^2,$$

where we have taken into account that the motion starts from rest (all initial values are zero). If v and x are given, this is a system of two equations with the unknowns a and t. This system of equations can be solved in different ways.

First method. For instance, one can express the time from the first equation, $t = v/a$, and substitute it to the second equation,

$$x = \frac{1}{2}a\left(\frac{v}{a}\right)^2 = \frac{v^2}{2a}.$$

From this single equation for a one finds

$$a = \frac{v^2}{2x}.$$

Now, one finds the time as

$$t = \frac{v}{a} = \frac{v}{v^2/(2x)} = \frac{2x}{v}.$$

Second method. Also, one can relate x to v as follows

$$x = \frac{1}{2}at \times t = \frac{1}{2}vt.$$

After that one finds

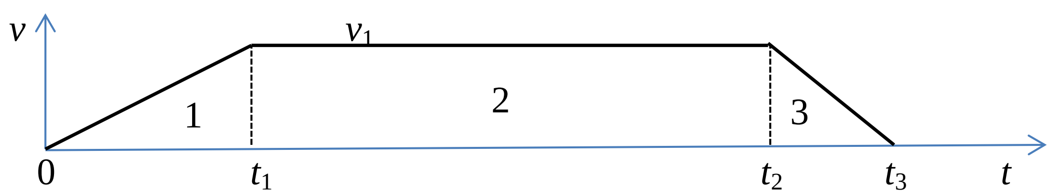
$$t = \frac{2x}{v},$$

and, further,

$$a = \frac{v}{t} = \frac{v}{2x/v} = \frac{v^2}{2x}.$$

4. A car trip (1D motion)

A car starts from the place with an acceleration $2\ \mathrm{m}/s^2$ and is accelerating during 10 seconds, then travels with the same speed for 30 seconds, then decelerates at the rate $3\ \mathrm{m}/s^2$ until stopping. Show the graph $v(t)$. Calculate the total time of the trip and the distance covered in each interval and the total distance covered by two methods: 1) Calculation of the area under the line $v(t)$; 2) Using the formula for the distance in the motion with constant acceleration.



Solution: First, we introduce missing notations: $a_1 = 2\ m/s^2$, $t_1 = 10\ s$, $\Delta t_2 \equiv t_2 - t_1 = 30\ s$, $a_3 = -3\ m/s^2$. The time dependence of the car's velocity is shown in the figure. In the interval 1 the car accelerates according to the formula

$$\text{Interval 1: } v = v_0 + a_1 t = a_1 t,$$

where we take into account that the initial velocity is zero: $v_0 = 0$. At the end of the first time interval, $t = t_1$, the velocity reaches the value

$$v_1 = a_1 t_1.$$

This expression is an instance of the formula above.

The velocity remains the same in the second interval of motion:

$$\text{Interval 2: } v = v_1.$$

The time at the end of the second interval is

$$t_2 = t_1 + \Delta t_2 = 10 + 30 = 40\ s.$$

In the third interval, the car decelerates according to

$$\text{Interval 3: } v = v_1 + a_3(t - t_2)$$

(this is the velocity formula with shifted time as the motion starts at $t = t_2$ rather than at $t = 0$). At the end of the motion the car stops that is described by the instance of the formula above with $v = 0$, that is,

$$0 = v_1 + a_3(t_3 - t_2)$$

that defines t_3. One obtains

$$\Delta t_3 \equiv t_3 - t_2 = -\frac{v_1}{a_3} = -\frac{a_1 t_1}{a_3} = -\frac{a_1}{a_3} t_1$$

and, further,

$$t_3 = t_2 + \Delta t_3 = t_1 + \Delta t_2 + \Delta t_3 = t_1 + \Delta t_2 - \frac{a_1}{a_3} t_1.$$

This is the analytical or symbolic or algebraic answer or formula for the total time. (This result will not be used, however). In this formula, the result is expressed through the quantities given in the formulation of the problem (this has to be checked each time before submitting the solution for grading!). Now, substituting given numbers, one obtains

$$t_3 = 10 + 30 - \frac{2}{-3} 10 = 10 + 30 + \frac{20}{3} = 46.7\ s.$$

The preparatory work done, let us now find the total distance covered. Using the first method, we find it as the area under the curve $v(t)$ that consists of two triangles and one rectangle, see the figure. The parameters of them have been calculated above. So we write

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 = \frac{1}{2} t_1 v_1 + \Delta t_2 v_1 + \frac{1}{2} \Delta t_3 v_1.$$

Here we must substitute the expressions for the quantities that are not given in the problem formulation: v_1 and Δt_3. We prefer not to factor v_1 to keep the contributions of each interval separately. The result reads

$$\Delta x = \frac{1}{2}a_1t_1^2 + \Delta t_2 a_1 t_1 + \frac{1}{2}\left(-\frac{a_1}{a_3}t_1\right)a_1t_1$$

or, finally,

$$\Delta x = \frac{1}{2}a_1t_1^2 + \Delta t_2 a_1 t_1 - \frac{1}{2}\frac{a_1^2}{a_3}t_1^2.$$

This is our symbolic answer for the distances covered in the motion.

Substituting the numerical values from the problem's formulation, one obtains

$$\Delta x = \frac{1}{2}2 \times 10^2 + 30 \times 2 \times 10 - \frac{1}{2}\frac{2^2}{(-3)}10^2 = 100 + 600 + 66.7 = 766.7m.$$

Now, let us find the total distance covered using the formula for the displacement in the motion with a constant acceleration

$$\Delta x \equiv x - x_0 = v_0\Delta t + \frac{1}{2}a(\Delta t)^2$$

in the form appropriate to each of the motion intervals. One has

$$\Delta x = \Delta x_1 + \Delta x_2 + \Delta x_3 = \frac{1}{2}a_1t_1^2 + v_1\Delta t_2 + \left[v_1\Delta t_3 + \frac{1}{2}a_3(\Delta t_3)^2\right]$$

$$= \frac{1}{2}a_1t_1^2 + v_1\Delta t_2 + \left[v_1 + \frac{1}{2}a_3\Delta t_3\right]\Delta t_3.$$

Substituting here the expressions for v_1 and Δt_3, one obtains

$$\Delta x = \frac{1}{2}a_1t_1^2 + \Delta t_2 a_1 t_1 + \left[a_1t_1 + \frac{1}{2}a_3\left(-\frac{a_1}{a_3}t_1\right)\right]\left(-\frac{a_1}{a_3}t_1\right)$$

$$= \frac{1}{2}a_1t_1^2 + \Delta t_2 a_1 t_1 + \left[a_1t_1 - \frac{1}{2}a_1t_1\right]\left(-\frac{a_1}{a_3}t_1\right)$$

$$= \frac{1}{2}a_1t_1^2 + \Delta t_2 a_1 t_1 + \frac{1}{2}a_1t_1\left(-\frac{a_1}{a_3}t_1\right)$$

$$= \frac{1}{2}a_1t_1^2 + \Delta t_2 a_1 t_1 - \frac{1}{2}\frac{a_1^2}{a_3}t_1^2$$

that coincides with the result obtained by the first method (the red formula).

5. Rocket motion (1D)

A rocket starts vertically up and moves with the acceleration 20 m/s^2 during 20 seconds. Then it continues its motion ballistically. Find the maximal height reached and the corresponding time. Find the time of hitting the ground and the corresponding speed.

Solution. First, we introduce the notations: the duration of the first stage (powered motion) $t_1 = 20\ s$, the acceleration in the first stage $a = 20\ m/s^2$. The initial velocity is zero.

We choose the origin of time $t = 0$ and put the origin of z-axis (directed up) at zero, so that the initial conditions are $z_0 = 0$ and $v_0 = 0$. The formulas for the motion with a constant acceleration at the first stage are

$$v = at, \quad z = \frac{1}{2}at^2.$$

At the end of the first stage, $t = t_1$, the velocity and the height read

$$v_1 = at_1, \quad z_1 = \frac{1}{2}at_1^2.$$

These are the initial conditions for the motion on the second stage. The formulas for the motion with the constant acceleration $-g$ on the second (ballistic) stage are

$$v = v_1 - g(t - t_1) \quad z = z_1 + v_1(t - t_1) - \frac{1}{2}g(t - t_1)^2.$$

Note that the second stage begins at $t = t_1$, thus we use the formulas with shifted time. The highest point can be found from the condition $v = 0$ that yields the equation for the time at which the maximal height is reached:

$$0 = v_1 - g(t_{max} - t_1).$$

The solution is

$$t_{max} - t_1 = \frac{v_1}{g} = \frac{at_1}{g} = \frac{a}{g}t_1$$

and, finally,

$$t_{max} = \frac{a}{g}t_1 + t_1 = \left(\frac{a}{g} + 1\right)t_1.$$

Substituting the numbers, one obtains

$$t_{max} = \left(\frac{20}{9.8} + 1\right)20 = 60.8\,s.$$

Now one can calculate the maximal height:

$$z_{max} = z_1 + v_1(t_{max} - t_1) - \frac{1}{2}g(t_{max} - t_1)^2.$$

Substituting here the quantities found above and simplifying, one obtains

$$\begin{aligned} z_{max} &= \frac{1}{2}at_1^2 + at_1\frac{a}{g}t_1 - \frac{1}{2}g\left(\frac{a}{g}t_1\right)^2 \\ &= \frac{1}{2}at_1^2 + \frac{a^2}{g}t_1^2 - \frac{1}{2}\frac{a^2}{g}t_1^2 \\ &= \frac{1}{2}at_1^2 + \frac{1}{2}\frac{a^2}{g}t_1^2 = \frac{1}{2}\left(1 + \frac{a}{g}\right)at_1^2. \end{aligned}$$

Substituting the numbers, one obtains

$$z_{max} = \frac{1}{2}\left(1 + \frac{20}{9.8}\right) 20 \times 20^2 = 12163\ m = 12.2\ km.$$

At the third stage, the rocket falls with the acceleration $-g$ from the height z_{max}. The formulas for its velocity and height at this stage are

$$v = -g(t - t_{max}), \qquad z = z_{max} - \frac{1}{2} g(t - t_{max})^2$$

(also formulas with shifted time). The final time (of hitting the ground) t_f is determined by $z = 0$. This gives the equation (an instance of the general formula)

$$0 = z_{max} - \frac{1}{2} g\left(t_f - t_{max}\right)^2.$$

From here one finds

$$t_f - t_{max} = \sqrt{\frac{2z_{max}}{g}} = \sqrt{\frac{2}{g}\frac{1}{2}\left(1 + \frac{a}{g}\right) a t_1^2} = \sqrt{\left(1 + \frac{a}{g}\right)\frac{a}{g}}\, t_1.$$

This will be needed to find the final velocity. For t_f itself one obtains

$$t_f = t_{max} + \sqrt{\left(1 + \frac{a}{g}\right)\frac{a}{g}}\, t_1 = \left(\frac{a}{g} + 1\right) t_1 + \sqrt{\left(1 + \frac{a}{g}\right)\frac{a}{g}}\, t_1.$$

Substituting the numbers, one obtains

$$t_f = \left(\frac{20}{9.8} + 1\right) 20 + \sqrt{\left(1 + \frac{20}{9.8}\right)\frac{20}{9.8}}\, 20 = 60.8 + 49.8 = 110.6\ s.$$

(Is it obvious that it takes a longer time for the rocket to reach the highest point than to fall back to the initial level? To understand this, sketch the function $z(t)$.)

Now, the final velocity can be found from the velocity formula:

$$v_f = -g\left(t_f - t_{max}\right) = -g\sqrt{\left(1 + \frac{a}{g}\right)\frac{a}{g}}\, t_1 = -\sqrt{\left(1 + \frac{a}{g}\right) a g}\, t_1 = -\sqrt{(a + g)a}\, t_1.$$

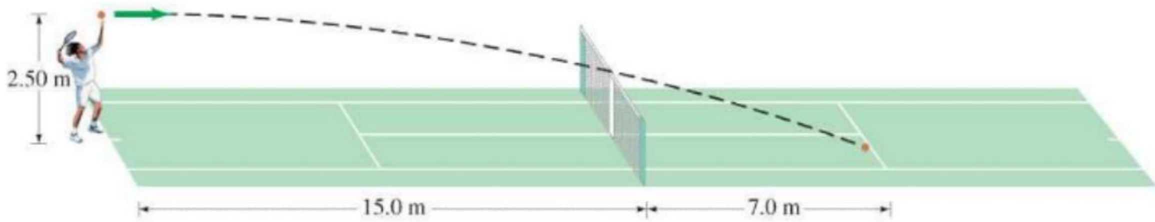
Substituting the numbers, one obtains

$$v_f = -\sqrt{(20 + 9.8)20} \times 20 = 488\ m/s.$$

6. Tennis serve (Giancoli Chapter 3)

64. At serve, a tennis player aims to hit the ball horizontally. What minimum speed is required for the ball to clear the 0.90-m-high net about 15.0 m from the server if the ball is "launched" from a height of 2.50 m? Where will the ball land if it just clears the net (and will it be "good" in the sense that it lands within 7.0 m of the net)? How long will it be in the air? See Fig. 3–45.

FIGURE 3–45 Problem 64.



Solution. First, we define the coordinate axes and introduce missing notations. The origin of the coordinate system is at the server's position, z-axis up and x-axis to the right. The initial height (serve height) $z_0 = 2.5\ m$, the height of the net $z_1 = 0.9\ m$, the height of the ground (the reference height) $0\ m$, distance server-net $x_1 = 15\ m$. Find v_{0x}.

First, use the x- and z-formulas to find v_{0x}:

$$x = v_{0x}t, \qquad z = z_0 - \frac{1}{2}gt^2.$$

The instance of these general formulas corresponding to the ball passing just above the net reads

$$x_1 = v_{0x}t_1, \qquad z_1 = z_0 - \frac{1}{2}gt_1^2.$$

This is a system of two equations with two unknowns: v_{0x} and t_1. The second equation is autonomous (contains only one unknown), so it can be solve to give

$$t_1 = \sqrt{\frac{2(z_0 - z_1)}{g}}.$$

Then, from the first equation one finds

$$v_{0x} = \frac{x_1}{t_1} = x_1\sqrt{\frac{g}{2(z_0 - z_1)}}.$$

Substituting the numbers into this formula, one obtains

$$v_{0x} = 15\sqrt{\frac{9.8}{2(2.5 - 0.9)}} = 26.3\frac{m}{s}.$$

Now we can find the distance from the server at which the ball lands. We use the instances of the general formulas above corresponding to the ball hitting the ground:

$$x_2 = v_{0x} t_2, \qquad 0 = z = z_0 - \frac{1}{2} g t_2^2.$$

One finds t_2 from the second equation:

$$t_2 = \sqrt{\frac{2z_0}{g}}.$$

From this formula, one can find the numerical value of t_2 that is the total time of the motion. Substituting the formula for t_2 into the first equation, one obtains

$$x_2 = v_{0x} t_2 = x_1 \sqrt{\frac{g}{2(z_0 - z_1)}} \sqrt{\frac{2z_0}{g}} = x_1 \sqrt{\frac{z_0}{z_0 - z_1}}.$$

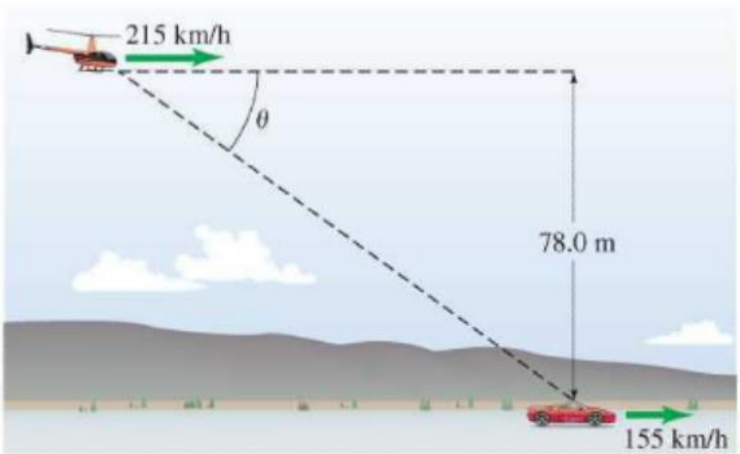
Substituting the numbers, one obtains

$$x_2 = 15 \sqrt{\frac{2.5}{2.5 - 0.9}} = 18.75\, m.$$

Now $x_2 - x_1 = 18.75 - 15 = 3.75\, m$ that is well below $7\, m$. Thus, the ball is "good".

7. Dropping a package from a copter into a moving car (Giancoli Chapter 3)

65. Spymaster Paul, flying a constant 215 km/h horizontally in a low-flying helicopter, wants to drop secret documents into his contact's open car which is traveling 155 km/h on a level highway 78.0 m below. At what angle (to the horizontal) should the car be in his sights when the packet is released (Fig. 3–46)?



Solution: First, we must introduce missing notations: the height of the copter $h = 78\, m$, the speed of the copter $v = 215 \frac{km}{h} = 215 \times \frac{1000}{3600} = 59.9 m/s$, the speed of the car $u = 155 \frac{km}{h} = \frac{155}{3.6} = 43.1 m/s$. Find the angle θ.

There are two solutions to this problem, in the laboratory frame and in the moving frame of the car.

Solution in the laboratory frame. Put the origin of the coordinate system on the ground below the copter. The initial x-coordinate of the car(when the package is dropped) is $x_{c,0}$. If it is found, then the angle θ can be expressed as

$$\tan\theta = \frac{h}{x_{c,0}}.$$

The formulas for the motion of the package and the car have the form

$$z_p = h - \frac{1}{2}gt^2, \qquad x_p = vt, \quad x_c = x_{c,0} + ut.$$

When the package lands into the car, the following conditions are fulfilled:

$$z_p = 0, \quad x_p = x_c.$$

Substituting these into the general equations, one obtains their instance

$$0 = h - \frac{1}{2}gt^2, \qquad vt = x_{c,0} + ut.$$

This is a system of two equations with two unknowns. The first equation is autonomous and yields the fall time

$$t = t_f = \sqrt{\frac{2h}{g}}.$$

Substituting this into the second equation, one obtains

$$x_{c,0} = (v-u)t_f = (v-u)\sqrt{\frac{2h}{g}}.$$

Now for the angle one obtains

$$\theta = \arctan\left(\frac{h}{v-u}\sqrt{\frac{g}{2h}}\right) = \arctan\left(\frac{1}{v-u}\sqrt{\frac{gh}{2}}\right).$$

Substituting the numbers, one obtains

$$\theta = \arctan\left(\frac{1}{(59.9-43.1)}\sqrt{\frac{9.8\times 78}{2}}\right) = \arctan(1.16) = 49°.$$

Solution in the moving frame (frame of the car). The absolute velocity of the copter can be represented as

$$v = v' + u,$$

where v' is the relative velocity of the copter with respect to the car, $v' = v - u$. The origin of the coordinate axes in the moving frame, O', is moving to the right with the velocity of the car u. At $t = 0$ the origins of the laboratory and moving frames coincide, $O' = O$. Thus the relation between the x-coordinate (absolute frame) and x'-coordinate (moving frame) is

$$x = x' + ut$$

or, conversely,

$$x' = x - ut$$

The formulas for the motion of the package in this frame have the form

$$z_p = h - \frac{1}{2}gt^2, \qquad x'_p = v't.$$

As for the car, it is at rest in its own frame:

$$x'_c(t) = x_{c,0}.$$

As in the first solution, one finds the fall time,

$$t_f = \sqrt{\frac{2h}{g}},$$

and substitutes it into the condition:

$$x'_p(t_f) = x'_c(t_f)$$

or

$$x'_p(t_f) = v't_f = (v-u)\sqrt{\frac{2h}{g}} = x'_c(t_f) = x_{c,0}.$$

The result for $x_{c,0}$ coincides with that obtained by the first method:

$$x_{c,0} = (v-u)\sqrt{\frac{2h}{g}}.$$

Then

$$\tan\theta = \frac{h}{x_{c,0}} = \frac{h}{v-u}\sqrt{\frac{g}{2h}} = \frac{1}{v-u}\sqrt{\frac{gh}{2}},$$

Wherefrom one finds θ.

8. Targeting angle (projectile motion)

A cannon launches missiles with the initial speed v_0. Find the targeting angles θ to hit the target at the distance d at the same height as the cannon.

Solution. The formulas for the projectile motion have the form

$$z = v_{0z}t - \frac{1}{2}gt^2, \qquad x = v_{0x}t.$$

The origin of the coordinate system is put at the location of the cannon, thus $x_0 = z_0 = 0$. The distance between the cannon and the landing point is defined by the fall time (or final time or flight time) t_f:

$$d = v_{0x} t_f.$$

The time t_f can be found from the first equation:

$$0 = v_{0z} t_f - \frac{1}{2} g t_f^2 = t_f \left(v_{0z} - \frac{1}{2} g t_f \right).$$

The first solution to this equation, $t_f = 0$, corresponds to the beginning of the motion and should be discarded. The landing time nullifies the expression in the brackets,

$$v_{0z} - \frac{1}{2} g t_f = 0,$$

wherefrom

$$t_f = \frac{2 v_{0z}}{g}.$$

Now

$$d = v_{0x} t_f = \frac{2 v_{0x} v_{0z}}{g}.$$

The components of the initial velocity can be expressed as

$$v_{0x} = v_0 \cos\theta, \qquad v_{0z} = v_0 \sin\theta,$$

so that

$$d = \frac{2 v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g},$$

where the trigonometric identity $\sin 2\theta \equiv 2 \sin\theta \cos\theta$ was used. As the maximal value of the sine function is 1 and it is reached for the argument equal to 90°, one can see that d reaches its maximum for $\theta = 45°$. One can rewrite

$$d = d_{max} \sin 2\theta, \qquad d_{max} = \frac{v_0^2}{g}.$$

This is an equation for θ if d has a prescribed value (the distance to the target). For the distance to the target $d > d_{max}$ the target cannot be hit. For $d < d_{max}$ the target can be hit in two different ways using two values of the targeting angle that satisfy

$$\sin 2\theta = \frac{d}{d_{max}}.$$

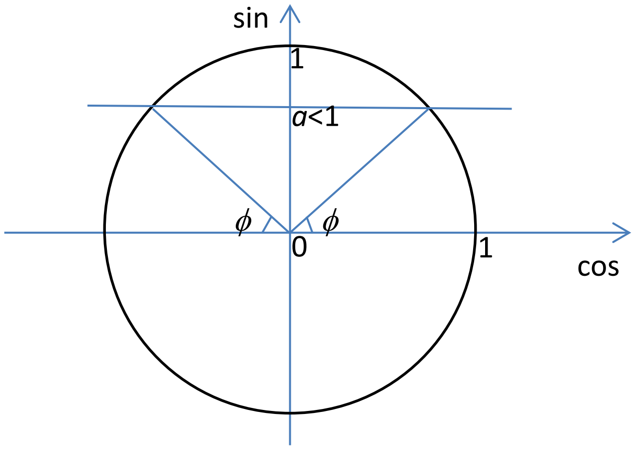
These solutions are

$$2\theta_1 = \arcsin\frac{d}{d_{max}} \quad \text{and} \quad 2\theta_2 = \pi - \arcsin\frac{d}{d_{max}},$$

that is,

$$\theta_1 = \frac{1}{2}\arcsin\frac{d}{d_{max}} \quad \text{and} \quad \theta_2 = \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{d}{d_{max}}.$$

The second solution is in radians, and π radians corresponds to 180°. For instance, for $\frac{d}{d_{max}} = \frac{1}{2}$ one has $\arcsin\frac{1}{2} = 30°$ and $\theta_1 = 15°$ and $\theta_2 = 90° - 15° = 75°$.



Equation: $\sin\phi = a$

Solutions:

$\phi_1 = \arcsin a$

$\phi_2 = \pi - \arcsin a$

One can check algebraically that the second expression is also a solution of the equation:

$$\sin\phi_2 = \sin(\pi - \arcsin a) = \sin(\arcsin a) = a.$$

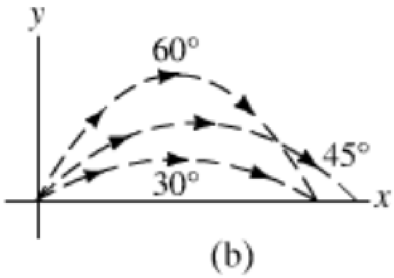
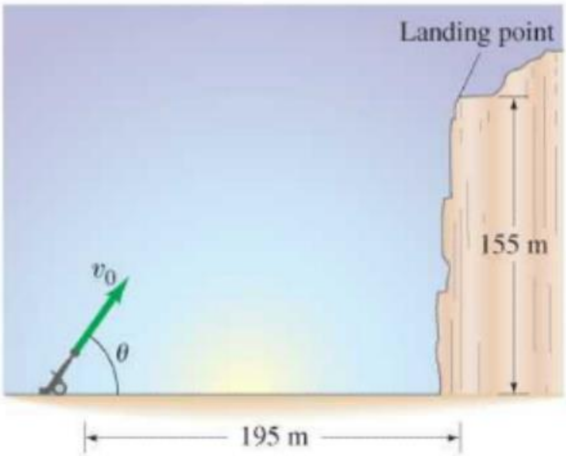


FIGURE 3–25 Example 3–8.
(a) The range R of a projectile;
(b) there are generally two angles θ_0 that will give the same range. Can you show that if one angle is θ_{01}, the other is $\theta_{02} = 90° - \theta_{01}$?

9. Hitting an elevated target (projectile motion, Giancoli, Chapter 3)

67. A projectile is launched from ground level to the top of a cliff which is 195 m away and 155 m high (see Fig. 3–47). If the projectile lands on top of the cliff 7.6 s after it is fired, find the initial velocity of the projectile (magnitude and direction). Neglect air resistance.



Solution. First, we introduce missing notations. The horizontal distance cannon-target $d = 195\ m$, the height of the target $h = 155\ m$, the missile flight time $t_f = 7.6\ s$. Find: v_0, θ.

The general formulas for the projectile motion have the form

$$z = v_{0z}t - \frac{1}{2}gt^2, \qquad x = v_{0x}t.$$

The origin of the coordinate system is put at the location of the cannon, thus $x_0 = z_0 = 0$.

The instance of these formulas, corresponding to the problem's formulation (hitting the target), is

$$h = v_{0z}t_f - \frac{1}{2}gt_f^2, \quad d = v_{0x}t_f.$$

From here, one finds the components of the initial velocity:

$$v_{0x} = \frac{d}{t_f}, \qquad v_{0z} = \frac{h + \frac{1}{2}gt_f^2}{t_f}.$$

Now

$$v_0 = \sqrt{v_{0x}^2 + v_{0z}^2} = \frac{\sqrt{d^2 + \left(h + \frac{1}{2}gt_f^2\right)^2}}{t_f}$$

and the angle θ is the solution of the equation

$$\tan\theta = \frac{v_{0z}}{v_{0x}}.$$

This equation has only one solution

$$\theta = \arctan\frac{v_{0z}}{v_{0x}} = \arctan\frac{h + \frac{1}{2}gt_f^2}{d}.$$

Substituting the numbers, one obtains…

10. Car jumping (Projectile motion, Giancoli, Chapter 3)

71. A stunt driver wants to make his car jump over eight cars parked side by side below a horizontal ramp (Fig. 3–49). (*a*) With what minimum speed must he drive off the horizontal ramp? The vertical height of the ramp is 1.5 m above the cars, and the horizontal distance he must clear is 20 m. (*b*) If the ramp is now tilted upward, so that "takeoff angle" is 10° above the horizontal, what is the new minimum speed?

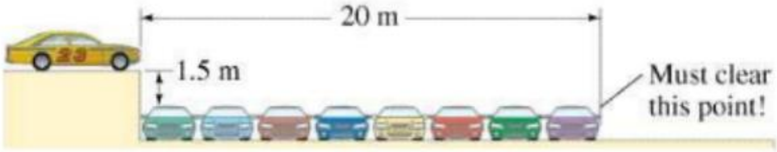


FIGURE 3–49 Problem 71.

Solution. First, we add missing notations. The horizontal distance $d = 20\ m$, the initial height $h = 1.5\ m$, the launching angle in (b) $\theta = 10°$.

We put the origin of the coordinate system at the foot of the "cliff" (below the end of the takeoff ramp at the level of the roofs of the standing cars). The formulas for the motion with a constant acceleration have the form

$$z = h + v_{0z}t - \frac{1}{2}gt^2, \qquad x = v_{0x}t.$$

When the jumping car clears the roof of the last standing car, one has (an instance of the formulas above)

$$0 = h + v_{0z}t_f - \frac{1}{2}gt_f^2, \qquad d = v_{0x}t_f. \qquad (1)$$

(a) In this case $v_{0z} = 0$ and $v_{0x} = v_0$. From the first equation one obtains

$$t_f = \sqrt{\frac{2h}{g}}.$$

Substituting this into the second equation, one obtains

$$v_0 = \frac{d}{t_f} = d\sqrt{\frac{g}{2h}}.$$

Substituting the numbers, one obtains

$$v_0 = 20\sqrt{\frac{9.8}{2 \times 1.5}} = 36\ m/s.$$

(b) In this case

$$v_{0x} = v_0 \cos\theta, \qquad v_{0z} = v_0 \sin\theta,$$

so that the general formulas at the clearing point, (1), take the form

$$0 = h + v_0 \sin\theta\, t_f - \frac{1}{2} g t_f^2, \qquad d = v_0 \cos\theta\, t_f.$$

This is again a system of two equations with two unknowns: v_0 and t_f. However, it is inconvenient to find t_f from the first equation, as above, because here one needs to sole a full quadratic equation. Thus we apply a slightly different method. Since we do not need t_f, we can eliminate if from the second equation [$t_f = d/(v_0 \cos\theta)$] and substitute it into the first equation that yields

$$0 = h + \frac{v_0 \sin\theta}{v_0 \cos\theta} d - \frac{1}{2} g \left(\frac{d}{v_0 \cos\theta}\right)^2.$$

After simplification, one obtains the equation for the car's speed

$$0 = h + d \tan\theta - \frac{g d^2}{2 \cos^2\theta} \frac{1}{v_0^2}$$

that is a quadratic equation without the linear term

$$(h + d \tan\theta) v_0^2 - \frac{g d^2}{2 \cos^2\theta} = 0.$$

Its solution reads

$$v_0 = \frac{d}{\cos\theta} \sqrt{\frac{g}{2(h + d \tan\theta)}}.$$

For $\theta = 0$, this formula simplifies to the solution obtained in (a). For small θ, one can use

$$\sin\theta \cong \theta, \qquad \tan\theta \cong \theta, \quad \cos\theta \cong 1 - \frac{1}{2}\theta^2 \cong 1,$$

(θ in radians) so that the value of v_0 decreases with θ because of the $\tan\theta$ term in the denominator. We have the angle, in radians,

$$\theta = 10^\circ \frac{2\pi}{360^\circ} = 0.175 \ll 1,$$

so that the angle is, indeed, small, and one can use the formulas for the small angles above.

Substituting the numbers, one obtains, approximately,

$$v_0 \cong 20 \sqrt{\frac{9.8}{2(1.5 + 20 \times 0.175)}} = 20 \sqrt{\frac{9.8}{2(1.5 + 3.5)}} = 19.80\ m/s.$$

Using the full expression yields

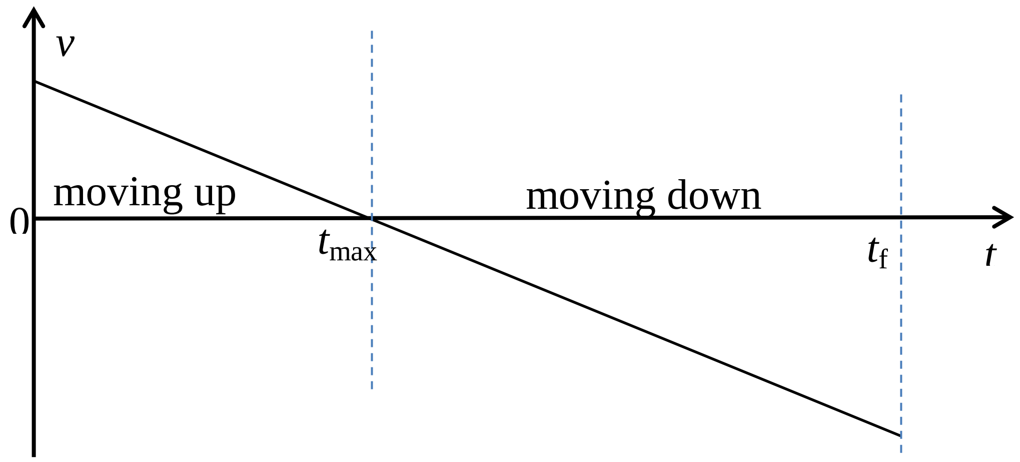
$$v_0 = \frac{20}{\cos 10^\circ} \sqrt{\frac{9.8}{2(1.5 + 20 \tan 10^\circ)}} = 20.05\ m/s.$$

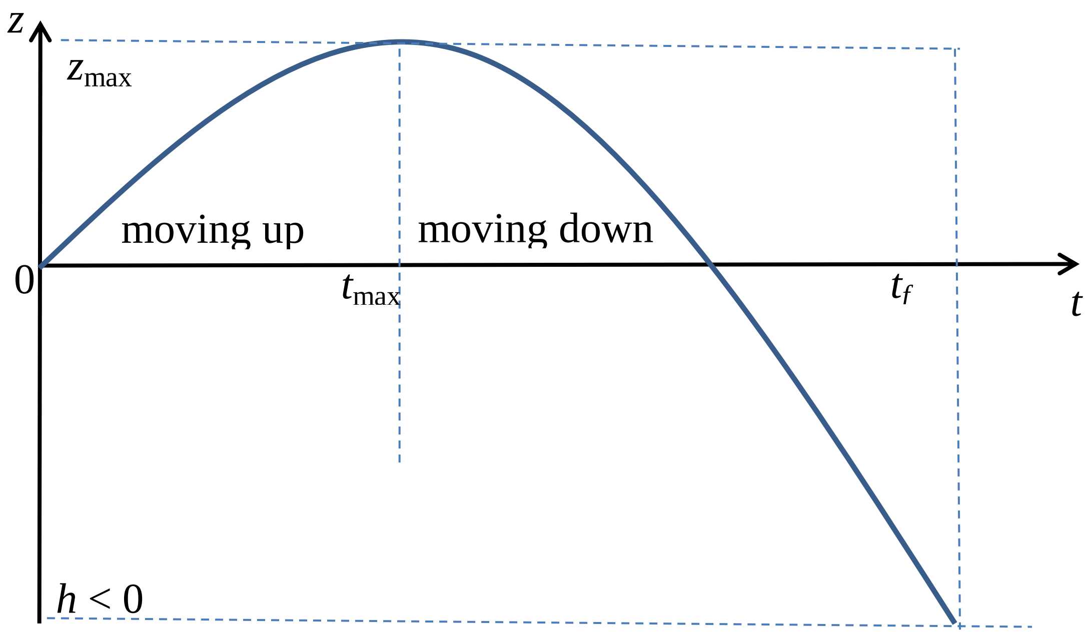
This is a serious decrease of the minimal speed in comparison to the case $\theta = 0$. The reason is that the small $\tan\theta$ is multiplied by the large d.

11. Vertical motion with gravity — full quadratic equation

A person standing on the edge of a cliff throws a rock straight upwards with an initial speed of 9 m/s. The cliff stands at a height of 105 meters from the bottom of the ravine. (a) Sketch a plot of velocity versus time and position versus time for the motion of the rock. (b) What will be the maximum height the rock reaches? (c) How long will it take to reach the ground? (d) How fast will it be traveling when it reaches the ground?

Solution: (a) Making a sketch is a guarantee of success in problem solving.





b) Let us introduce missing notations. Initial velocity $v_0 = 9\ m/s$, the height of the bottom of the ravine $h = -105\ m$. The reference level is the edge of the cliff.

The time dependences of the velocity and displacement in the motion with constant acceleration $a = -g$ are given by the formulas

$$v(t) = v_0 - gt; \qquad z(t) = v_0 t - \frac{1}{2} g t^2. \qquad (1)$$

Finding the maximum of $z(t)$ directly from the second formula requires using the calculus. However, one can use the physical argument and point out that when the height reaches its maximum, the vertical velocity must vanish. Thus, from the first equation one obtains

$$0 = v_0 - g t_{max} \quad \Longrightarrow \quad t_{max} = \frac{v_0}{g} = \frac{9}{9.8} = 0.92\ s.$$

After that, one finds the maximal height from the height formula substituting $t \Rightarrow t_{max}$, that is,

$$z_{max} \equiv z(t_{max}) = v_0 t_{max} - \frac{1}{2} g t_{max}^2 = v_0 \frac{v_0}{g} - \frac{1}{2} g \left(\frac{v_0}{g}\right)^2 = \frac{v_0^2}{2g}.$$

Substituting numbers, one obtains

$$z_{max} = \frac{9^2}{2 \times 9.8} = 4.1\ m$$

c) The time to reach the ground, that is, the fall time t_f, can be found from the height equation (1) substituting $z \Rightarrow h$:

$$h = v_0 t_f - \frac{1}{2} g t_f^2.$$

This is a quadratic equation that can be rewritten into the canonical form

$$g t_f^2 - 2 v_0 t_f + 2h = 0.$$

The two solutions of this equation are

$$t_f = \frac{1}{g}\left(v_0 \pm \sqrt{v_0^2 - 2gh}\right).$$

If $0 < h < z_{max}$, both solutions are positive and both make sense. The object thrown up crosses the level $z = h$ twice. The smaller t_f value (with minus) corresponds to crossing the level $z = h$ moving up. The larger t_f value (with plus) corresponds to crossing the level $z = h$ moving down. For $h < 0$ the object crosses the level of the negative height only once. The negative solution for the time should be discarded on physical grounds (negative times are not acceptable). Substituting the numbers, one obtains

$$t_f = \frac{1}{9.8}\left(9 + \sqrt{9^2 + 2 \times 9.8 \times 105}\right) = 5.64\ s.$$

d) The velocity at the end of the fall can be obtained from the velocity equation (1) as

$$v(t_f) = v_0 - g \frac{1}{g}\left(v_0 + \sqrt{v_0^2 - 2gh}\right) = -\sqrt{v_0^2 - 2gh}.$$

This velocity is negative as it is directed down. The value given by the square root can be found from the energy conservation law in a shorter way. Substituting the numbers, one obtains

$$v(t_f) = -\sqrt{9^2 + 2 \times 9.8 \times 105} = -46.2\frac{m}{s}.$$

12. Targeting angle for different heights (projectile motion)

A missile launched from a cannon with the initial speed v_0 targets an object at the linear distance d from the cannon and at the height h with respect to the cannon. Investigate the possibility of hitting the object and the launching angles.

Solution (?). The formulas for the motion of the missile have the form (motion with constant acceleration)

$$z = v_{0z}t - \frac{1}{2}gt^2, \qquad x = v_{0x}t.$$

The instance of these general formulas corresponding to hitting the target is

$$h = v_{0z}t_f - \frac{1}{2}gt_f^2, \qquad d = v_{0x}t_f.$$

From the first equation one finds t_f as in the preceding problem,

$$t_f = \frac{1}{g}\left(v_{0z} \pm \sqrt{v_{0z}^2 - 2gh}\right).$$

For $h > 0$, there are two positive-time solutions. The smaller time (with the – sign in the formula) corresponds to hitting the target while moving upward. The larger time (with the + sign in the formula) corresponds to hitting the target while moving downward. For $h < 0$, there is only the second solution. The solutions exists only for $v_{0z}^2 > gh$, otherwise, the missile cannot reach the required height. The time of the motion should satisfy the second equation above,

$$d = v_{0x}t_f = \frac{v_{0x}}{g}\left(v_{0z} \pm \sqrt{v_{0z}^2 - 2gh}\right).$$

Substituting

$$v_{0x} = v_0 \cos\theta, \qquad v_{0z} = v_0 \sin\theta,$$

one obtains the equation for the targeting angle θ

$$d = \frac{v_0 \cos\theta}{g}\left(v_0 \sin\theta \pm \sqrt{v_0^2 \sin^2\theta - 2gh}\right).$$

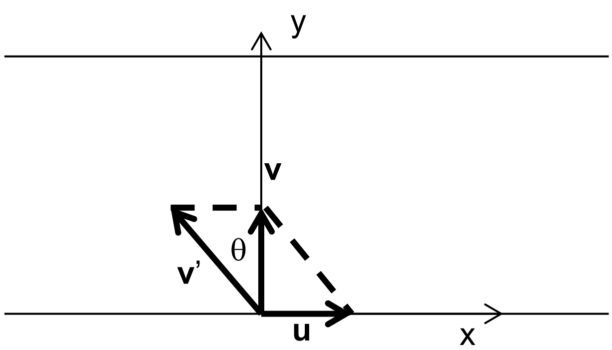
For $h = 0$, this equation simplifies and one obtains the well-known formula from which one finds θ. In this case, the missile can hit the target only moving downward, so, using the solution of the quadratic equation with the sign (+), one obtains the known result

$$d = \frac{v_0 \cos\theta}{g}(v_0 \sin\theta + v_0 \sin\theta) = \frac{2v_0^2 \sin\theta \cos\theta}{g} = \frac{v_0^2 \sin(2\theta)}{g}$$

In the general case, this is a complicated trigonometric equation for θ that does not have an analytical solution and has to be solved numerically.

13. Boat in the river (relative motion, Giancoli, Chapter 3)

66. The speed of a boat in still water is v. The boat is to make a round trip in a river whose current travels at speed u. Derive a formula for the time needed to make a round trip of total distance D if the boat makes the round trip by moving (*a*) upstream and back downstream, (*b*) directly across the river and back. We must assume $u < v$; why?



Solution. The speed of the boat with respect to water is $v' > u$, where u is the speed of the water. The velocity of the boat in the laboratory system is

$$\mathbf{v} = \mathbf{v}' + \mathbf{u}.$$

a) The boat goes along the river. When it is going upstream, its absolute velocity is $v = v' - u$. When the boat goes downstream, its absolute velocity is $v = v' + u$. The distances are $d = D/2$ upstream and the same downstream. The total trip time is given by

$$t_{tot,a} = t_{up} + t_{down} = \frac{d}{v' - u} + \frac{d}{v' + u} = \frac{D}{2}\left(\frac{1}{v' - u} + \frac{1}{v' + u}\right)$$
$$= \frac{D}{2}\frac{v' + u + v' - u}{v'^2 - u^2} = \frac{Dv'}{v'^2 - u^2}.$$

If the boat is traveling in a motionless water (a lake or a sea), then $u = 0$ and the time of the trip is given by the obvious formula

$$t_{tot} = \frac{D}{v'}.$$

One can see that this time is shorter than in the case (a).

b) The boat goes straight across the river, as shown in the sketch. In this case, it is essential to consider the velocities as vectors. Projected onto the coordinate axes, the expression for the absolute velocity reads

$$v_x = v'_x + u_x = -v' \sin\theta + u$$
$$v_y = v'_y + u_y = v' \cos\theta$$

The condition that the boat crosses the river straight is $v_x = 0$. From this, using the first equation, one obtains

$$0 = -v' \sin\theta + u \quad \rightarrow \quad \sin\theta = \frac{u}{v'}.$$

Now from the second equation one obtains

$$v_y = v'\sqrt{1 - \sin^2\theta} = v' \sqrt{1 - \left(\frac{u}{v'}\right)^2} = \sqrt{v'^2 - u^2}.$$

Now the total time of the trip is

$$t_{tot,b} = \frac{D}{v_y} = \frac{D}{\sqrt{v'^2 - u^2}}.$$

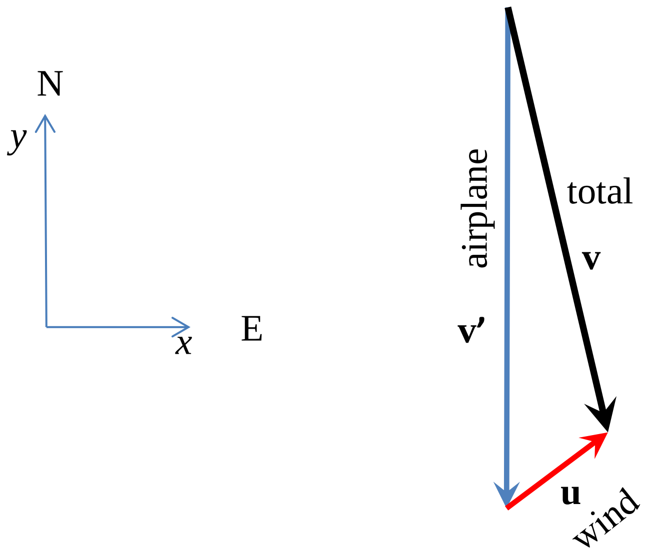
What time is longer? Both are diverging if $v' \rightarrow u$ but $t_{tot,a}$ diverges stronger. Thus, in the limit of a slow boat, $t_{tot,a} > t_{tot,b}$. In the limit $v' \gg u$, both times become D/v'. Then, most probably, $t_{tot,a} \geq t_{tot,b}$ holds always. To investigate the problem thoroughly, one can consider

$$\left(\frac{t_{tot,a}}{t_{tot,b}}\right)^2 = \frac{\frac{v'^2}{\left(v'^2 - u^2\right)^2}}{\frac{1}{v'^2 - u^2}} = \frac{v'^2}{v'^2 - u^2} \geq 1.$$

Thus, indeed, $t_{tot,a} \geq t_{tot,b}$ holds always.

14. Airplane flying in the wind (relative motion, Giancoli, Chapter 3)

* **41.** (II) An airplane is heading due south at a speed of 600 km/h. If a wind begins blowing from the southwest at a speed of 100 km/h (average), calculate: (*a*) the velocity (magnitude and direction) of the plane relative to the ground, and (*b*) how far from its intended position will it be after 10 min if the pilot takes no corrective action. [*Hint*: First draw a diagram.]



Solution.

a) Velocity of the airplane with respect to the air:

$$\mathbf{v}' = (0, -600)\frac{km}{h}.$$

Velocity of the air (of the wind):

$$\mathbf{u} = (100\cos 45°, 100\sin 45°) = \left(\frac{100}{\sqrt{2}}, \frac{100}{\sqrt{2}}\right)\frac{km}{h}.$$

Absolute velocity of the airplane:

$$\mathbf{v} = \mathbf{v}' + \mathbf{u} = \left(\frac{100}{\sqrt{2}}, -600 + \frac{100}{\sqrt{2}}\right) = (70.7, -529.3)\ \frac{km}{h}.$$

b) Let the considered time of the flight be $t_f = 10\ min = \frac{10}{60}h = \frac{1}{6}h$. We put the origin of the coordinate system to the point of departure. Intended position:

$$\mathbf{r}_{intended} = \mathbf{v}'t_f.$$

Actual position:

$$\mathbf{r}_{actual} = \mathbf{v}t_f.$$

Displacement from the intended position:

$$\mathbf{d} = \mathbf{r}_{actual} - \mathbf{r}_{intended} = (\mathbf{v} - \mathbf{v}')t_f = \mathbf{u}t_f.$$

Substituting numbers, one obtains

$$\mathbf{d} = \left(\frac{100}{\sqrt{2}}, \frac{100}{\sqrt{2}}\right)\frac{1}{6} = (11.8{,}11.8)\ km.$$

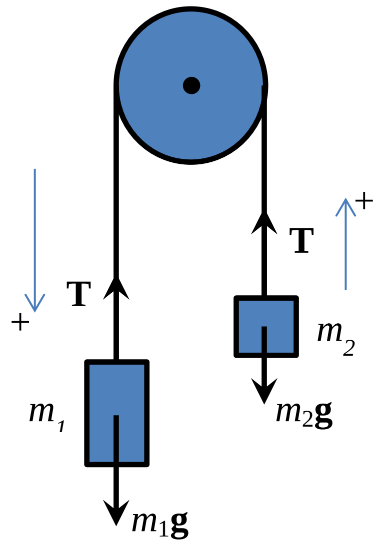
Distance from the intended point:

$$d = |\mathbf{d}| = \sqrt{d_x^2 + d_y^2} = \frac{100}{6}\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \frac{100}{6}\sqrt{\frac{1}{2} + \frac{1}{2}} = \frac{100}{6} = 16.7\ km.$$

Dynamics: Newton's laws

15. Two masses on a massless block

A massless cord goes over a massless block, and the masses m_1 and m_2 are suspended at the ends of the cord. Find the acceleration of the masses and the tension of the cord.



Solution. To write down Newton's second law for both masses, it is essential to choose the positive direction of motion that is down for one of the masses and up for the other, as shown on the sketch. As the block and the cord are massless, the tension forces on both sides of the cord are the same. As the masses are connected by the cord, their acceleration is the same. The equations of motion for the masses (Newton's second law), with explicit signs, are as follows

$$m_1 g - T = m_1 a$$
$$-m_2 g + T = m_2 a.$$

This is a system of two linear equations with two unknowns: a and T. Adding these equations, one can eliminate T that yields

$$(m_1 - m_2)g = (m_1 + m_2)a.$$

Thus

$$a = \frac{m_1 - m_2}{m_1 + m_2} g.$$

If $m_1 > m_2$, the acceleration is positive and the masses are accelerating in the directions indicated in the sketch. If $m_2 = 0$, then $a = g$, as expected. If $m_1 = 0$, then $a = -g$, as expected.

Tension force can now be found from one of the equations, for instance, from the first one:

$$T = m_1 g - m_1 a = m_1 g \left(1 - \frac{m_1 - m_2}{m_1 + m_2}\right) = m_1 g \frac{m_1 + m_2 - m_1 + m_2}{m_1 + m_2}$$

and, finally,

$$T = \frac{2m_1 m_2}{m_1 + m_2} g.$$

This formula is symmetric in m_1 and m_2, as it should be. If one of the masses is small, tension force is also small, as expected. If $m_1 = m_2 = m$, then

$$T = \frac{2m^2}{m + m} g = mg,$$

as expected.

16. Dangling watch in the airplane (Giancoli, chapter 4)

75. Francesca, who likes physics experiments, dangles her watch from a thin piece of string while the jetliner she is in takes off from JFK Airport (Fig. 4–58). She notices that the string makes an angle of 25° with respect to the vertical as the aircraft accelerates for takeoff, which takes about 18 s. Estimate the takeoff speed of the aircraft.

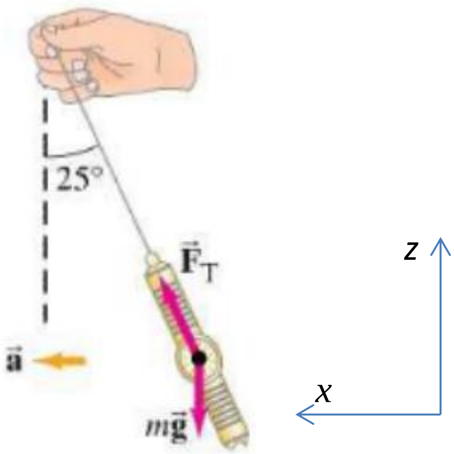


FIGURE 4–58 Problem 75.

Solution. First, we introduce missing notations: deviation angle $\theta = 25°$, takeoff time $t_f = 18\ s$.

Newton's second law for the dangling watch reads

$$m\mathbf{g} + \mathbf{F}_T = m\mathbf{a}.$$

In projections onto the axes this becomes

$$\text{"x"}: \quad F_T \sin\theta = ma$$

$$\text{"z"}: \quad F_T \cos\theta - mg = 0.$$

From the second equation one finds the tension force:

$$F_T = \frac{mg}{\cos\theta}.$$

Substituting this into the first equation, one finds the acceleration:

$$a = \frac{F_T \sin\theta}{m} = \frac{mg}{\cos\theta}\frac{\sin\theta}{m} = g\tan\theta.$$

Now the takeoff speed can be found as

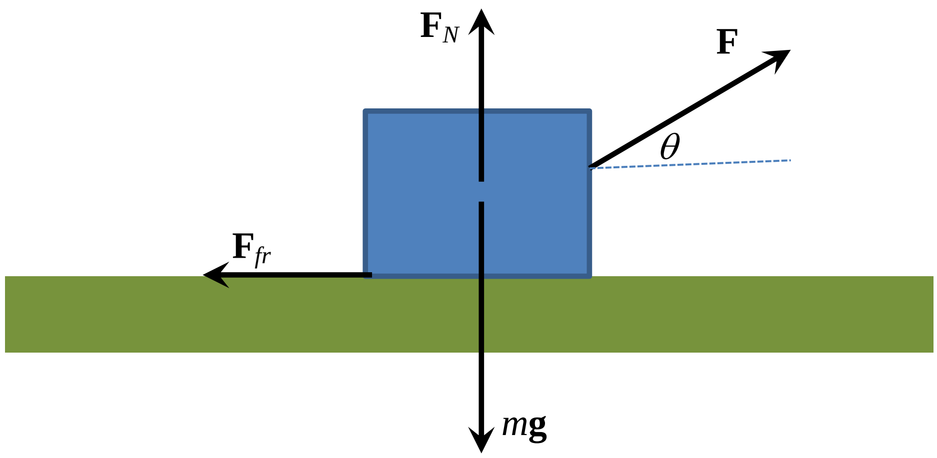
$$v_f = at_f = g\tan\theta\, t_f.$$

Substituting the numbers, one obtains

$$v_f = 9.8 \times \tan 25° \times 18 = 82.3\frac{m}{s} = 82.3\frac{\frac{1}{1000}km}{\frac{1}{3600}hour} = 82.3 \times 3.6\frac{km}{h} = 296\ \frac{km}{h}.$$

17. Pulling a block with friction

A worker is pulling a block at the angle θ to the horizontal without acceleration. The mass of the block is m, the coefficient of dry friction is μ. What is the value of the force F the worker is applying? What is the optimal value of θ?



Solution: As the acceleration of the block is zero (the motion is quasistatic), Newton's second law has the form of the equilibrium condition

$$\mathbf{F}_{total} = m\mathbf{g} + \mathbf{F}_N + \mathbf{F}_{fr} + \mathbf{F} = \mathbf{0}.$$

As the block is moving, the friction force is given by $F_{fr} = \mu F_N$. Components x,z of this equation (with explicit signs) are

$$\text{"x"}:\ -\mu F_N + F\cos\theta = 0$$
$$\text{"z"}:\ -mg + F_N + F\sin\theta = 0.$$

This is a system of two equations with two unknowns: F and F_N. There are many ways to solve this system of linear algebraic equations. As we do not need F_N, we can eliminate it by multiplying the second equation by μ and then adding the two equations. This yields a single equation for F:

$$-\mu mg + F\cos\theta + \mu F\sin\theta = 0.$$

The solution of this equation reads

$$F = \frac{\mu m g}{\cos\theta + \mu \sin\theta}. \qquad (1)$$

For $\theta = 0$ one has the obvious result $F = \mu m g$. Increasing θ from zero leads to the increasing of the denominator as, for small θ, one has $\cos\theta \cong 1 - \theta^2/2$ and $\sin\theta \cong \theta$. This leads to decreasing the applied force. Thus, to minimize the force, one has to use some nonzero pulling angle θ. The physics of this is the following: the vertical component of the pulling force decreases the normal reaction force and thus the friction force.

To find the optimal condition for pulling, one can transform the denominator to a single trigonometric function with the help of the following trick.

$$\cos\theta + \mu\sin\theta = \sqrt{1+\mu^2}\left(\frac{1}{\sqrt{1+\mu^2}}\cos\theta + \frac{\mu}{\sqrt{1+\mu^2}}\sin\theta\right).$$

Here the coefficients in front of $\cos\theta$ and $\sin\theta$ can be interpreted as sine and cosine of an angle φ, as the sum of their squares is one. For instance

$$\cos\varphi = \frac{1}{\sqrt{1+\mu^2}}, \qquad \sin\varphi = \frac{\mu}{\sqrt{1+\mu^2}}.$$

Then, using the formula

$$\cos\varphi\cos\theta + \sin\varphi\sin\theta = \cos(\theta - \varphi),$$

one obtains

$$\cos\theta + \mu\sin\theta = \sqrt{1+\mu^2}\cos(\theta-\varphi), \qquad \varphi = \arccos\frac{1}{\sqrt{1+\mu^2}}$$

and can rewrite our final result, Eq. (1), as

$$F = \frac{1}{\sqrt{1+\mu^2}}\frac{\mu m g}{\cos(\theta-\varphi)}.$$

Thus, the minimal value of the pulling force F is reached for $\theta = \varphi$ and is given by

$$F_{min} = \frac{\mu m g}{\sqrt{1+\mu^2}} < \mu m g.$$

For example, for $\mu = 1$ one has $F_{min} = \mu m g/\sqrt{2} = 0.71\mu m g$ that is achieved for

$$\theta = \varphi = \arccos\frac{1}{\sqrt{2}} = 45°.$$

In this case, the beneficial effect of pulling at the angle to the horizontal is substantial.

For $\mu = 1/2$ one has $F_{min} = (2/\sqrt{5})\mu m g = 0.89\mu m g$ that is achieved for

$$\theta = \varphi = \arccos\frac{2}{\sqrt{5}} = 26.6°.$$

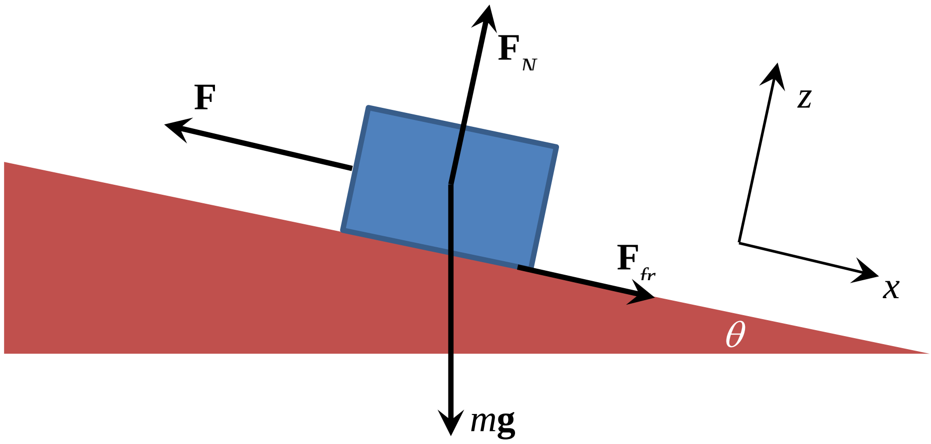
In this case, the gain is smaller.

For $\mu = 0.1$, the gain is very small, $F_{min} = 0.995\mu mg$, that is, 0.5%. The corresponding pulling angle reads

$$\theta = \varphi = \arccos 0.995 = 5.7°.$$

18. Pulling a block uphill with friction

A worker is pulling a block of mass m up a slope with the angle θ. The pulling force is applied parallel to the slope. The friction coefficients are μ and μ_s. What pulling force has to be applied to set the block in motion? What pulling force is needed to stationary pull the block uphill? Calculate the forces for $m = 30\ kg$, $\theta = 10°$, friction wood on wood.



Solution: In the absence of acceleration (quasistatic regime) Newton's second law has the form

$$\mathbf{F}_{total} = m\mathbf{g} + \mathbf{F}_N + \mathbf{F}_{fr} + \mathbf{F} = \mathbf{0}.$$

Components x,z of this equation (with explicit signs) are

$$\text{"x"}: \quad F_{fr} - F + mg\sin\theta = 0$$
$$\text{"z"}: \quad F_N - mg\cos\theta = 0.$$

If the pulling force is too small and the block is not moving, then from the first equation one obtains

$$F_{fr} = F - mg\sin\theta,$$

or, for the pulling force,

$$F = F_{fr} + mg\sin\theta.$$

From the second equation follows $F_N = mg\cos\theta$. The friction force satisfies $F_{fr} \leq \mu_s F_N$. The block begins to move when the friction force reaches its maximal value,

$$F_{fr} = \mu_s F_N = \mu_s mg\cos\theta.$$

Substituting this into the formula for F, one obtains

$$F = mg(\sin\theta + \mu_s\cos\theta).$$

One can see that the worker has to work both against gravity and against friction. For wood on wood, $\mu_s = 0.4$ and $\mu = 0.2$. Numerically, the force required to start the block moving is

$$F = 30 \times 9.8 \times (\sin 10° + 0.4 \cos 10°) = 294 \times (0.174 + 0.394) = 167\ N.$$

In this case, the contribution of the friction force is greater than that of gravity.

If the block is moving, then the friction force is given by $F_{fr} = \mu F_N$. In a similar way, one obtains

$$F = mg(\sin\theta + \mu \cos\theta).$$

Numerically, the force required to pull the block stationary uphill is

$$F = 30 \times 9.8 \times (\sin 10° + 0.2 \cos 10°) = 294 \times (0.174 + 0.197) = 109\ N.$$

In this case, the contributions of gravity and friction are comparable.

19. Two boxes with different frictions on the incline (Giancoli, chapter 4)

87. Two boxes, $m_1 = 1.0$ kg with a coefficient of kinetic friction of 0.10, and $m_2 = 2.0$ kg with a coefficient of 0.20, are placed on a plane inclined at $\theta = 30°$. (*a*) What acceleration does each box experience? (*b*) If a taut string is connected to the boxes (Fig. 4–64), with m_2 initially farther down the slope, what is the acceleration of each box? (*c*) If the initial configuration is reversed with m_1 starting lower with a taut string, what is the acceleration of each box?

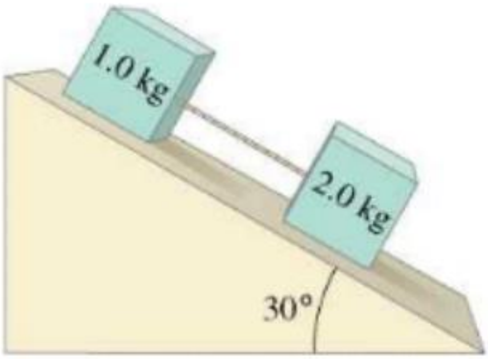


FIGURE 4–64 Problem 87.

Solution. First, we introduce missing notations: $\mu_1 = 0.1$, $\mu_2 = 0.2$.

In lecture notes, the formula for the acceleration of a block sliding on the incline was obtained:

$$a = g(\sin\theta - \mu \cos\theta).$$

In the case a), boxes are not connected by the string and move independently from each other. The acceleration of the block 1 is higher as it has less friction.

In the case b), the solution is the same as in a) as the string does not resist shortening its length. As the acceleration of box 1 is higher than that of box 2, the distance between them will be decreasing until they come in contact. After that, they will be moving together as a system, and the regime changes.

c) In this case, the distance between the boxes would increase with time but the string does not allow it. Thus, both boxes are moving as a system with a common acceleration. The shortest way to find the acceleration is to consider the motion of the system of two blocks along the *x*-axis parallel to the slope, as usual. In this solution, one does not have to consider the tension of the string. The projection of Newton's second law onto this axis reads

$$-F_{fr,1} - F_{fr,2} + (m_1 + m_2)g \sin\theta = (m_1 + m_2)a.$$

Here,

$$F_{fr,1} = \mu_1 F_{N,1} = \mu_1 m_1 g \cos\theta, \qquad F_{fr,2} = \mu_2 F_{N,2} = \mu_2 m_2 g \cos\theta.$$

Substituting this into the equation above and dividing by $m_1 + m_2$, one obtains the result

$$a = g\left(\sin\theta - \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2}\cos\theta\right).$$

The fraction here is the effective friction coefficient for the system of two connected blocks:

$$\mu_{eff} = \frac{\mu_1 m_1 + \mu_2 m_2}{m_1 + m_2}.$$

If the masses are equal to each other, μ_{eff} is just the average of the two friction coefficients. In our case,

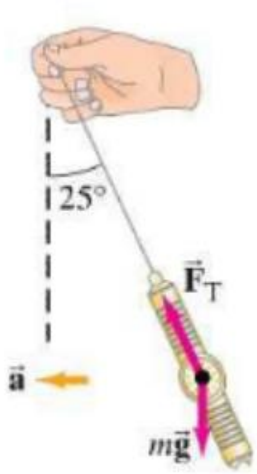
$$\mu_{eff} = \frac{0.1 \times 1 + 0.2 \times 2}{1 + 2} = 0.167.$$

Circular motion

20. A lamp dangling in a train (Giancoli, chapter 5)

79. A train traveling at a constant speed rounds a curve of radius 235 m. A lamp suspended from the ceiling swings out to an angle of 17.5° throughout the curve. What is the speed of the train?

This problem uses the same idea as that of the dangling watch in the airplane above, thus, one can use the same drawing



Let us introduce missing notations: $R = 235\ m$, $\theta = 17.5°$.

Newton's second law for the dangling lamp reads

$$m\mathbf{g} + \mathbf{F}_T = m\mathbf{a}.$$

In projections onto the axes this becomes

$$\text{"x"}: \quad F_T \sin\theta = ma$$

$$\text{"z"}: \quad F_T \cos\theta - mg = 0.$$

From the second equation one finds the tension force:

$$F_T = \frac{mg}{\cos\theta}.$$

Substituting this into the first equation, one finds the acceleration:

$$a = \frac{F_T \sin\theta}{m} = \frac{\frac{mg}{\cos\theta}\sin\theta}{m} = g\tan\theta.$$

This is the centripetal acceleration due to the curvature of the train's trajectory:

$$a_c = \frac{v^2}{R}.$$

Equating these two expressions, one finds the train's speed as

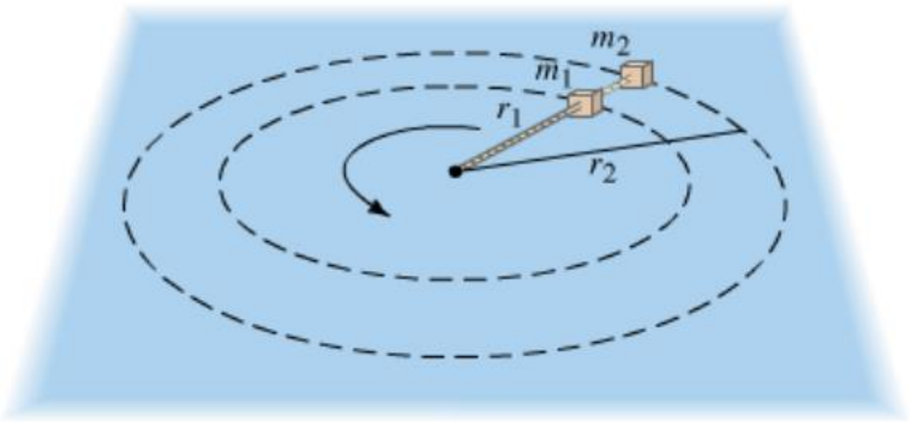
$$v = \sqrt{Rg \tan\theta}.$$

Substituting the numbers, one obtains

$$v = \sqrt{235 \times 9.8 \times \tan 17.5°} = 27\ m/s = 27 \times 3.6\ km/h = 97\ km/h.$$

21. Two masses on a string (Giancoli, chapter 5)

23. (III) Two blocks, of masses m_1 and m_2, are connected to each other and to a central post by cords as shown in Fig. 5–37. They rotate about the post at a frequency f (revolutions per second) on a frictionless horizontal surface at distances r_1 and r_2 from the post. Derive an algebraic expression for the tension in each segment of the cord.



Solution. Let the tension force acting on m_1 from the central part of the cord be F and let the tension of the cord between the two masses be T. With the radial axis directed towards the center, Newton's second law has the form

$$m_1 a_{c,1} = m_1\omega^2 r_1 = F - T, \qquad m_2 a_{c,2} = m_2\omega^2 r_2 = T.$$

From the second equation one immediately finds the value of T. Adding these equations, one eliminates T and obtains the value of F:

$$F = m_1\omega^2 r_1 + m_2\omega^2 r_2 = (m_1 r_1 + m_2 r_2)\omega^2.$$

The angular velocity ω is defined as

$$\omega = \frac{\Delta\theta}{\Delta t},$$

where θ is the rotation angle in radians. The frequency of rotations f is defined as the number of rotations per second,

$$f = \frac{number\ of\ rotations}{\Delta t}.$$

As one rotations corresponds to 2π radians, the number of rotations in the angle $\Delta\theta$ is given by $\Delta\theta/(2\pi)$. Thus

$$f = \frac{\Delta\theta/(2\pi)}{\Delta t} = \frac{1}{2\pi}\frac{\Delta\theta}{\Delta t} = \frac{\omega}{2\pi}$$

and $\omega = 2\pi f$. Thus, in terms of what in given in the problem's formulation, the results have the form

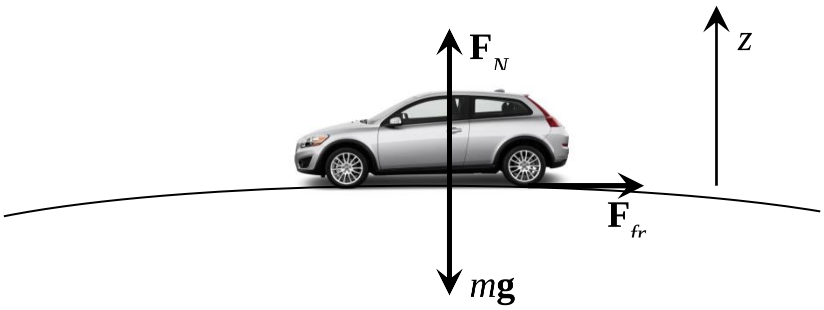
$$T = m_2\omega^2 r_2 = m_2 r_2 (2\pi f)^2$$

and

$$F = (m_1 r_1 + m_2 r_2)(2\pi f)^2.$$

22. A car going over a hilltop

A car is going over a hilltop with the curvature radius R at speed v. Suddenly the driver sees an obstacle behind the hilltop and needs to brake. What is the maximal deceleration if the static friction coefficient is μ_S?



Solution. The maximal deceleration is set by the maximal friction force $F_{fr,max} = \mu_s F_N$. When the car is going over a hilltop, the normal force decreases so that it does not compensate for the gravity force, and this creates the centripetal force. For the projections of the forces on x-axis one has

$$F_N - mg = -F_c = -m\frac{v^2}{R}$$

(centripetal acceleration is directed down towards the center of curvature). From here one obtains

$$F_N = m\left(g - \frac{v^2}{R}\right) < mg.$$

Now the maximal deceleration is given by

$$a_{max} = \frac{F_{fr,max}}{m} = \frac{\mu_s F_N}{m} = \mu_s\left(g - \frac{v^2}{R}\right).$$

It is smaller than on the flat road, thus hilltops are dangerous.

23. The satellite (Giancoli, chapter 5)

90. A satellite of mass 5500 kg orbits the Earth (mass = 6.0×10^{24} kg) and has a period of 6200 s. Find (*a*) the magnitude of the Earth's gravitational force on the satellite, (*b*) the altitude of the satellite.

Solution. Let us introduce missing notations. The satellite's mass $m = 5500\ kg$, the mass of the Earth $M = 0.6 \times 10^{25}\ kg$, the period of the satellite's orbiting around the Earth $T = 6200\ s$.

The gravitational force acting on the satellite is given by the gravitation law and plays the role of the centripetal force:

$$F = G\frac{mM}{R_s^2} = ma_c = m\omega^2 R_s,$$

where R_s is the radius of the satellite's orbit. The angular velocity of the satellite's rotation can be expressed via the orbiting period as follows:

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Thus, one can solve task (b) of finding R_s. Canceling the satellite's mass in the first equation, one obtains

$$G\frac{M}{R_s^2} = \omega^2 R_s,$$

or

$$GM = \omega^2 {R_s}^3,$$

(the third Kepler's law) where from

$$R_s = \left(\frac{GM}{\omega^2}\right)^{1/3} = \left(\frac{GMT^2}{4\pi^2}\right)^{1/3}.$$

Substituting the numbers, one obtains

$$R_s = \left(\frac{0.667 \times 10^{-10} \times 0.6 \times 10^{25} \times 6200^2}{4\pi^2}\right)^{1/3} = 7.3 \times 10^6\ m = 7300\ km.$$

Since the radius of the Earth is $R = 6400\ km$, the satellite's altitude h is

$$h = R_s - R = 7300 - 6400 = 900\ km$$

above the Earth's surface.

Now one can solve task (a) to find the gravitational force on the satellite. Using the first formula and the result for R_s, one obtains

$$F = G\frac{mM}{R_S^2} = mGM\left(\frac{4\pi^2}{GMT^2}\right)^{2/3} = m(GM)^{1-2/3}\left(\left(\frac{2\pi}{T}\right)^2\right)^{2/3} = m(GM)^{1/3}\left(\frac{2\pi}{T}\right)^{4/3}.$$

Substituting the numbers, one obtains

$$F = 5500 \times (0.667 \times 10^{-10} \times 0.6 \times 10^{25})^{1/3}\left(\frac{2\pi}{6200}\right)^{4/3} = 41258\ N.$$

24. Gravity and apparent gravity on Jupiter (Giancoli, chapter 5)

80. Jupiter is about 320 times as massive as the Earth. Thus, it has been claimed that a person would be crushed by the force of gravity on a planet the size of Jupiter since people can't survive more than a few *g*'s. Calculate the number of *g*'s a person would experience at the equator of such a planet. Use the following data for Jupiter: mass = 1.9×10^{27} kg, equatorial radius = 7.1×10^4 km, rotation period = 9 hr 55 min. Take the centripetal acceleration into account.

Solution. First, introduce missing notations: mass of Jupiter $M = 1.9 \times 10^{27}\ kg$, radius of Jupiter $R = 7.1 \times 10^4\ km = 7.1 \times 10^7\ m$, Jupiter's rotation period $T = 9\ hr\ 55\ min$. One can see that Jupiter rotates fast, thus the apparent gravity on Jupiter's equator can be significantly smaller than the real gravity because of the effect of the centripetal force. We immediately convert the period into seconds: $T = 9 \times 3600 + 55 \times 60 = 35700\ s$.

On Jupiter's surface, the gravity can be calculated using the law of gravitation for any mass m:

$$mg = G\frac{mM}{R^2},$$

thus

$$g = G\frac{M}{R^2}.$$

Substituting the numbers, one obtains

$$g = 0.667 \times 10^{-10}\frac{1.9 \times 10^{27}}{(7.1 \times 10^7)^2} = 25.2\ m/s^2.$$

This is slightly more than twice the value of g on the Earth, and a person having the mass $50\ kg$ would feel like having $50 \times 25.2/9.8 = 129\ kg$. While it is questionable whether such a high gravity would crush a human immediately, it is clear that living with such a high gravity is difficult.

Let us now consider the apparent gravity defined via the normal reaction force F_N acting on the person from the ground. According to Newton's second law,

$$mg - F_N = ma_c = m\omega^2 R.$$

Using the relations between the angular velocity ω, frequency f, and period T,

$$\omega = 2\pi f = \frac{2\pi}{T},$$

one obtains

$$F_N = m(g - a_c) = m(g - \omega^2 R) = m\left(g - \left(\frac{2\pi}{T}\right)^2 R\right).$$

One can write $F_N = mg_{apparent}$, where

$$g_{apparent} = g - \left(\frac{2\pi}{T}\right)^2 R = G\frac{M}{R^2} - \left(\frac{2\pi}{T}\right)^2 R$$

is smaller than the actual g. Substituting the numbers, one obtains

$$g_{apparent} = 25.2 - \left(\frac{2\pi}{35700}\right)^2 \times 7.1 \times 10^7 = 23.0\ m/s^2.$$

This only slightly smaller than the actual g on Jupiter.

25. A tape planet (Giancoli, chapter 5)

* **65.** (III) A science-fiction tale describes an artificial "planet" in the form of a band completely encircling a sun (Fig. 5–40). The inhabitants live on the inside surface (where it is always noon). Imagine that this sun is exactly like our own, that the distance to the band is the same as the Earth–Sun distance (to make the climate temperate), and that the ring rotates quickly enough to produce an apparent gravity of g as on Earth. What will be the period of revolution, this planet's year, in Earth days?

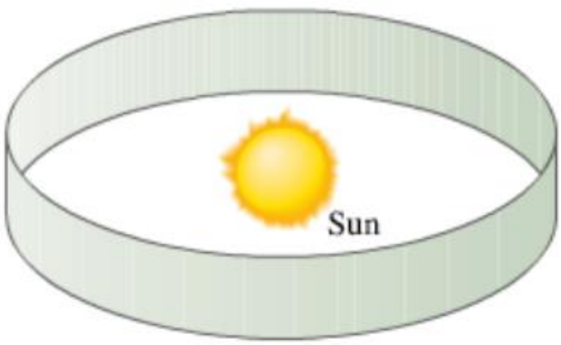


FIGURE 5–40 Problem 65.

Solution. In this case, the apparent gravity is created by the rotation of the tape planet that creates the normal reaction force $\mathbf{F}_N$ imitating the gravity $m\mathbf{g}$. That is,

$$F_N = ma_c = m\omega^2 R = mg,$$

where R is the radius of the tape planet. From this one can find the angular velocity

$$\omega = \sqrt{\frac{g}{R}}$$

The angular velocity can be expressed via the orbiting period as follows:

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Thus the period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R}{g}}.$$

Substituting the numbers, one obtains

$$T = 2\pi\sqrt{\frac{1.5 \times 10^{11}}{9.8}} = 777343\ s = \frac{777343}{24 \times 3600} = 9.0\ days.$$

This is very fast. However, rotation of this planet around the sum does not lead to the change of seasons.

26. Orbiting of the sum around the center of our galaxy (Giancoli, chapter 5)

* **88.** The Sun rotates around the center of the Milky Way Galaxy (Fig. 5–46) at a distance of about 30,000 light-years from the center $(1\ \text{ly} = 9.5 \times 10^{15}\ \text{m})$. If it takes about 200 million years to make one rotation, estimate the mass of our Galaxy. Assume that the mass distribution of our Galaxy is concentrated mostly in a central uniform sphere. If all the stars had about the mass of our Sun $(2 \times 10^{30}\ \text{kg})$, how many stars would there be in our Galaxy?

Solution. Let us introduce missing notations. The distance from the Sun to the center of our galaxy (the radius of Sun's orbit) is $R = 30000\ ly$, The period of the sun's orbiting is $T = 200 \times 10^6\ years$, the mass of the sun is $m = 2 \times 10^{30}\ kg$. The mass is the galaxy is M.

According to Newton's second law, the gravitational force causes the centripetal acceleration of the Sun,

$$G\frac{mM}{R^2} = ma_c = m\omega^2 R.$$

From here one finds the mass of the galaxy

$$M = \frac{\omega^2 R^3}{G}.$$

The angular velocity can be expressed via the orbiting period as follows:

$$\omega = 2\pi f = \frac{2\pi}{T}.$$

Thus

$$M = \left(\frac{2\pi}{T}\right)^2 \frac{R^3}{G}.$$

If all stars in the galaxy have approximately the mass of the Sun, the number of the stars can be estimated as

$$N = \frac{M}{m} = \left(\frac{2\pi}{T}\right)^2 \frac{R^3}{mG}.$$

To calculate the numerical result, first one has to convert all numbers from special units to the SI units. The light year (ly) is defined as the distance covered by the light during one year. Using the speed of light $c = 3 \times 10^8\ m/s$, one obtains

$$1\ ly = 3 \times 10^8 \times \frac{m}{s} \times 1\ year = 3 \times 10^8 \times \frac{m}{s} \times 365 \times 24 \times 3600\ s = 9.46 \times 10^{15}\ m.$$

Thus

$$R = 30000\ ly = 30000 \times 9.46 \times 10^{15} = 2.84 \times 10^{20}\ m$$

The orbiting period of the Sun is

$$T = 200 \times 10^6\ years = 200 \times 10^6 \times 365 \times 24 \times 3600 = 6.31 \times 10^{15}\ s.$$

Now the mass of the galaxy is

$$M = \left(\frac{2\pi}{6.31 \times 10^{15}}\right)^2 \frac{(2.84 \times 10^{20})^3}{0.667 \times 10^{-10}} = 2.10 \times 10^{43}\ kg.$$

The number of stars in the galaxy is

$$N = \frac{M}{m} = \frac{2.10 \times 10^{43}}{2 \times 10^{30}} = 1.05 \times 10^{13} \approx 10^{13}.$$

Work and energy

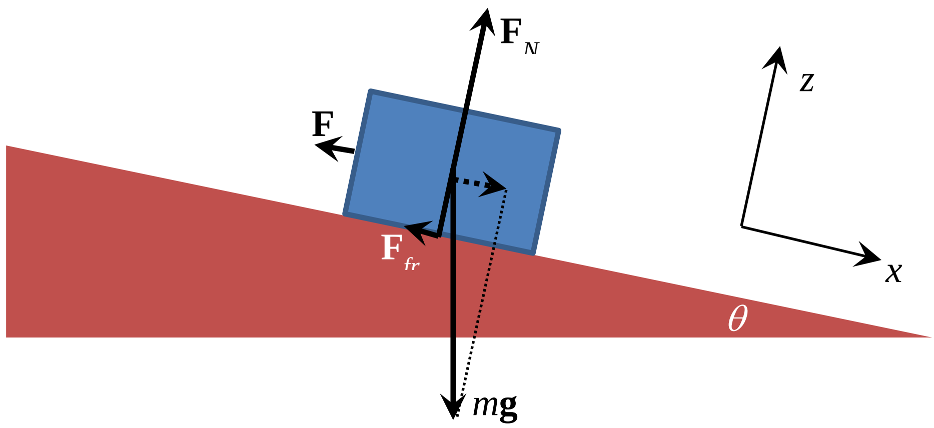
27. Sliding piano (Giancoli, chapter 6)

8. (II) A 330-kg piano slides 3.6 m down a 28° incline and is kept from accelerating by a man who is pushing back on it *parallel to the incline* (Fig. 6–36). The effective coefficient of kinetic friction is 0.40. Calculate: (*a*) the force exerted by the man, (*b*) the work done by the man on the piano, (*c*) the work done by the friction force, (*d*) the work done by the force of gravity, and (*e*) the net work done on the piano.

FIGURE 6–36
Problem 8.

Solution. First, we introduce missing notations. The mass of the piano $m = 330\ kg$, the incline's angle $\theta = 28°$, kinetic friction coefficient $\mu = 0.4$, the sliding distance $d = 3.6\ m$.

This is a problem about the incline. We use the sketch from one of the preceding problems.



In the absence of acceleration (quasistatic regime) Newton's second law has the form

$$\mathbf{F}_{total} = m\mathbf{g} + \mathbf{F}_N + \mathbf{F}_{fr} + \mathbf{F} = \mathbf{0}.$$

Components x,z of this equation (with explicit signs) are

$$\begin{aligned} \text{"x"}: &\quad -F_{fr} - F + mg\sin\theta = 0 \\ \text{"z"}: &\quad F_N - mg\cos\theta = 0. \end{aligned}$$

As the piano is sliding, the dry friction force is given by

$$F_{fr} = \mu F_N.$$

This is a system of three equations for three unknowns: F, F_{fr}, and F_N. Finding $F_N = mg\cos\theta$ from the "z" equation, one then finds the friction force as

$$F_{fr} = \mu F_N = \mu mg\cos\theta.$$

After that, from the "x" equation one finds the pushing force:

$$F = mg\sin\theta - F_{fr} = mg(\sin\theta - \mu\cos\theta).$$

Let us calculate the work done by different forces now. The normal force is perpendicular to the displacement that is along *x*-axis and is not doing any work. The gravity force makes the angle $\theta' = 90° - \theta$ with x-axis and doing a positive work:

$$W_G = mgd\cos\theta' = mgd\cos(90° - \theta) = mgd\sin\theta\,.$$

Substituting the numbers, one obtains

$$W_G = 330 \times 9.8 \times 3.6 \times \sin 28° = 5466\ \text{J}.$$

Both pushing force and friction force are directed oppositely to the displacement and thus are doing a negative work. For the friction force the work is given by

$$W_{fr} = \mu mg\cos\theta\ \ d\ \ \cos 180° = -\mu mgd\cos\theta.$$

Substituting the numbers, one obtains

$$W_{fr} = -0.4 \times 330 \times 9.8 \times 3.6 \times \cos 28° = -4112\ \text{J}.$$

Finally, the work done by the pushing force is given by

$$W = Fd\cos 180° = -mgd(\sin\theta - \mu\cos\theta).$$

Substituting the numbers, one obtains

$$W = -330 \times 9.8 \times 3.6 \times (\sin 28° - 0.4 \times \cos 28°) = -1354\ \text{J}.$$

The total work is zero:

$$W_G + W_{fr} + W = mgd\sin\theta - \mu mgd\cos\theta - mgd(\sin\theta - \mu\cos\theta) = 0$$

because the sum of projections of all forces on x-axis is zero ("x" equation).

28. Accident skid mark (Giancoli, chapter 6)

22. (II) At an accident scene on a level road, investigators measure a car's skid mark to be 88 m long. The accident occurred on a rainy day, and the coefficient of kinetic friction was estimated to be 0.42. Use these data to determine the speed of the car when the driver slammed on (and locked) the brakes. (Why does the car's mass not matter?)

Solution. First, we introduce missing notations: the length of the skid mark (distance traveled) $d = 88\ m$, kinetic friction coefficient $\mu = 0.42$.

In this problem it is illustrated that the energy is the ability to do work. In the initial state, the car possesses the kinetic energy that is then wasted into the heat via dry friction in the process of braking /skidding. More precisely, in the process, the friction force is doing a negative work on the car decreasing its kinetic energy to zero. We use the work-energy relation

$$W = E_f - E_i \qquad (1).$$

Here

$$E_i = \frac{mv^2}{2}, \qquad E_f = 0$$

and the work of the friction force is given by

$$W = -F_{fr}d.$$

Using the formula for the friction force

$$F_{fr} = \mu F_N = \mu mg$$

and putting everything together, one obtains the work-energy balance in the form

$$-\mu mgd = 0 - \frac{mv^2}{2}.$$

From this equation one finds

$$v = \sqrt{2\mu gd}.$$

Substituting the numbers, one obtains

$$v = \sqrt{2 \times 0.42 \times 9.8 \times 88} = 26.9\ m/s = 26.9 \times 3.6 = 96.8\ km/h.$$

Notes:

- Police officers need to know physics.
- A tutor solving this problem for you would typically start with the formula
  $$F_{fr}d = \frac{mv^2}{2}.$$
  Although this relation is correct and leads to the correct result, it hides the fact that friction is doing a negative work and the kinetic energy decreases. It distorts the fundamental work-energy relation, Eq. (1). Thus, in our course, such a solution is inacceptable.

29. Looped track (Giancoli, chapter 6)

40. (II) A block of mass m slides without friction along the looped track shown in Fig. 6–39. If the block is to remain on the track, even at the top of the circle (whose radius is r), from what minimum height h must it be released?

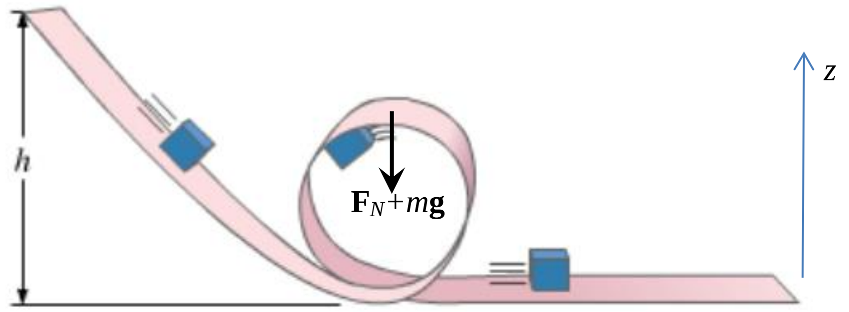


FIGURE 6–39 Problems 40 and 75.

Solution. This is a problem about energy conservation and circular motion. At the top of the loop, the speed should be sufficiently high so that a normal force must add to the gravity force to ensure the needed centripetal acceleration. In projection on the vertical-up z-axis, Newton's second law reads

$$-F_N - mg = -ma_c = -m\frac{v^2}{r},$$

where from

$$F_N = m\left(\frac{v^2}{r} - g\right).$$

To preserve the contact of the sliding block with the structure, the condition $F_N > 0$ should be satisfied. This requires that the speed v is above a minimal value,

$$v^2 \geq v_{min}^2 = rg.$$

The speed can be found from the energy conservation law. At the highest point in the loop the height is $z = 2r$. The energy conservation $E_i = E_f$ becomes

$$mgh = mg2r + \frac{mv^2}{2},$$

where from

$$v^2 = 2g(h - 2r).$$

Substituting this into the inequality above, one obtains

$$2g(h - 2r) \geq rg,$$

that is,

$$h \geq \frac{r}{2} + 2r = \frac{5}{2}r \equiv h_{min}.$$

It is noteworthy that the result does not depend on m and g.

30. Spring cut in half (Giancoli, chapter 6)

89. A spring with spring stiffness constant k is cut in half. What is the spring stiffness constant for each of the two resulting springs?

What do you think before solving this problem? What does your intuition tell you?

Solution. If the spring is pulled with the force F, any part of the spring is acting on the neighboring part with the same force F that is related to the spring's deformation x by the Hooke's law

$$F = kx.$$

This is the result of the absence of acceleration and Newton's laws. In particular, if we consider a half of the spring, it is pulled by the other half with the same force F but its elongation is $x' = x/2$. For this half of the spring, the Hooke's law has the form

$$F = k'x' = k'x/2.$$

Comparing with the first formula, one obtains

$$kx = k'x/2,$$

that is, $k' = 2k$. Is it counterintuitive that cutting a spring in two makes it stronger?

The same result can be obtained from the energy argument. The energy of the deformed spring is

$$E = \frac{kx^2}{2}.$$

Considering the two halves of the spring, one can write the same energy as the sum of the energies of both halves:

$$E = \frac{k'x'^2}{2} + \frac{k'x'^2}{2}.$$

Substituting here $x' = x/2$ and equating to the preceding formula, one obtains

$$E = k'x'^2 = \frac{k'x^2}{4} = \frac{kx^2}{2}.$$

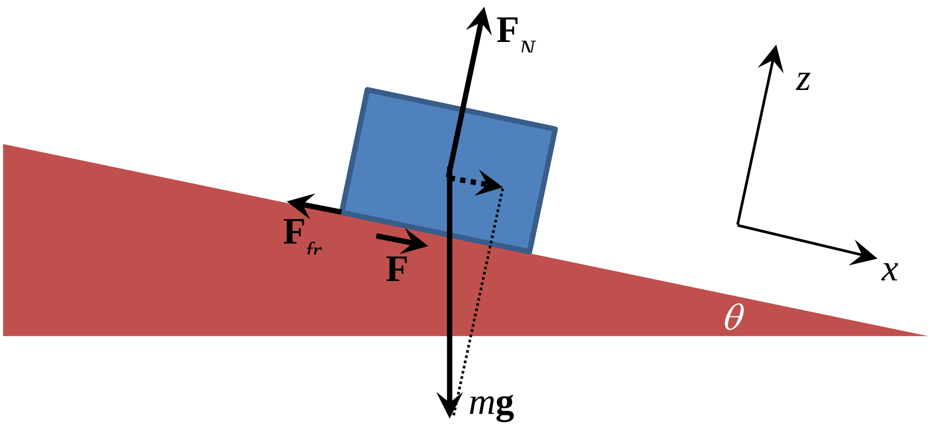
From here follows $k' = 2k$.

31. Power of the patient on a treadmill (Giancoli, chapter 6)

92. In a common test for cardiac function (the "stress test"), the patient walks on an inclined treadmill (Fig. 6–46). Estimate the power required from a 75-kg patient when the treadmill is sloping at an angle of 15° and the velocity is 3.3 km/h. (How does this power compare to the power rating of a lightbulb?)

FIGURE 6–46 Problem 92.

Solution. First, we introduce missing notations. Mass of the patient $m = 75\ kg$, the slope of the treadmill $\theta = 15°$, the speed of the patient $v = 3.3\ km/h = 3.3/3.6 = 0.917\ m/s$.



The mechanical power is given by

$$P = \mathbf{F} \cdot \mathbf{v}.$$

The question is of how to apply this formula correctly.

The solution in the Giancoli book is conceptually wrong, although it leads to the correct final result. Giancoli assumes that the patient generates a pushing force $F_p = mg\sin\theta$ (to compensate for the gravity) that is applied to himself and multiplies this by the speed, obtaining the power $P = mgv\sin\theta$. However, the speed of the patient is zero, and this approach in fact yields a zero power.

In this setting, mechanical work is not done on the patient. Rather, the patient does work on the treadmill pushing it back with his feet with the force $\mathbf{F}$**,** as shown in the sketch. This force is applied to the treadmill and is opposite to the friction force applied from the treadmill to the patient, according to Newton's third law,

$$\mathbf{F} = -\mathbf{F}_{fr}$$

As there is no acceleration, the total force on the patient is zero:

$$\mathbf{F}_{fr} + \mathbf{F}_N + m\mathbf{g} = \mathbf{0}.$$

Thus, the friction force compensates for the *x*-component of the gravity force,

$$F_{fr} = mg\sin\theta.$$

Now, the treadmill band is moving down the slope and the force $\mathbf{F}$ is applied to it from the patient in the same direction, so that the power developed by the patient is given by

$$P = Fv = F_{fr}v = mgv\sin\theta.$$

Substituting the numbers, one obtains

$$P = 75 \times 9.8 \times 0.917 \times \sin 15° = 174\ W$$

that is comparable with the power of the electric bulb (an old-style one with a wolfram filament).

If a person is just walking up the incline that is at rest, then the work is done on the person by the person him- or herself. This problem should have the same answer and the person should develop the same power. However, this problem is conceptually more difficult as it is not easy to identify the force that is doing work. One needs to consider the human body as consisting of different parts, so that different parts are doing work on each other (feet on legs, legs on the torso, etc.)

Linear momentum

32. Inelastic collision

Mass $m_1 = 2\ kg$ moving North-West with the velocity $v_1 = 4\ m/s$ collides inelastically with mass $m_2 = 3\ kg$ moving North-East with the velocity $v_2 = 5\ m/s$. Find the velocity of the system u after the collision and the lost energy.

Solution: In the collision the momentum of the system is conserved and in the final state both bodies are moving together, thus

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{u}.$$

From here one obtains

$$\mathbf{u} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}.$$

Substituting the numbers into this formula, one obtains

$$\mathbf{u} = \frac{2 \times (4\cos 135°\,, 4\sin 135°) + 3 \times (5\cos 45°\,, 5\sin 45°)}{2+3}$$

$$= \frac{2 \times \left(4\left(-\frac{1}{\sqrt{2}}\right), 4\frac{1}{\sqrt{2}}\right) + 3 \times \left(5\frac{1}{\sqrt{2}}, 5\frac{1}{\sqrt{2}}\right)}{2+3}$$

$$= \frac{(-2 \times 4 + 3 \times 5,\ \ 2 \times 4 + 3 \times 5)}{5\sqrt{2}} = \frac{(7, 23)}{5\sqrt{2}} = (0.99, 3.25)m/s$$

The energy lost in the collision if defined as

$$E_{lost} \equiv E_i - E_f,$$

where

$$E_i = \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2}, \qquad E_f = \frac{(m_1 + m_2)\mathbf{u}^2}{2}.$$

Substituting here $\mathbf{u}$ found above, one obtains

$$E_{lost} \equiv \frac{2 \times 4^2}{2} + \frac{3 \times 5^2}{2} - \frac{(2+3) \times (0.99^2 + 3.25^2)}{2}$$
$$= 43.5 - 40.4 = 3.1\ J.$$

33. Energy lost in the inelastic collision (general)

Derive a general formula for the energy lost in the inelastic collision. Calculate the energy lost and the fraction of the energy lost in the collision of the mass $m_1 = 1\ kg$ moving with the speed $v_{1,i} = 2\ m/s$ and the mass $m_2 = 2\ kg$ moving with the speed $v_{2,i} = 1\ m/s$, if in the initial state the masses are moving perpendicularly to each other (the angle between their velocities being 90°.

Solution. In the inelastic collision, the two masses stick together as the result of the collision, so that the conservation of the linear momentum has the form

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 = (m_1 + m_2)\mathbf{u},$$

where $\mathbf{u} \equiv \mathbf{v}_f$ is the velocity of the system in the final state. From here, one finds

$$\mathbf{u} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}. \qquad (1)$$

Now, the energies in the initial and final states are given by

$$E_i = \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2}, \qquad E_f = \frac{(m_1 + m_2)\mathbf{u}^2}{2}.$$

Here and elsewhere, the square of a vector is defined as the dot-product of the vector with itself, for instance:

$$\mathbf{u}^2 \equiv \mathbf{u}\cdot\mathbf{u} = |\mathbf{u}||\mathbf{u}|\cos 0° = |\mathbf{u}|^2 = u^2. \qquad (2)$$

The energy lost in the collision is defined as

$$E_{lost} \equiv E_i - E_f,$$

so that here

$$E_{lost} = \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2} - \frac{(m_1 + m_2)\mathbf{u}^2}{2}. \qquad (3)$$

Substituting the solution for $\mathbf{u}$, one obtains

$$\begin{aligned} E_{lost} &= \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2} - \frac{m_1 + m_2}{2}\left(\frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}\right)^2 \\ &= \frac{m_1\mathbf{v}_1^2}{2} + \frac{m_2\mathbf{v}_2^2}{2} - \frac{m_1^2\mathbf{v}_1^2 + 2m_1m_2\mathbf{v}_1\cdot\mathbf{v}_2 + m_2^2\mathbf{v}_2^2}{2(m_1 + m_2)}. \end{aligned}$$

Bringing this expression into the form with the common denominator, one proceeds as follows

$$\begin{aligned} E_{lost} &= \frac{(m_1 + m_2)(m_1\mathbf{v}_1^2 + m_2\mathbf{v}_2^2) - m_1^2\mathbf{v}_1^2 - 2m_1m_2\mathbf{v}_1\cdot\mathbf{v}_2 - m_2^2\mathbf{v}_2^2}{2(m_1 + m_2)} \\ &= \frac{m_1m_2\mathbf{v}_1^2 + m_1m_2\mathbf{v}_2^2 - 2m_1m_2\mathbf{v}_1\cdot\mathbf{v}_2}{2(m_1 + m_2)} = \frac{m_1m_2(\mathbf{v}_1 - \mathbf{v}_2)^2}{2(m_1 + m_2)}. \qquad (4) \end{aligned}$$

In this calculation, the terms $m_1^2\mathbf{v}_1^2$ and $m_2^2\mathbf{v}_2^2$ cancel that leads to a great simplification. The formula obtained is very elegant and can be checked on particular cases. If one of the masses is zero or the initial velocities are equal to each other, there is actually no collision and the lost energy is zero. Of course one can calculate the lost energy using Eq.(3) in which $\mathbf{u}$ is given by Eq.(1). However, using the formula above is more satisfying.

For the example to consider, one can choose the axes so that the first mass is moving in the positive *x*-direction and the second mass is moving in the positive *y*-direction, that is

$$\mathbf{v}_1 = (2,0,0)\ m/s, \qquad \mathbf{v}_2 = (0,1,0)\ m/s.$$

(we take into account the *z*-component, too). Thus,

$$\mathbf{v}_1 - \mathbf{v}_2 = (2,0,0) - (0,1,0) = (2,-1,0)\ m/s$$

and, according to Eq.(2),

$$(\mathbf{v}_1 - \mathbf{v}_2)^2 = |\mathbf{v}_1 - \mathbf{v}_2|^2.$$

Substituting the numbers, one obtains

$$(\mathbf{v}_1 - \mathbf{v}_2)^2 = |(2,-1,0)|^2 = 2^2 + (-1)^2 + 0^2 = 5\ m^2/s^2.$$

Then, the lost energy given by Eq.(4) becomes

$$E_{lost} = \frac{1 \times 2 \times 5}{2 \times (1+2)} = \frac{5}{3} = 1.67\ J.$$

The initial energy is

$$E_i = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} = \frac{1 \times 2^2}{2} + \frac{2 \times 1^2}{2} = 2 + 1 = 3\ J.$$

Thus, the fraction of the energy lost in the collision is

$$\eta \equiv \frac{E_{lost}}{E_i} = \frac{5/3}{3} = \frac{5}{9} = 0.556.$$

34. Recoil

In the recoil of an object into two parts with masses m_1 and m_2, the energy ΔE is released. Find the velocities of the parts 1 and 2.

Solution. Conservation laws for the linear momentum in this case has the form

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = \mathbf{0}.$$

As the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (the velocities in the final state) are proportional to each other, that is, directed along the same line (that has a random direction), one can choose the *x*-axis along this line. Then one can discard vectors and write

$$m_1 v_1 + m_2 v_2 = 0,$$

where v_1 and v_2 are projections of the velocity vectors onto the *x*-axis, that can be positive or negative. The energy balance in the process has the form

$$\Delta E = E_1 + E_2 = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2},$$

where ΔE is the energy released and converted into mechanical energy. From the first equation, one obtains

$$\frac{v_2}{v_1} = -\frac{m_1}{m_2}.$$

This implies that the lighter part has a higher speed (think about the rifle and the bullet).

Expressing v_2 via v_1,

$$v_2 = -v_1 \frac{m_1}{m_2},$$

and substituting this into the energy equation, one obtains

$$\Delta E = \frac{m_1 v_1^2}{2} + \frac{m_2}{2}\left(-v_1 \frac{m_1}{m_2}\right)^2 = \frac{m_1 v_1^2}{2} + \frac{m_1^2 v_1^2}{2m_2} = \frac{m_1(m_1 + m_2)v_1^2}{2m_2}.$$

From this one finds

$$v_1 = \sqrt{\frac{2m_2 \Delta E}{m_1(m_1 + m_2)}} = \sqrt{\frac{2\Delta E}{m_1 + m_2}} \sqrt{\frac{m_2}{m_1}}.$$

The last form of the result separates the parts symmetric and non-symmetric in 1 and 2. Now v_2 can be found using the formula for v_2 above:

$$v_2 = -v_1 \frac{m_1}{m_2} = -\sqrt{\frac{2\Delta E}{m_1 + m_2}} \sqrt{\frac{m_2}{m_1}} \frac{m_1}{m_2} = -\sqrt{\frac{2\Delta E}{m_1 + m_2}} \sqrt{\frac{m_1}{m_2}}.$$

In fact, this formula could be obtained immediately from the formula for v_1 by just exchanging $1 \rightleftharpoons 2$ and changing the sign.

35. Explosion of an object (recoil, Giancoli, chapter 7)

74. An object at rest is suddenly broken apart into two fragments by an explosion. One fragment acquires twice the kinetic energy of the other. What is the ratio of their masses?

Solution. Conservation laws for the linear momentum in this case has the form

$$m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 = \mathbf{0}.$$

As the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ (the velocities in the final state) are proportional to each other, that is, directed along the same line, one can choose the *x*-axis along this line. Then one can discard vectors and write

$$m_1 v_1 + m_2 v_2 = 0,$$

where v_1 and v_2 are projections of the velocity vectors onto the *x*-axis, that can be positive or negative. The energy balance in the process has the form

$$\Delta E = E_1 + E_2 = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2},$$

where ΔE is the energy released in the explosion. It turns out that the energy equation is not needed, however.

The condition in the problem's formulation is

$$\alpha \equiv \frac{E_1}{E_2} = 2.$$

Inserting here the kinetic energies, one obtains

$$\alpha = \frac{m_1 v_1^2}{m_2 v_2^2} = \beta \left(\frac{v_1}{v_2}\right)^2, \qquad \beta \equiv \frac{m_1}{m_2},$$

where the mass ratio, β, has to be found. From the linear-momentum conservation above follows

$$\frac{v_1}{v_2} = -\frac{m_2}{m_1} = -\frac{1}{\beta}.$$

Substituting this into the equation above, one obtains

$$\alpha = \beta \frac{1}{\beta^2} = \frac{1}{\beta}.$$

From this one finds

$$\beta = \frac{1}{\alpha} = \frac{1}{2}.$$

We have found that the smaller mass, here m_1, receives a larger energy in the process of recoil. This is an important result. In shooting a rifle or a gun, the most energy of the burning powder goes to the bullet and only a very small energy goes to the rifle itself, as the ratio of the masses m_{rifle}/m_{bullet} is very large.

36. The center of mass of a system of point masses

Find the position of the CM of a system of three masses: $m_1 = 1\ kg$ at $\mathbf{r}_1 = (1,1,-1)\ m$, $m_2 = 2\ kg$ at $\mathbf{r}_2 = (2,-2,2)\ m$, and $m_3 = 3\ kg$ at $\mathbf{r}_3 = (-3,3,3)\ m$.

Solution. The position of the CM is defined by

$$\mathbf{r}_{CM} = \frac{1}{M} \sum_i m_i\, \mathbf{r}_i.$$

Substituting the given values, one obtains

$$\begin{aligned} \mathbf{r}_{CM} &= \frac{1}{1+2+3} [1 \times (1,1,-1) + 2 \times (2,-2,2) + 3 \times (-3,3,3)] \\ &= \frac{(1+4-9, 1-4+9, -1+4+9)}{6} = \frac{(-4,6,12)}{6} = \left(-\frac{2}{3}, 1,2\right)\ m. \end{aligned}$$

37. People exchanging seats in a boat (Giancoli, chapter 7)

72. Two people, one of mass 75 kg and the other of mass 60 kg, sit in a rowboat of mass 80 kg. With the boat initially at rest, the two people, who have been sitting at opposite ends of the boat 3.2 m apart from each other, now exchange seats. How far and in what direction will the boat move?

Solution. Let us introduce notations: $m_1 = 75\ kg$, $m_2 = 60\ kg$, the mass of the boat $M = 80\ kg$, $L = 3.2\ m$.

Neglecting the waves produced by the process, one can consider the system of two persons + the boat as isolated. Then, the center of mass of this system remains at the same position. This problem is effectively one-dimensional, so we use the *x*-axis directed along the axis of the boat. The *X*-coordinate of the center of mass (CM) in the laboratory system (related to the ground) is given by

$$X_{CM} = \frac{MX_B + m_1X_1 + m_2X_2}{M + m_1 + m_2}.$$

Here X_B is the coordinate of the center of the boat in the laboratory system (the ground or the water) and X_1 and X_2 are the positions of the persons in the laboratory system. The latter can be expressed via their positions in the boat frame (with respect to the boat) x_1 and x_2 as

$$X_1 = x_1 + X_B, \qquad X_2 = x_2 + X_B,$$

resulting in

$$X_{CM} = \frac{MX_B + m_1(x_1 + X_B) + m_2(x_2 + X_B)}{M + m_1 + m_2}.$$

According to the problem's formulation,

$$x_2 - x_1 = L \qquad (1)$$

(the larger mass m_1 is shifting in the positive direction, from x_1 to x_2). In the initial state, after collecting terms, one has

$$X_{CM,i} = X_{B,i} + \frac{m_1x_1 + m_2x_2}{M + m_1 + m_2}.$$

In the final state, the persons have the interchanged positions and the boat is shifted, so that

$$X_{CM,f} = X_{B,f} + \frac{m_1x_2 + m_2x_1}{M + m_1 + m_2}.$$

The positions of the CM in both states are the same: $X_{CM,f} = X_{CM,i}$. Subtracting the first equation from the second one yields

$$0 = X_{B,f} - X_{B,i} + \frac{m_1(x_2 - x_1) + m_2(x_1 - x_2)}{M + m_1 + m_2}.$$

From here, using Eq. (1), one finds the displacement of the boat,

$$\Delta X_B \equiv X_{B,f} - X_{B,i} = -\frac{m_1 L - m_2 L}{M + m_1 + m_2} = -\frac{m_1 - m_2}{M + m_1 + m_2} L.$$

Substituting the numbers, one obtains

$$\Delta X_B = -\frac{75 - 60}{80 + 75 + 60} 3.2 = -0.22\, m.$$

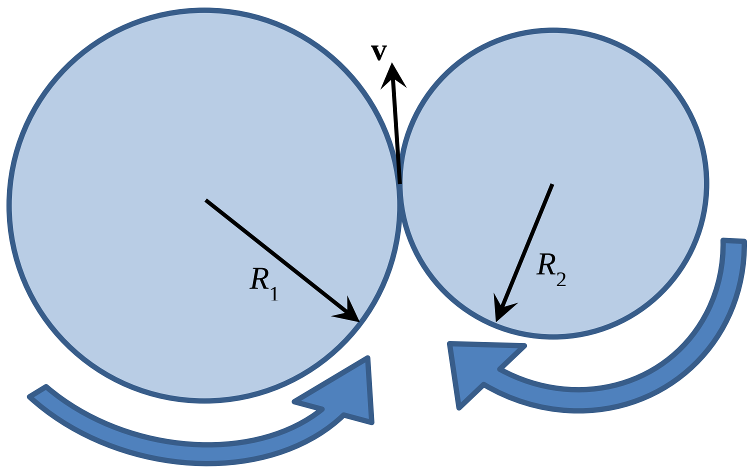
That is, the boat is shifting in the negative direction, opposite to the direction of shifting of the larger mass m_1. The direction of the displacement of the boat must be opposite to the direction of the displacement of the heavier person, to keep the CM at the same position.

The key point in this solution is using the moving frame of the boat to specify the positions of the persons. This is suggested by the fact that the persons interchange their positions in the frame of the boat, not in the laboratory frame.

Rotational motion

38. Two coupled rotating disks (Giancoli, chapter 8)

13. (II) A turntable of radius R_1 is turned by a circular rubber roller of radius R_2 in contact with it at their outer edges. What is the ratio of their angular velocities, ω_1/ω_2?



Solution. The two discs are not slipping with respect to each other, thus the velocity of the contact point $\mathbf{v}$ is the same for both disks. Using the linear-angular velocity relation $v = \omega R$, one writes for both discs

$$\omega_1 R_1 = v = \omega_2 R_2.$$

From this one finds

$$\frac{\omega_1}{\omega_2} = \frac{R_2}{R_1}.$$

The smaller disk is rotating faster.

39. The torque on the seesaw (Giancoli, chapter 8)

25. (II) Two blocks, each of mass m, are attached to the ends of a massless rod which pivots as shown in Fig. 8–40. Initially the rod is held in the horizontal position and then released. Calculate the magnitude and direction of the net torque on this system.

FIGURE 8–40 Problem 25.

Solution. With respect to the support point, the total torque is the sum of two torque due to gravity forces on the two masses:

$$\tau = mgL_1 - mgL_2 = mg(L_1 - L_2).$$

We have taken into account that the torque rotating clockwise is negative and the torque rotating counterclockwise is positive. If the rod has the mass M and is uniform, then its center of mass is in the middle of the rod at the distance from the support

$$L_M = \frac{L_1 + L_2}{2} - L_1 = \frac{L_2 - L_1}{2}.$$

In this case, the total torque on the system is

$$\tau = mg(L_1 - L_2) - MgL_M = \left(m + \frac{M}{2}\right) g(L_1 - L_2).$$

For $L_1 = L_2$ the torque is zero due to the symmetry.

40. Torque and work on a lever

A worker is lifting a heavy block of the mass M with the help of the lever. The far end of the lever is put on the solid floor, and the other end is slowly lifted by the worker. The length of the lever is L. The heavy block is lying on the lever at the distance x from the far end. What force does the worker apply? Prove that the total work by all forces in this process is zero. Prove that the work by the worker is converted into the potential energy of the heavy block.

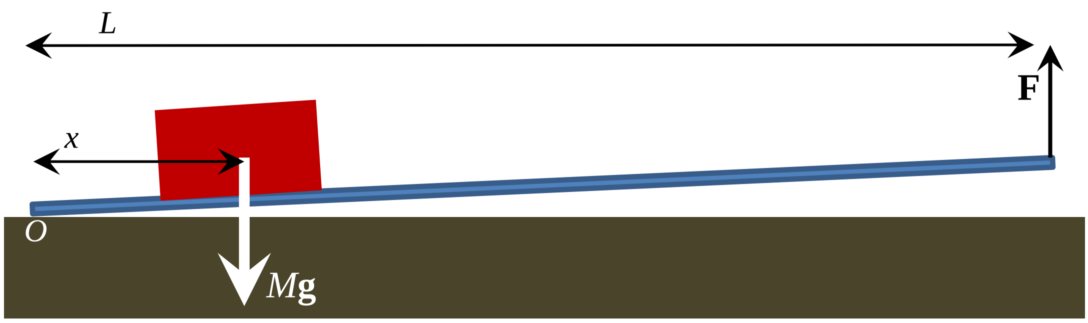
Solution. For the lever in the horizontal position (that is tacitly assumed in the problem's formulation) the rotational equilibrium condition has the form

$$\tau = FL - Mgx = 0.$$

From this one finds the force applied by the worker:

$$F = \frac{x}{L} Mg < Mg.$$

One can see that the lever allows gaining in the force.



Let us consider the work now. If the lever turns by a small angle $\Delta\theta$ out of the horizontal position, the elevations of the worker's end of the lever and the heavy block are

$$\Delta z = L\Delta\theta, \qquad \Delta z_M = x\Delta\theta,$$

respectively. The total work done by the worker and by the gravity force is

$$\Delta W = F\Delta z - Mg\Delta z_M.$$

Substituting the elevations listed above, one finds

$$\Delta W = FL\Delta\theta - Mgx\Delta\theta = (FL - Mgx)\Delta\theta.$$

The expression in the brackets is the total torque that is zero. Thus, the total work is zero, too. This is how the work-energy principle is working for the lever.

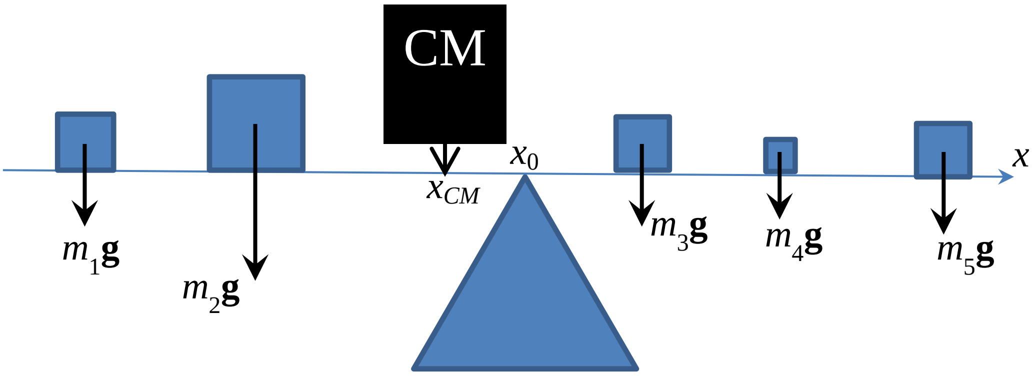
One can consider the work-energy balance as the equality between the work done by the worker and the increase of the potential energy of the heavy block:

$$\Delta E_{pot} = Mg\Delta z_M = F\Delta z,$$

where the last term in the work done by the worker.

41. The gravity force is applied to the CM

For a system of masses m_i placed at positions x_i on the horizontal axis prove that torque of the gravity force with respect to the CM is zero. This allows one to consider the gravity force as applied at the CM.



Solution. First, the position of the CM is given by

$$x_{CM} = \frac{1}{M}\sum_i m_i x_i\,, \qquad M = \sum_i m_i.$$

Then, with respect to support point x_0, the lever arms of the individual gravity forces are

$$L_i = x_i - x_0.$$

The total torque is given by

$$\tau = \sum_i m_i g L_i = g\sum_i m_i(x_i - x_0) = g\sum_i m_i x_i - g x_0 \sum_i m_i.$$

Using the definitions of x_{CM} and M above, one obtains

$$\tau = gMx_{CM} - gx_0 M = Mg(x_{CM} - x_0) = MgL_{CM},$$

where we have introduced the lever arm of the CM

$$L_{CM} \equiv x_{CM} - x_0.$$

Thus, the total torque can be calculated in a simplified way by considering the CM instead of all elementary masses:

$$\tau = \sum_i m_i g L_i \Rightarrow MgL_{CM}.$$

This means that, in the calculation of torques, one can consider the total gravity force Mg applied to the center of mass.

We now put the support point at the center of mass of the system, $x_0 = x_{CM}$ and thus $L_{CM} = 0$, then the torque of the gravity force with respect to this pivot point will be zero and the system will be in rotational equilibrium.

Using similar approach, one can prove that the gravity force can be considered as applied to the CM for any arrangement of masses in 3D. For this, one has to put the support under the

CM of the system, and check that the total gravity torques with respect to x and y axes are zero.

42. Calculation of the moment of inertia of a system of point masses (Giancoli, chapter 8)

31. (II) Calculate the moment of inertia of the array of point objects shown in Fig. 8–43 about (*a*) the vertical axis, and (*b*) the horizontal axis. Assume $m = 1.8\,\text{kg}$, $M = 3.1\,\text{kg}$, and the objects are wired together by very light, rigid pieces of wire. The array is rectangular and is split through the middle by the horizontal axis. (*c*) About which axis would it be harder to accelerate this array?

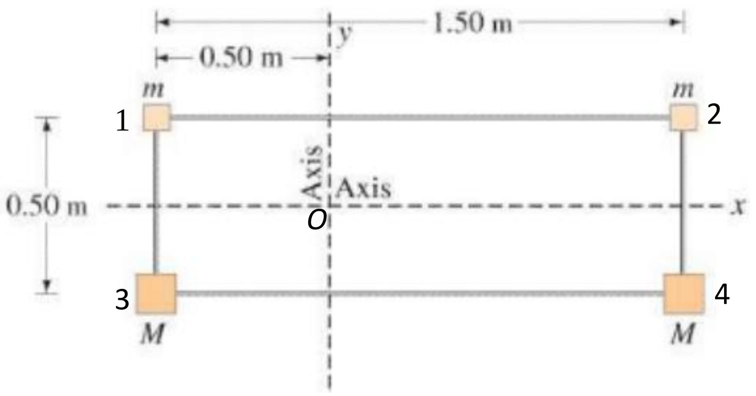


FIGURE 8–43 Problem 31.

Solution. This problem has to be prepared for solving, so that one gets a connection to the general formulas. First, we rewrite the problem in the intelligible way by numbering the masses as shown in the picture. One obtains

$$m_1 = m_2 = m = 1.8\,kg, \qquad m_3 = m_4 = M = 3.1\,kg.$$

After putting the origin of the coordinate system O at the intersection of the x and y axes the coordinates of the masses become

$$\mathbf{r}_1 = (-0.5, 0.25)\,m, \qquad \mathbf{r}_2 = (1, 0.25)\,m$$
$$\mathbf{r}_3 = (-0.5, -0.25)\,m, \qquad \mathbf{r}_4 = (1, -0.25)\,m.$$

The moment of inertia with respect to the x axis is given by

$$I_x = \sum_i m_i {y_i}^2 = m_1 {y_1}^2 + m_2 {y_2}^2 + m_3 {y_3}^2 + m_4 {y_4}^2.$$

Substituting the numbers, one obtains

$$I_x = 1.8 \times 0.25^2 + 1.8 \times 0.25^2 + 3.1 \times (-0.25)^2 + 3.1 \times (-0.25)^2 = 0.6125\,kg\,m^2.$$

The moment of inertia with respect to the y axis is given by

$$I_y = \sum_i m_i {x_i}^2.$$

Substituting the numbers, one obtains

$$I_y = 1.8 \times (-0.5)^2 + 1.8 \times 1^2 + 3.1 \times (-0.5)^2 + 3.1 \times 1^2 = 6.125\ kg\ m^2$$

that is exactly ten time greater than I_x. One could investigate what is special about the numbers in this problem.

Finally, one can calculate the moment of inertia with respect to z axis that is perpendicular to the x and y axes and goes through their intersection O. One has

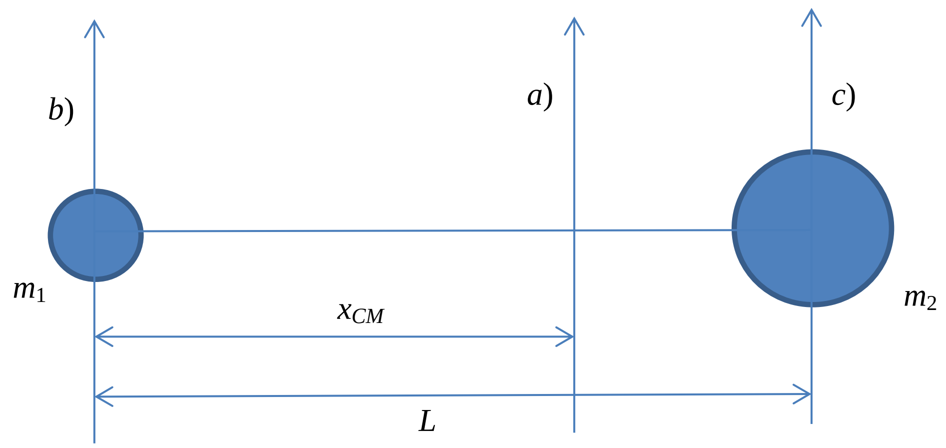
$$I_z = \sum_i m_i({x_i}^2 + {y_i}^2) = I_y + I_x.$$

Using the values calculated above, one obtains $I_z = 0.6125 + 6.125 = 6.7375\ kg\ m^2$.

<u>Note</u>: The relation $I_z = I_x + I_y$ holds for any arrangement of masses within the xy plane.

43. Checking the Steiner theorem for a system of two masses.

Consider the moment of inertia of a system of two masses, m_1 and m_2, with respect to the axes perpendicular to the line connecting the two masses. Consider the following cases: a) the axis goes through the CM of the system; b) the axis goes through the mass m_1; c) the axis goes through the mass m_2. Check if the Steiner theorem holds.



Solution. First, one has to find the position of the center of mass x_{CM}. Choosing the position of the mass m_1 as the origin of the x axis, one obtains

$$x_{CM} = \frac{0m_1 + Lm_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2}L,$$

as well as

$$L - x_{CM} = \frac{m_1}{m_1 + m_2}L.$$

Now, the moment of inertia with respect to the CM is given by

$$I_{CM} = m_1 x_{CM}^2 + m_2(L - x_{CM})^2.$$

Substituting here the expressions for x_{CM} and $L - x_{CM}$ above, one finds

$$I_{CM} = m_1 \left(\frac{m_2}{m_1 + m_2} L \right)^2 + m_2 \left(\frac{m_1}{m_1 + m_2} L \right)^2 = \frac{m_1 m_2}{m_1 + m_2} L^2.$$

The moment of inertia with respect to the axis going through the mass m_1 is

$$I_1 = m_2 L^2.$$

The moment of inertia with respect to the axis going through the mass m_2 is

$$I_2 = m_1 L^2.$$

Let us now obtain I_1 using the Steiner theorem, that is,

$$I_1 = I_{CM} + (m_1 + m_2) x_{CM}^2.$$

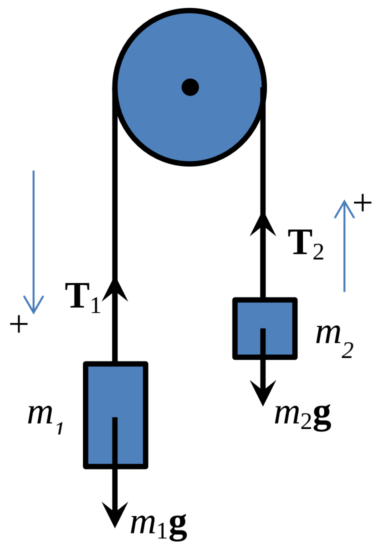
One obtains

$$\begin{aligned} I_1 &= \frac{m_1 m_2}{m_1 + m_2} L^2 + (m_1 + m_2) \left(\frac{m_2}{m_1 + m_2} L \right)^2 \\ &= \frac{m_1 m_2}{m_1 + m_2} L^2 + \frac{m_2^2}{m_1 + m_2} L^2 = m_2 L^2 \end{aligned}$$

that coincides with the result above obtained directly. The same calculation can be done for I_2.

44. Two masses on a massive block

A massless cord goes over a massive block of radius R and the moment of inertia I, and the masses m_1 and m_2 are suspended at the ends of the cord. Find the acceleration of the masses and the tension of the cord.



$$\begin{aligned} m_1 g - T_1 &= m_1 a \\ -m_2 g + T_2 &= m_2 a \\ \tau = (T_1 - T_2) R &= I\alpha \end{aligned}$$

$$a = \alpha R$$

Solution. A similar problem with a massless block was solved above in the section on dynamics. Here, the tension forces on the two sides of the block must be different to provide the torque on the block needed to give it the required angular acceleration. The latter satisfies the constraint tying it to the linear acceleration of the loads a:

$$a = \alpha R.$$

Again, to write down Newton's second law for both masses and the block, it is essential to choose the positive direction of motion that is down for one of the masses and up for the other, as shown on the sketch. The equations of motion for the masses (Newton's second law) and for the block, with explicit signs, are as follows

$$\begin{aligned} m_1 g - T_1 &= m_1 a \\ -m_2 g + T_2 &= m_2 a \\ \tau = (T_1 - T_2)R &= I\alpha. \end{aligned}$$

It is convenient to divide the last equation by R and express $\alpha = a/R$. This yields a system of three linear equations with three unknowns: T_1, T_2, and a:

$$\begin{aligned} m_1 g - T_1 &= m_1 a \\ -m_2 g + T_2 &= m_2 a \\ T_1 - T_2 &= \frac{I}{R^2} a. \end{aligned}$$

Adding these equations, one can eliminate the tension forces that yields

$$(m_1 - m_2)g = \left(m_1 + m_2 + \frac{I}{R^2}\right) a.$$

Thus

$$a = \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}} g.$$

One can see that the total mass of the system increases because of the massive block that contributes I/R^2 as its effective mass. If $m_1 > m_2$, the acceleration is positive and the masses are accelerating in the directions indicated in the sketch. Tension forces can now be found from the first and second equations of motion. One obtains

$$T_1 = m_1 g - m_1 a = m_1 g \left(1 - \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}}\right) = \frac{m_1\left(2m_2 + \frac{I}{R^2}\right)}{m_1 + m_2 + \frac{I}{R^2}} g$$

and

$$T_2 = m_2 a + m_2 g = m_2 g \left(\frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}} + 1\right) = \frac{m_2\left(2m_1 + \frac{I}{R^2}\right)}{m_1 + m_2 + \frac{I}{R^2}} g.$$

The difference of the tension forces is

$$T_1 - T_2 = \frac{I}{R^2} \frac{m_1 - m_2}{m_1 + m_2 + \frac{I}{R^2}} g = \frac{I}{R^2} a.$$

This is just the equation of motion for the block listed above. In the limit of the massless block, $I = 0$, the tension forces become the same, $T_1 = T_2$.

45. Rotational vs translational kinetic energy of a rolling body

A solid cylinder is rolling on a horizontal surface without slipping. Find the ratio of its rotational to translational kinetic energies.

Solution. The formulas for both parts of the kinetic energy have the form

$$E_{tr} = \frac{Mv^2}{2}, \qquad E_{rot} = \frac{I\omega^2}{2}.$$

The angular and linear velocities are related by

$$v = \omega R.$$

With the help of this (in the form $\frac{\omega}{v} = \frac{1}{R}$), for the ratio of the energies one obtains

$$\frac{E_{rot}}{E_{tr}} = \frac{I\omega^2}{Mv^2} = \frac{I}{MR^2}.$$

For the solid sphere, one has

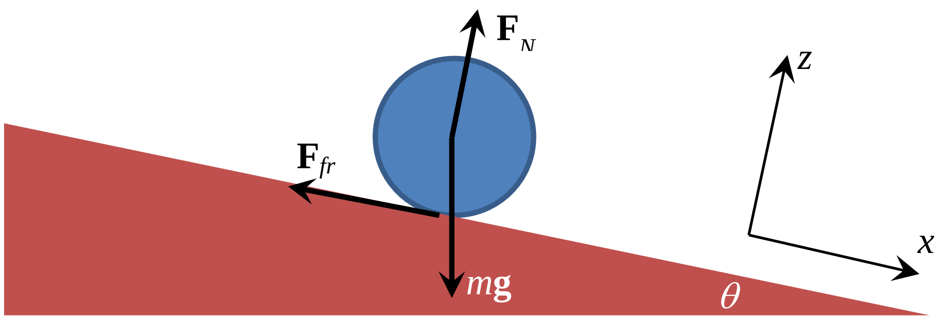
$$I = \frac{2}{5}MR^2.$$

Thus, in this case,

$$\frac{E_{rot}}{E_{tr}} = \frac{2}{5}.$$

46. Rolling down the incline

A solid sphere is put on the incline with the angle θ. Find the acceleration of the center of mass of the sphere if it is rolling without slipping.



Solution. The three forces acting on the rolling sphere are shown in the sketch above. With respect to the center of the sphere, only the friction force has a nonzero torque, and this force creates the angular acceleration in the direction of rolling. This is why the friction force is directed back. Because of this, the acceleration of the center of the rolling object is always smaller than that of a body sliding down the incline without friction, $g \sin\theta$. The equations of motion are Newton's second law for translational motion of the center of mass and for the rotation around the CM. The former projected on the x-axis reads

$$mg\sin\theta - F_{fr} = ma,$$

the same as for the body sliding with friction. The rotational Newton's law reads

$$\tau = F_{fr}R = I\alpha,$$

where R is the radius of the sphere. Finally, there is a constraint equation relating α and a in the condition of non-slipping:

$$a = \alpha R.$$

Thus, we have a system of three equations with three unknowns: F_{fr}, a, and α, that is well defined and can be solved. Dividing the rotational equation by R and expressing α via a, one obtains

$$F_{fr} = \frac{I}{R^2}a.$$

Substituting this into the first equation and solving for a, one obtains

$$a = \frac{mg\sin\theta}{m + \frac{I}{R^2}} = \frac{g\sin\theta}{1 + \frac{I}{mR^2}}$$

that is smaller than $a = g\sin\theta$ for the body sliding without friction. The formula above is good for any rolling body: a wheel, a cylinder, a solid or a hollow sphere. For the solid sphere one has

$$I = \frac{2}{5}mR^2,$$

thus, in this case,

$$a = \frac{g\sin\theta}{1 + \frac{2}{5}} = \frac{5}{7}g\sin\theta.$$

47. The minimal friction coefficient for rolling down the incline without slipping

In the problem above, find the minimal value of the static friction coefficient, for which rolling without slipping is possible.

Solution. The no-slipping condition requires

$$F_{fr} \leq \mu_s F_N.$$

Substituting

$$F_{fr} = \frac{I}{R^2}a = \frac{I}{R^2}\frac{g\sin\theta}{1 + \frac{I}{mR^2}} = \frac{mg\sin\theta}{\frac{mR^2}{I} + 1}, \qquad F_N = mg\cos\theta,$$

one obtains the condition

$$\frac{mg\sin\theta}{\frac{mR^2}{I}+1} \leq \mu_s mg\cos\theta$$

from which follows

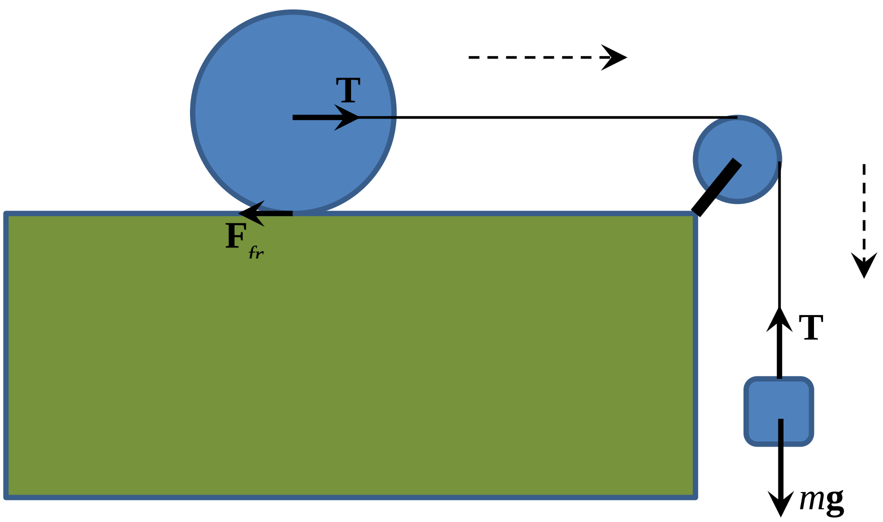
$$\mu_s \geq \frac{\tan\theta}{\frac{mR^2}{I}+1}.$$

For the solid sphere, this condition becomes

$$\mu_s \geq \frac{\tan\theta}{\frac{5}{2}+1} = \frac{2}{7}\tan\theta.$$

48. Rolling cylinder pulled at its center

There is a cylinder of mass M that can roll without slipping on a horizontal surface. It is pulled by its center by a cord going over a massless block and then down to the mass m suspended on the cord. Find the linear acceleration of the system, the tension of the cord, and the friction force acting on the cylinder.



Solution. As the block is massless, the tension forces on its sides are the same. With the obvious choice of the positive direction of motion, Newton's second law for the suspended mass has the form

$$mg - T = ma.$$

For the rolling cylinder, there are two equations of motion for its translational and rotational motion:

$$T - F_{fr} = Ma$$
$$\tau = F_{fr}R = I\alpha.$$

Dividing the rotational equation by R and expressing the angular acceleration via the linear acceleration as $\alpha = a/R$, for the rolling cylinder one obtains

$$T - F_{fr} = Ma$$
$$F_{fr} = \frac{I}{R^2}a.$$

Summing the equation for the suspended mass and thsese two equations, one eliminates T and F_{fr} and obtains

$$mg = \left(m + M + \frac{I}{R^2}\right)a,$$

where the expression in the brackets is the effective mass of the system. From here one finds the acceleration of the system:

$$a = \frac{m}{m + M + \frac{I}{R^2}}g.$$

Now, the friction force can be obtained from the rotational equation:

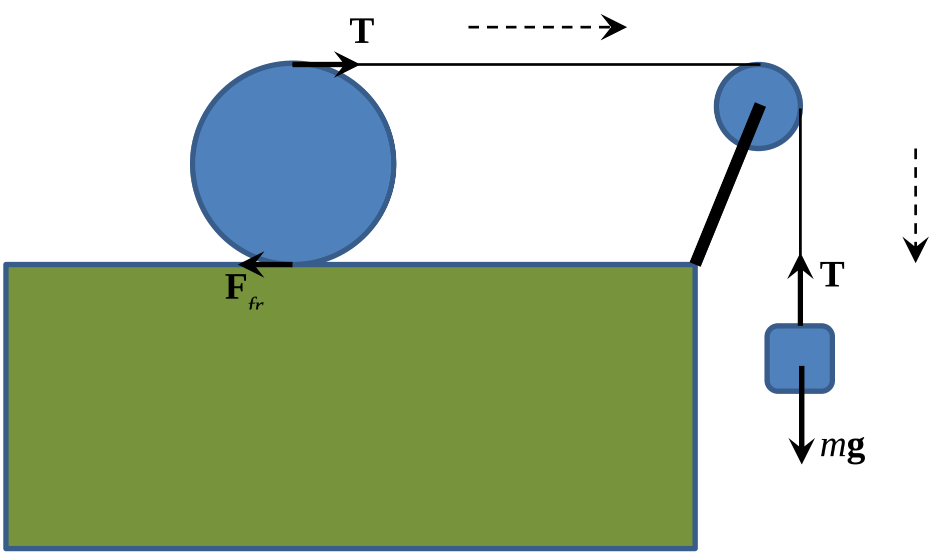
$$F_{fr} = \frac{I}{R^2}a = \frac{m\frac{I}{R^2}}{m + M + \frac{I}{R^2}}g$$

The cord tension force can be found from the equation for the suspended mass:

$$T = mg - ma = mg\left(1 - \frac{m}{m + M + \frac{I}{R^2}}\right) = \frac{m\left(M + \frac{I}{R^2}\right)}{m + M + \frac{I}{R^2}}g.$$

49. Rolling cylinder pulled at its top

There is a cylinder of mass M that can roll without slipping on a horizontal surface. It is pulled by its top by a cord going over a massless block and then down to the mass m suspended on the cord. Find the linear acceleration of the system, the tension of the cord, and the friction force acting on the cylinder.



Solution. As the block is massless, the tension forces on its sides are the same. With the obvious choice of the positive direction of motion, Newton's second law for the suspended mass has the form

$$mg - T = ma.$$

For the rolling cylinder, there are two equations of motion for its translational and rotational motion:

$$T - F_{fr} = Ma$$
$$\tau = F_{fr}R + TR = I\alpha.$$

Dividing the rotational equation by R and expressing the angular acceleration via the linear acceleration as $\alpha = a/R$, for the rolling cylinder one obtains

$$T - F_{fr} = Ma$$
$$F_{fr} + T = \frac{I}{R^2}a.$$

Summing the doubled equation for the suspended mass and these two equations, one eliminates T and F_{fr} and obtains

$$2mg = \left(2m + M + \frac{I}{R^2}\right)a.$$

From here one finds the acceleration of the system:

$$a = \frac{2m}{2m + M + \frac{I}{R^2}}g = \frac{m}{m + \frac{1}{2}\left(M + \frac{I}{R^2}\right)}g.$$

The mass combination in the denominator is the effective mass of the syste. It is smaller than in the preceding problem because the contribution of the rolling body enters with the factor ½.

The cord tension force can be found from the equation for the suspended mass:

$$T = mg - ma = mg\left(1 - \frac{2m}{2m + M + \frac{I}{R^2}}\right) = \frac{m\left(M + \frac{I}{R^2}\right)}{2m + M + \frac{I}{R^2}}g.$$

The friction force can be obtained by subtracting translational equation for the cylinder from the rotational equation:

$$F_{fr} + T - \left(T - F_{fr}\right) = 2F_{fr} = \frac{I}{R^2}a - Ma = \left(\frac{I}{R^2} - M\right)a.$$

From this one obtains

$$F_{fr} = \frac{m\left(\frac{I}{R^2} - M\right)}{2m + M + \frac{I}{R^2}}g$$

For the solid cylinder,

$$I = \frac{1}{2}MR^2,$$

so that the friction force becomes

$$F_{fr} = -\frac{\frac{mM}{2}}{2m + \frac{3M}{2}} g = -\frac{mM}{4m + 3M} g.$$

Note that the friction force is negative. This means that it is directed not to the left, as shown in the sketch, but to the right. This is the case for most round bodies having the factor smaller than 1 in the moment of inertia. For the hollow cylinder

$$I = MR^2,$$

and the friction force is zero!

50. Angular momentum conservation (Giancoli, chapter 8)

60. (II) A uniform disk turns at 2.4 rev/s around a frictionless spindle. A nonrotating rod, of the same mass as the disk and length equal to the disk's diameter, is dropped onto the freely spinning disk, Fig. 8–49. They then both turn around the spindle with their centers superposed. What is the angular frequency in rev/s of the combination?

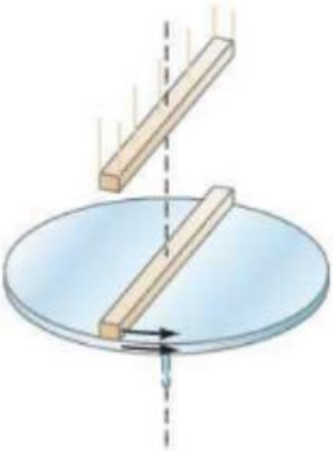


FIGURE 8–49
Problem 60.

Solution. As there are no external torque acting on the system disk+rod, its algular momentum is conserved, $L = const$, or, in terms of the initial and final states,

$$L = I_{disk}\omega_i = (I_{disk} + I_{rod})\omega_f.$$

Thus, the final angular velocity ω_f is

$$\omega_f = \frac{I_{disk}}{I_{disk} + I_{rod}}\omega_i.$$

The moments of inertia of the disk is given by

$$I_{disk} = \frac{1}{2}MR^2.$$

The moment of inertia of the rod (with respect to its center) is

$$I_{rod} = \frac{1}{12}Ml^2 = \frac{1}{12}M(2R)^2 = \frac{1}{3}MR^2.$$

Substituting these expressions, one obtains

$$\omega_f = \frac{\frac{1}{2}MR^2}{\frac{1}{2}MR^2 + \frac{1}{3}MR^2}\omega_i = \frac{3}{5}\omega_i.$$

Substituting the numbers, one obtains

$$\omega_f = \frac{3}{5} \times 2.3\frac{rev}{s} = 1.38\frac{rev}{s}.$$

Physics part II

Electrostatics

51. Electric field from a collection of charges

Electric charges $Q_1 = Q$, $Q_2 = 2Q$, and $Q_3 = 3Q$ are placed at $\mathbf{r}_1 = (1,0,0)a$, $\mathbf{r}_2 = (0,1,0)a$, and $\mathbf{r}_3 = (0,0,1)a$. Find the electric field $\mathbf{E}$ at $\mathbf{r} = (1,1,1)a$.

Solution. This problem deals the electric field in the general vector form in 3D with vectors given by their components, such as $\mathbf{A} = (A_x, A_y, A_z)$. In the notations above a is a length in meters. The basic formula for the electric field of a point charge Q following from the Coulomb's law reads

$$\mathbf{E} = k\frac{Q}{r^2}\frac{\mathbf{r}}{r}.$$

Here the first part gives the magnitude of the electric-field vector $\mathbf{E}$ while the unit vector $\mathbf{r}/r$ gives its direction. (Check that it is indeed a unit vector by calculating its magnitude that should be 1). This formula implies that the charge Q is in the origin of a coordinate system and the position of the observation point is given by the vector $\mathbf{r}$ that goes from the origin to the observation point. However, if there are several charges, one cannot put them all in the origin. A more general formula for the electric field $\mathbf{E}_1$ created by the charge Q_1 that is not necessarily in the origin reads

$$\boldsymbol{E}_1 = k\frac{Q_1}{|\mathbf{r}-\mathbf{r}_1|^2}\frac{\mathbf{r}-\mathbf{r}_1}{|\mathbf{r}-\mathbf{r}_1|}.$$

Here the vector $\mathbf{r}-\mathbf{r}_1$ goes from the charge Q_1 located at $\mathbf{r}_1$ to the observation point $\mathbf{r}$. This formula can be generalized for several charges put at different positions:

$$\mathbf{E} = \sum_i k\frac{Q_i}{|\mathbf{r}-\mathbf{r}_i|^2}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|} = k\sum_i Q_i\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3}.$$

This formula can be used as a starting point for solving our problem. It is convenient to pre-calculate

$$\mathbf{r}-\mathbf{r}_1 = (1,1,1)a - (1,0,0)a = (0,1,1)a$$
$$\mathbf{r}-\mathbf{r}_2 = (1,1,1)a - (0,1,0)a = (1,0,1)a$$
$$\mathbf{r}-\mathbf{r}_3 = (1,1,1)a - (0,0,1)a = (1,1,0)a$$

and

$$|\mathbf{r}-\mathbf{r}_1| = \sqrt{0^2+1^2+1^2}a = \sqrt{2}a$$
$$|\mathbf{r}-\mathbf{r}_2| = \sqrt{2}a$$
$$|\mathbf{r}-\mathbf{r}_3| = \sqrt{2}a.$$

Now

$$\mathbf{E} = k\left[\frac{Q}{a^2}\frac{(0,1,1)}{2^{\frac{3}{2}}} + \frac{2Q}{a^2}\frac{(1,0,1)}{2^{\frac{3}{2}}} + \frac{3Q}{a^2}\frac{(1,1,0)}{2^{\frac{3}{2}}}\right]$$

$$= k\frac{Q}{a^2}\frac{(0,1,1) + 2(1,0,1) + 3(1,1,0)}{2^{\frac{3}{2}}}$$
$$= k\frac{Q}{a^2}\frac{(0 + 2 + 3,1 + 0 + 3,1 + 2 + 0)}{2^{\frac{3}{2}}}$$
$$= k\frac{Q}{a^2}\frac{(5,4,3)}{2^{\frac{3}{2}}}.$$

Understanding this problem gives a student a tool to find the electric field from a collection of point charges in the most general form.

52. Forces on charges *Q* put in corners of a rectangle with sides *a* and *b*

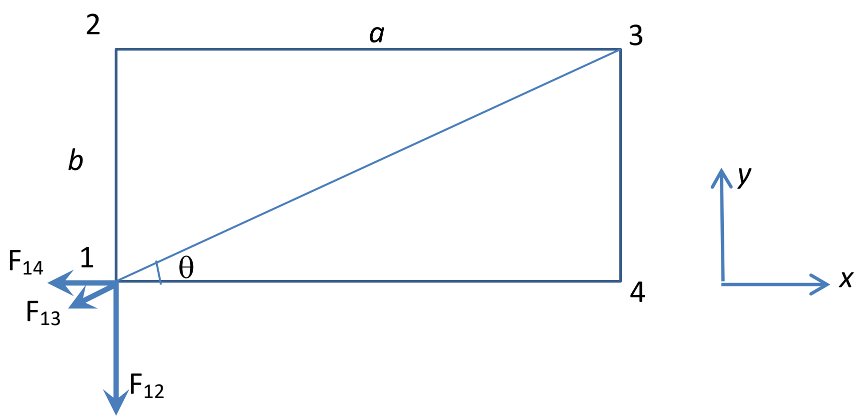
Charges Q are put in the corners of a rectangle with sides a and b. Find the magnitude of the force acting on each charge.

Solution. Here we use the Coulomb's law for the interaction of two point charges Q_1 and Q_2 at positions $\mathbf{r}_1$ and $\mathbf{r}_2$

$$\mathbf{F}_{12} = k\frac{Q_1Q_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$

This is the force acting on charge 1 from charge 2. The force is directed along the line connecting the two charges. Vector $\mathbf{r}_1 - \mathbf{r}_2$ goes from charge 2 to charge 1. It is repulsive if the charges have the same sign and attractive if the charges have different signs.

In the problem, position vectors are not explicitly given but we know that the lines connecting the charges are sides and diagonals of the rectangle. Thus, we do not need the full form of the formula above.



We have to introduce the axes x and y and project all forces on these axes to and add the forces component by component. The magnitudes of the forces acting on each of the charges are the same, only their directions are different. We will consider the force acting on the charge 1,

$$\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_{14}.$$

The force from charge 2 has only y component, while the force from charge 4 has only x-component. The force from charge 3 has both *x*- and *y*-components and is defined by the distance $\sqrt{a^2+b^2}$, as well as by

$$\cos\theta = \frac{a}{\sqrt{a^2+b^2}}, \qquad \sin\theta = \frac{b}{\sqrt{a^2+b^2}}.$$

One obtains

$$F_{1,x} = -k\frac{Q^2}{a^2} - k\frac{Q^2}{a^2+b^2}\cos\theta = -k\frac{Q^2}{a^2} - k\frac{Q^2 a}{(a^2+b^2)^{\frac{3}{2}}}$$
$$F_{1,y} = -k\frac{Q^2}{b^2} - k\frac{Q^2}{a^2+b^2}\sin\theta = -k\frac{Q^2}{b^2} - k\frac{Q^2 b}{(a^2+b^2)^{\frac{3}{2}}}.$$

Now the magnitude of the force is given by

$$F_1 = \sqrt{{F_{1,x}}^2 + {F_{1,y}}^2} = kQ^2\sqrt{\left(\frac{1}{a^2}+\frac{a}{(a^2+b^2)^{\frac{3}{2}}}\right)^2 + \left(\frac{1}{b^2}+\frac{b}{(a^2+b^2)^{\frac{3}{2}}}\right)^2}.$$

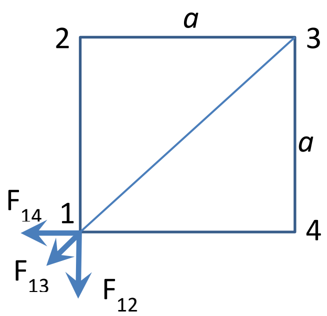
Forces acting on other charges have the same magnitude, so that one can discard the subscript 1. In the case *b*=*a* the result simplifies to

$$F = k\frac{Q^2}{a^2}\sqrt{2\left(1+\frac{1}{2^{3/2}}\right)^2} = k\frac{Q^2}{a^2}\left(\sqrt{2}+\frac{1}{2}\right).$$

This result can be obtained directly for the problem with the square instead of the rectangle that is much simpler (the next problem).

53. Forces on charges *Q* put in corners of a square

This problem is a particular case of the more general preceding problem. In the case of the square with all charges the same, there is a symmetry that allows one to obtain the solution without invoking the full vector formalism.



Solution. By the symmetry it is clear that the force on charge 1 is directed along the diagonal shown in the drawing. Thus, one has to project all three forces on this direction that yields

$$F = 2k\frac{Q^2}{a^2}\cos 45° + k\frac{Q^2}{(\sqrt{2}a)^2}.$$

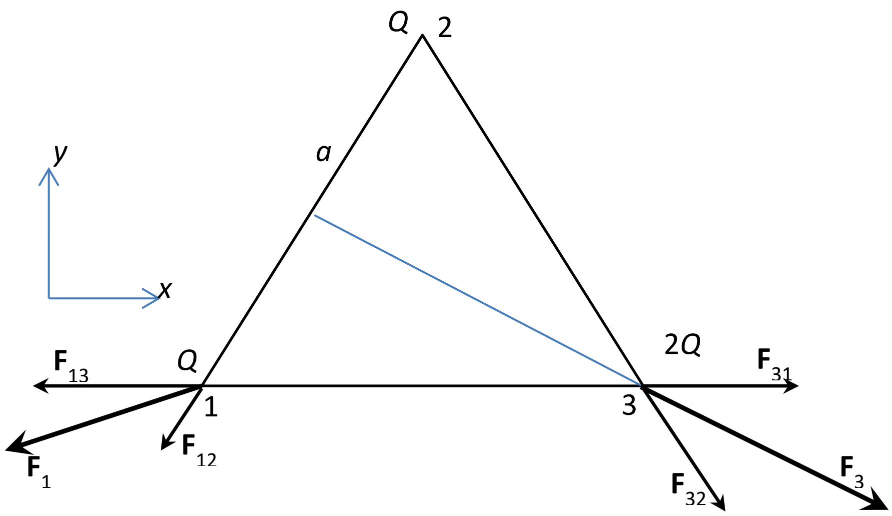
Here the factor 2 accounts for the two contributions from charges 2 and 4. The second term accounts for charge 3. With $\cos 45° = \sqrt{2}/2$ one obtains

$$F = k\frac{Q^2}{a^2}\left(\sqrt{2} + \frac{1}{2}\right).$$

This result has been obtained as a particular case of the preceding problem and it can be used for testing the validity of the latter.

54. Forces on different charges at an equilateral triangle

Charges Q, Q, and $2Q$ are put at the corners of an equilateral triangle with side a. Find the magnitudes of the forces acting on each charge.



Solution. Here the system has symmetry with respect to the triangle's bisectrix shown in the drawing. By the symmetry, the force on charge $2Q$,

$$\mathbf{F}_3 = \mathbf{F}_{31} + \mathbf{F}_{32},$$

is directed along the bisectrix. The magnitude of this force is given by

$$F_3 = k\frac{Q \times 2Q}{a^2} 2\cos 30° = k\frac{Q^2}{a^2} 2\sqrt{3}.$$

To the contrary, there is no symmetry that could help to simplify the forces acting on charges Q. Thus, one has to add the vectors in

$$\mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13},$$

component by component using x- and y-axes shown in the drawing. One obtains

$$F_{1,x} = F_{12,x} + F_{13,x} = -k\frac{Q^2}{a^2}\cos 60° - k\frac{2Q^2}{a^2} = -k\frac{Q^2}{a^2}\left(\frac{1}{2} + 2\right) = -k\frac{Q^2}{a^2}\frac{5}{2}$$

and

$$F_{1,y} = F_{12,y} + F_{13,y} = -k\frac{Q^2}{a^2}\sin 60° + 0 = -k\frac{Q^2}{a^2}\frac{\sqrt{3}}{2}.$$

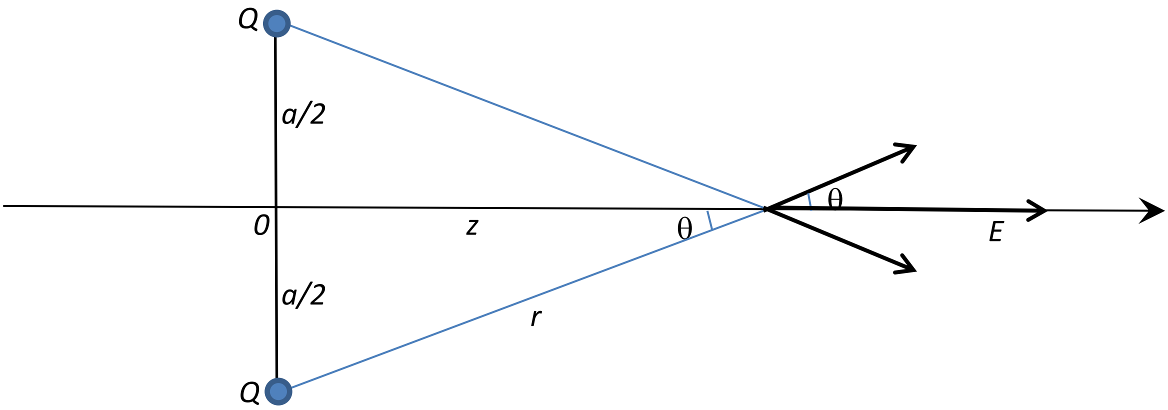
Now the magnitude of $\mathbf{F}_2$ is given by

$$F_1 = \sqrt{{F_{1,x}}^2 + {F_{1,y}}^2} = k\frac{Q^2}{a^2}\sqrt{\left(\frac{5}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = k\frac{Q^2}{a^2}\frac{1}{2}\sqrt{25+3} = k\frac{Q^2}{a^2}\frac{\sqrt{28}}{2} = k\frac{Q^2}{a^2}\sqrt{7}.$$

The force acting on the other charge Q has the same magnitude: $F_2 = F_1$.

55. Electric field at the center line between two equal charges

Find the electric field $E(z)$ along the straight line going between two equal charges Q put at the distance a from each other. Analyze particular cases.



Solution. Put the origin of the coordinate system O in the point between the two charges and direct z-axis in the right horizontal direction. From the symmetry it follows that the electric field on the center line is directed horizontally. It is given by

$$E = 2k\frac{Q}{r^2}\cos\theta,$$

where the factor 2 takes care for the two contributions into the result from the upper and lower charges. Using

$$r = \sqrt{z^2 + \left(\frac{a}{2}\right)^2}, \qquad \cos\theta = \frac{z}{r},$$

one finally obtains

$$E = k\frac{2Qz}{\left(z^2 + \left(\frac{a}{2}\right)^2\right)^{3/2}}.$$

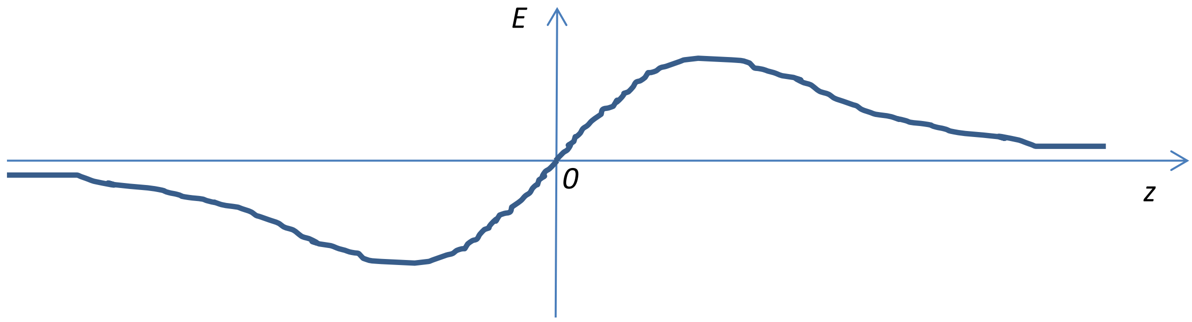
One particular case is the point between the charges, z=0. Here the electric field from the two charges cancel each other and E=0. The formula above reproduces this result that serves as its check.

Another particular case is the region far away from the charges, $z \gg a$. Here one can neglect the term $(a/2)^2$ in the denominator of the formula after which the result becomes

$$E = k\frac{2Q}{z^2}.$$

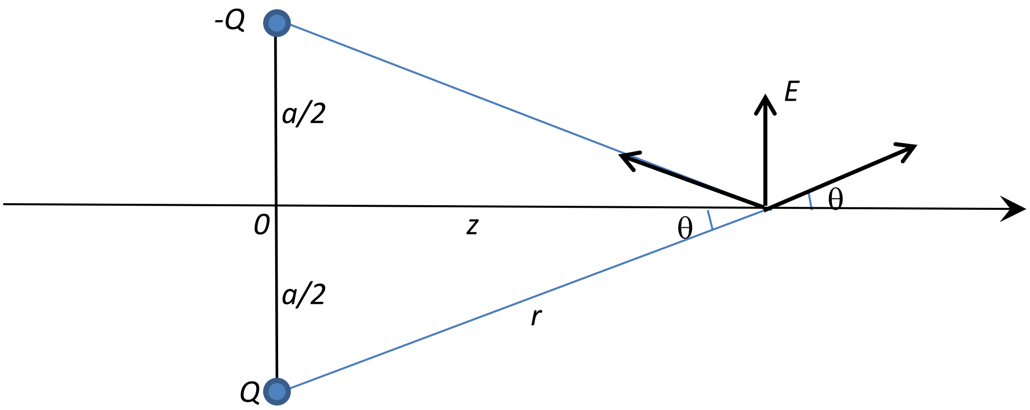
This is is nothing else than the electric field of the charge 2Q at the distance z. This is an expected result as from large distances the two charges close to each other are looking as one double charge. This is another check on our general formula.

The plot of the dependence $E(z)$ is shown below. There is a maximum of E at the distance z of order a.



56. Electric field at the center line between two opposite charges

Find the electric field E(z) along the straight line going between two opposite charges Q and $-Q$ put at the distance a from each other. Analyze particular cases.



Solution. As the upper charge is negative, its electric field at the observation point is directed toward it, as shown in the drawing. The resulting electric field **E** is thus directed up. It is given by

$$E = 2k\frac{Q}{r^2}\sin\theta.$$

Using

$$r = \sqrt{z^2 + \left(\frac{a}{2}\right)^2}, \qquad \sin\theta = \frac{a/2}{r},$$

one finally obtains

$$E = k\frac{Qa}{\left(z^2 + \left(\frac{a}{2}\right)^2\right)^{3/2}}.$$

One particular case of this formula is z=0, the observation point between the charges. In this case one obtains

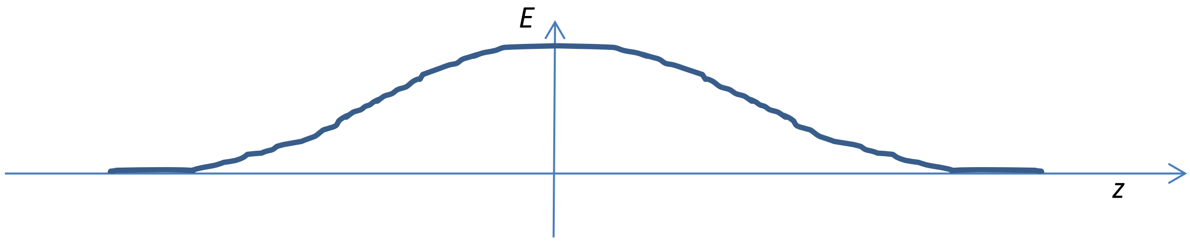
$$E = k\frac{8Q}{a^2}$$

that is the doubled electric field at the distance $a/2$ from the charge Q.

At large distances , $z>>a$, one can neglect the term $(a/2)^2$ in the denominator of the formula after which the result becomes

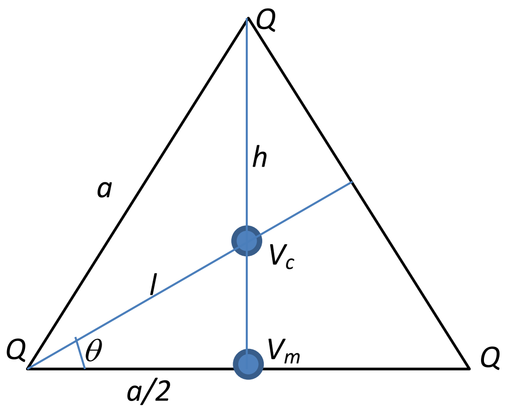
$$E = k\frac{Qa}{z^3}.$$

As the power in the denominator is three rather than two, one concludes that the electric field produced by two opposite charges decreases faster at large distances than the field produces by one charge. Indeed, looking from large distances, one cannot see the system of two opposite charges as one effective charge (as in the preceding problem) because the sum of the two charges is zero. Such a system is called "electric dipole".



57. Electric potentials in the center of the equilateral triangle of charges and in the middle of a side.

Consider an equilateral triangle with side a having point charges Q at its corners. Compare electric potentials in the center of the triangle and in the middle of its side. What is your expectation? Which potential is higher?



Solution. Let us denote the electric potential in the center of the triangle V_c and the electric potential in the middle of the side V_m. At the center one has

$$V_c = 3k\frac{Q}{l},$$

while in the middle of the side

$$V_m = 2k\frac{Q}{a/2} + k\frac{Q}{h}.$$

With

$$h = a\cos 30° = a\frac{\sqrt{3}}{2}, \qquad l = \frac{a/2}{\cos 30°} = \frac{a}{\sqrt{3}}$$

one finally obtains

$$V_c = k\frac{Q}{a}3\sqrt{3} \approx k\frac{Q}{a} \times 5.196$$

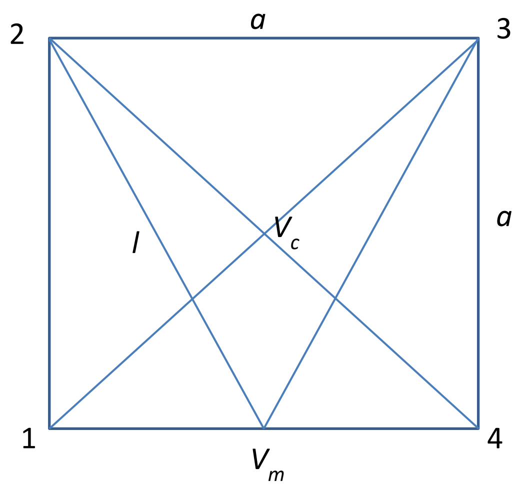
and

$$V_m = k\frac{Q}{a}\left(4 + \frac{2}{\sqrt{3}}\right) \approx k\frac{Q}{a} \times 5.155 < V_c.$$

These two values are so close to each other that no intuition in the world can figure out what potential is higher without the actual calculating the potentials.

58. Electric potentials in the center of a square of charges and in the middle of a square's side.

Consider a square with side *a* having point charges *Q* at its corners. Compare electric potentials in the center of the square and in the middle of its side. What is your expectation? Which potential is higher?



Solution. Let us denote the electric potential in the center of the triangle V_c and the electric potential in the middle of the side V_m. At the center one has

$$V_c = 4k\frac{Q}{a/\sqrt{2}} = k\frac{Q}{a}4\sqrt{2} = k\frac{Q}{a} \times 5.657$$

In the middle of a site there are two different contributions:

$$V_m = 2k\frac{Q}{a/2} + 2k\frac{Q}{l}.$$

The distance l can be obtained from the Pythagoras' theorem:

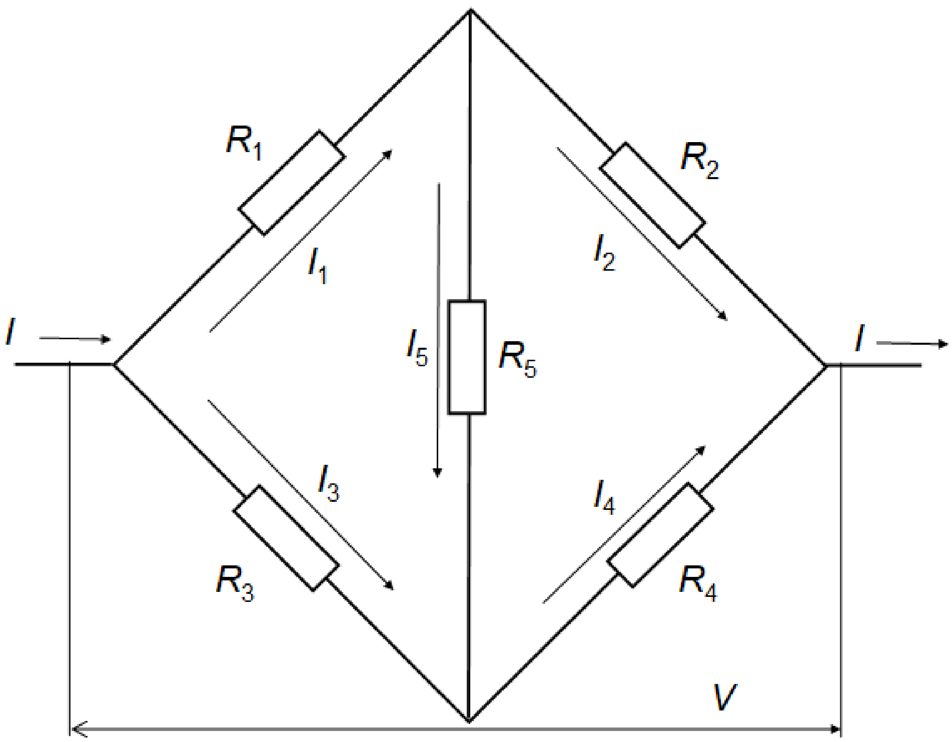
$$l = \sqrt{a^2 + \left(\frac{a}{2}\right)^2} = a\frac{\sqrt{5}}{2}.$$

Finally,

$$V_m = k\frac{Q}{a}4\left(1 + \frac{1}{\sqrt{5}}\right) = k\frac{Q}{a} \times 5.789 > V_{c.}$$

Electric circuits

59. The Wheatstone bridge



Not all circuits can be calculated using the formulas for the serial and parallel connection of resistors. The simplest example is the so-called Wheatstone bridge. The task is to calculate the effective resistance of the bridge as $R = V/I$. For this, one has to write down the Kirchhoff's equations and solve them (using computer algebra) for I for a given V. In the limits $R_5 = 0$ and $R_5 \to \infty$ the problem simplifies, and these solutions can be used to check the general formula for R.

Solution. The Kirchhoff equations for the currents are

$$I = I_1 + I_3, \qquad I_1 = I_2 + I_5, \qquad I_3 + I_5 = I_4$$

and Kirchhoff equations for the voltages, combined with the Ohm's law, have the form

$$R_1 I_1 + R_2 I_2 = V, \qquad R_3 I_3 + R_4 I_4 = V, \qquad R_1 I_1 + R_5 I_5 + R_4 I_4 = V.$$

One can add more equations but they are not independent and follow from the equations written above. There are six unknowns, all currents, and six equations, so that the system of these linear equations is well defined and can be solved. It is difficult to do by hand but one can use computer algebra. Surprisingly, one obtains a formula compact enough:

$$R = \frac{V}{I} = \frac{(R_1 + R_2)(R_3 + R_4)R_5 + (R_1 + R_3)R_2 R_4 + (R_2 + R_4)R_1 R_3}{(R_1 + R_2 + R_3 + R_4)R_5 + (R_1 + R_3)(R_2 + R_4)}.$$

For $R_5 = 0$, one can set this in the formula above and obtain

$$R = \frac{R_1 R_3}{R_1 + R_3} + \frac{R_2 R_4}{R_2 + R_4}.$$

For $R_5 \to \infty$, one can neglect in the general formula the terms in the numerator and the denominator that do not contain R_5. After this R_5 cancels and one obtains

$$R = \frac{(R_1 + R_2)(R_3 + R_4)}{R_1 + R_2 + R_3 + R_4}.$$

These two limits can be considered independently. For $R_5 = 0$, the upper and the lower corners of the circuit are short-circuited, thus one has the parallel connection of resistors R_1 and R_3 and the parallel connection of resistors R_2 and R_4. These two groups of resistors are connected serially. Thus

$$R = \frac{1}{\frac{1}{R_1} + \frac{1}{R_3}} + \frac{1}{\frac{1}{R_2} + \frac{1}{R_4}} = \frac{R_1 R_3}{R_1 + R_3} + \frac{R_2 R_4}{R_2 + R_4}.$$

For $R_5 \to \infty$, there is simply no resistor R_5 in the circuit. Then we have resistors R_1 and R_2 connected serially, same for R_3 and R_4, and these two groups are connected in parallel. Thus one obtains

$$R = \frac{1}{\frac{1}{R_1 + R_2} + \frac{1}{R_3 + R_4}} = \frac{(R_1 + R_2)(R_3 + R_4)}{R_1 + R_2 + R_3 + R_4}.$$

One more feature of the Bridge is the following. If $R_1 = R_3$ and $R_2 = R_4$, the circuit is symmetric with respect to the horizontal central line. Thus, in this case, the current through R_5 does not flow and one can remove R_5 (make a break or short-circuiting at its place). One obtains

$$R = \frac{R_1 + R_2}{2}$$

that also follows from all the formulas above as a particular case.

More complicated electric circuits can be treated similarly. Typically, the system of linear equations has to be solved numerically as the analytical solution becomes too cumbersome.

60. Problem 83, end of Chapter 19 of the Giancoli book, 6th edition

83. The current through the 20-Ω resistor in Fig. 19–68 does not change whether the two switches S_1 and S_2 are both open or both closed. Use this clue to determine the value of the unknown resistance R.

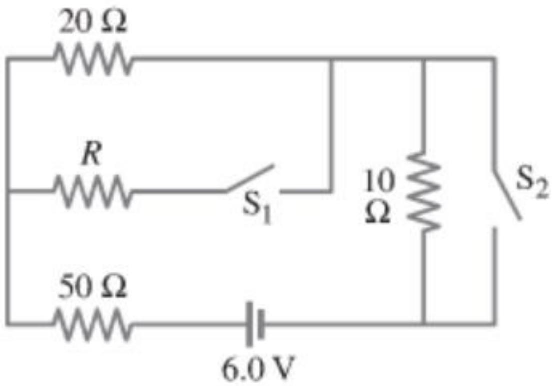


FIGURE 19–68 Problem 83.

Solution. Without redrawing the circuit with general notations for resistors, one can write down the equations as follows. For both switches open one has

$$(R_{50} + R_{20} + R_{10})I = \mathcal{E}, \qquad I_{20} = I.$$

For both switches closed for the total current I one has

$$\left(R_{50} + \frac{1}{\frac{1}{R_{20}} + \frac{1}{R}}\right)I = \mathcal{E}.$$

The voltage V_R on the group of parallel resistors R_{20} and R is

$$V_R = \frac{1}{\frac{1}{R_{20}} + \frac{1}{R}} I = \frac{1}{\frac{1}{R_{20}} + \frac{1}{R}} \times \frac{\mathcal{E}}{R_{50} + \frac{1}{\frac{1}{R_{20}} + \frac{1}{R}}} = \frac{\mathcal{E}}{R_{50}\left(\frac{1}{R_{20}} + \frac{1}{R}\right) + 1}.$$

The current I_{20} is given by

$$I_{20} = \frac{V_R}{R_{20}} = \frac{\mathcal{E}}{R_{50}\left(\frac{1}{R_{20}} + \frac{1}{R}\right) + 1} \frac{1}{R_{20}}.$$

Equating it with I_{20} found from the first equation, one obtains

$$\frac{\mathcal{E}}{R_{50} + R_{20} + R_{10}} = \frac{\mathcal{E}}{R_{50}\left(\frac{1}{R_{20}} + \frac{1}{R}\right) + 1} \frac{1}{R_{20}}.$$

This is the equation for R. Canceling Ɛ and simplifying the fractions, one obtains

$$\left(R_{50}\left(\frac{1}{R_{20}}+\frac{1}{R}\right)+1\right)R_{20}=R_{50}+R_{20}+R_{10}$$

and further

$$R_{20}R_{50}\left(\frac{1}{R_{20}}+\frac{1}{R}\right)=R_{50}+R_{10}$$

and

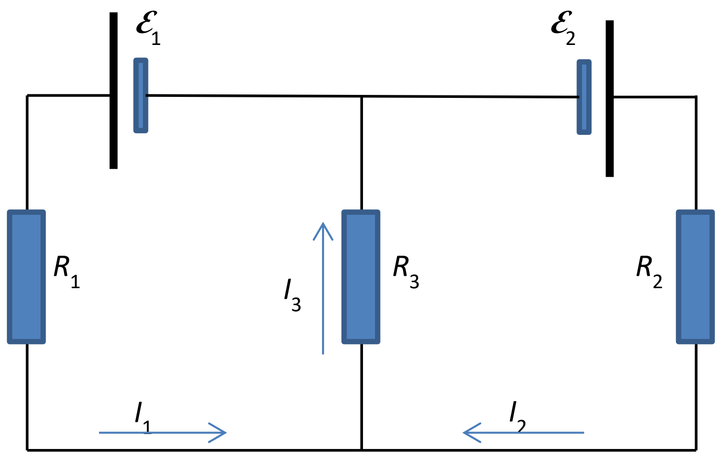
$$\frac{1}{R}=-\frac{1}{R_{20}}+\frac{R_{50}+R_{10}}{R_{20}R_{50}}=\frac{1}{R_{20}}\left(-1+\frac{R_{50}+R_{10}}{R_{50}}\right)=\frac{-R_{50}+R_{50}+R_{10}}{R_{20}R_{50}}=\frac{R_{10}}{R_{20}R_{50}}.$$

Finally,

$$R=\frac{R_{20}R_{50}}{R_{10}}=\frac{20\times 50}{10}=100\ \Omega.$$

61.A circuit with two batteries and three resistors

Find the currents and voltages for each of three resistors in the following circuit



Solution. Choose the positive directions of the currents according to the directions of the EMF's of the batteries. According to the first Kirchhoff's law,

$$I_3=I_1+I_2$$

(charges are not accumulating in the nodes). The second Kirchhoff's law states that for each closed loop in the circuit the sum of voltages is zero that reflects the fact that electric potential is defined unambiguously (and the work of the electric field over each closed trajectory is zero):

$$\sum_i V_i=0.$$

To the Kirchhoff's laws, one has to add the Ohm's law

$$V_i=R_iI_i$$

for each resistor. On the top of it, there can be EMF's acting within resistors (batteries have their own internal resistance and thus can be considered as resistors) and pushing the current through them. With the EMF's, the Ohms law becomes

$$V_i + \mathcal{E}_i = R_i I_i.$$

Substituting $V_i = R_i I_i - \mathcal{E}_i$ into the second Kirchhoff's law, one obtains

$$\sum_i R_i I_i = \sum_i \mathcal{E}_i.$$

In the circuit above, we neglect the internal resistances of the batteries. The Kirchhoff-Ohm's law above for the left and right loops becomes

$$R_1 I_1 + R_3(I_1 + I_2) = \mathcal{E}_1$$
$$R_2 I_2 + R_3(I_1 + I_2) = \mathcal{E}_2.$$

This is a system of two linear equations with two unknowns that has a solution. To solve this system of equations in a most elegant way, one can first rewrite it in the form collecting terms with I_1 and I_2

$$(R_1+R_3)I_1 + R_3 I_2 = \mathcal{E}_1$$
$$R_3 I_1 + (R_2+R_3)I_2 = \mathcal{E}_2.$$

Then one can eliminate, say, I_2 by multiplying the first equation by R_2+R_3, the second equation by R_3 and then subtracting the second equation from the first one. This yields

$$(R_1 + R_3)(R_2 + R_3)I_1 - R_3^2 I_1 = (R_2 + R_3)\mathcal{E}_1 - R_3\mathcal{E}_2.$$

From here one finds

$$I_1 = \frac{(R_2 + R_3)\mathcal{E}_1 - R_3\mathcal{E}_2}{R_1R_2 + (R_1+R_2)R_3}.$$

Since this circuit is symmetric, one can obtain the formula for I_2 without calculations just by replacing $1 \rightarrow 2, 2 \rightarrow 1$. This yields

$$I_2 = \frac{(R_1 + R_3)\mathcal{E}_2 - R_3\mathcal{E}_1}{R_1R_2 + (R_1+R_2)R_3}.$$

Now

$$I_3 = I_1 + I_2 = \frac{R_2\mathcal{E}_1 + R_1\mathcal{E}_2}{R_1R_2 + (R_1+R_2)R_3}.$$

Voltages on the three resistors can be now found from Ohm's law. One can see that the currents can flow both in positive and negative directions. For instance, if $\mathcal{E}_1$=$\mathcal{E}_2$, then both currents are positive. If $\mathcal{E}_1$ is sufficiently stronger than $\mathcal{E}_2$ (work out the exact condition!), then I_1>0 but I_2<0. If $\mathcal{E}_2$ is sufficiently stronger than $\mathcal{E}_1$, then I_2>0 but I_1<0.

In the particular case R_3=0 (short circuiting) there are two independent circuits for which one obtains

$$I_1 = \frac{\mathcal{E}_1}{R_1}, \qquad I_2 = \frac{\mathcal{E}_2}{R_2}$$

that follows from the formulas above if one sets R_3=0. This provides a check for the formulas obtained.

If one removes R_3, there is only one loop for which one obtains

$$I_1 \equiv I = \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2}.$$

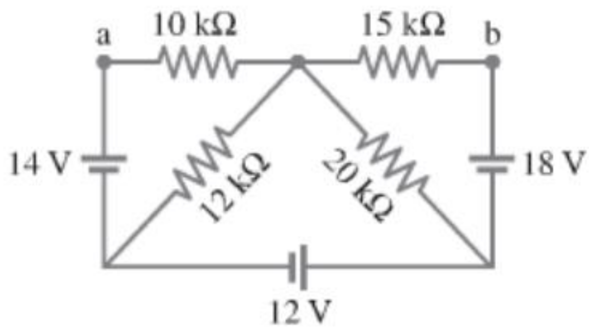
This result follows from the limit $R_3 \to \infty$ as follows

$$I_1 = \frac{(R_2/R_3 + 1)\mathcal{E}_1 - \mathcal{E}_2}{R_1 R_2/R_3 + R_1 + R_2} \to \frac{\mathcal{E}_1 - \mathcal{E}_2}{R_1 + R_2}.$$

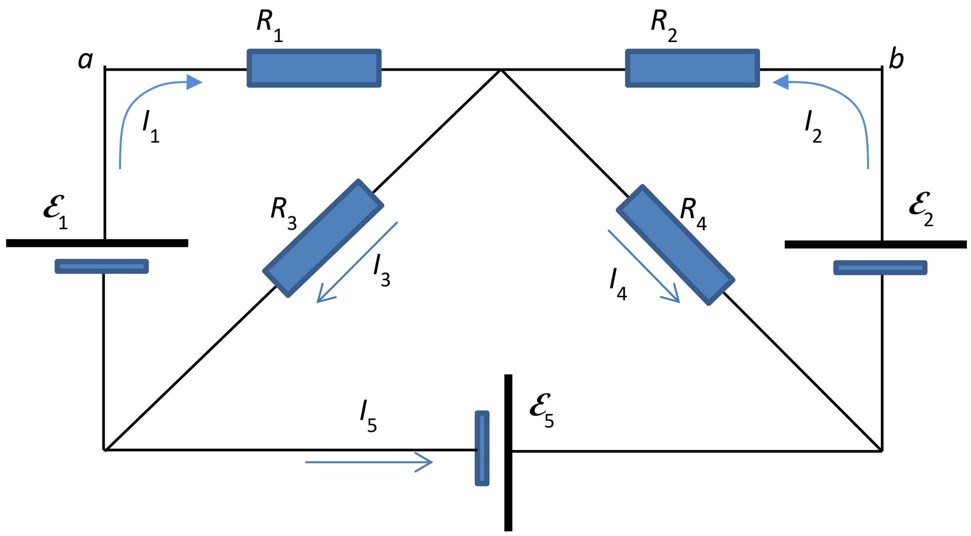
(one discards the terms containing R_3 in the denominator). This is another check of our main result.

62. Problem 79, end of Chapter 19 of the Giancoli book, 6th edition

79. For the circuit shown in Fig. 19–65, determine (*a*) the current through the 14-V battery and (*b*) the potential difference between points a and b, $V_a - V_b$.



Solution. To use algebra, first we introduce the notations for the resistors and currents, making another sketch of the circuit, trying to choose the notations as symmetric as possible



First, one can eliminate I_3 and I_4 from the 1-st Kirchhoff's law:

$$I_3 = I_1 + I_5, \qquad I_4 = I_2 - I_5.$$

Then, one can write the 2-nd Kirchhoff's law combined with the generalized Ohm's law for the three loops: $\mathcal{E}_1 R_1 R_3$, $\mathcal{E}_2 R_2 R_4$, and $\mathcal{E}_1 \mathcal{E}_2 \mathcal{E}_5 R_2 R_1$(counterclockwise). One obtains

$$\begin{aligned}
&R_1 I_1 + R_3(I_1 + I_5) = \mathcal{E}_1\\
&R_2 I_2 + R_4(I_2 - I_5) = \mathcal{E}_2\\
&-R_1 I_1 + R_2 I_2 = -\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_5.
\end{aligned}$$

In the third equation, there is a minus sign in front of $R_1 I_1$ and $\mathcal{E}_1$ because we are moving counterclockwise in the direction opposite to the chosen positive direction of I_1 and against the EMF of the first battery. There are three equations for three unknowns, the currents I_1, I_2, and I_5. Thee equations are close to symmetric, except for the minuses. The first step in solving this system of equations is to group the terms with the same currents in the first two equations:

$$\begin{aligned}
&(R_1 + R_3) I_1 + R_3 I_5 = \mathcal{E}_1\\
&(R_2 + R_4) I_2 - R_4 I_5 = \mathcal{E}_2\\
&-R_1 I_1 + R_2 I_2 = -\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_5. \qquad (1)
\end{aligned}$$

Now, one can eliminate I_1 and I_2 from the first two equations,

$$I_1 = \frac{\mathcal{E}_1 - R_3 I_5}{R_1 + R_3}, \qquad I_2 = \frac{\mathcal{E}_2 + R_4 I_5}{R_2 + R_4}, \qquad (2)$$

and substitute these expression into the third equation:

$$-R_1 \frac{\mathcal{E}_1 - R_3 I_5}{R_1 + R_3} + R_2 \frac{\mathcal{E}_2 + R_4 I_5}{R_2 + R_4} = -\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_5.$$

Regrouping the terms, one obtains

$$\frac{R_1 R_3}{R_1 + R_3} I_5 + \frac{R_2 R_4}{R_2 + R_4} I_5 = \frac{R_1}{R_1 + R_3} \mathcal{E}_1 - \frac{R_2}{R_2 + R_4} \mathcal{E}_2 - \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_5.$$

On the right side, one can add the similar terms that yields

$$\left(\frac{R_1R_3}{R_1+R_3}+\frac{R_2R_4}{R_2+R_4}\right)I_5=-\frac{R_3}{R_1+R_3}\mathcal{E}_1+\frac{R_4}{R_2+R_4}\mathcal{E}_2+\mathcal{E}_5.$$

Finally,

$$I_5=\frac{-\frac{R_3}{R_1+R_3}\mathcal{E}_1+\frac{R_4}{R_2+R_4}\mathcal{E}_2+\mathcal{E}_5}{\frac{R_1R_3}{R_1+R_3}+\frac{R_2R_4}{R_2+R_4}}.$$

The currents I_1 and I_2 can be found by substituting this expression into the formulas for I_1 and I_2 above in Eq. (2). However, the formulas for I_1 and I_2 become too cumbersome and it does not make sense to write them down. At this point, the numbers can be plugged into the formulas. One obtains

$$I_5=\frac{-\frac{12}{10+12}14+\frac{20}{15+20}18+12}{\frac{10\times 12}{10+12}+\frac{15\times 20}{15+20}}10^{-3}=\frac{47}{45}10^{-3}A\approx 1.0444\times 10^{-3}\,A.$$

$$I_1=\frac{14-12\frac{47}{45}}{10+12}10^{-3}=\frac{1}{15}10^{-3}A\approx 0.0666\ \times 10^{-3}A=0.667\times 10^{-4}A$$

$$I_2=\frac{18+20\frac{47}{45}}{15+20}10^{-3}=\frac{10}{9}\ 10^{-3}A\approx 1.1111\times 10^{-3}\,A.$$

The factor 10^{-3} arises because the resistances are in KΩ. One can see that all the current are positive, thus they all flow in the directions shown in the sketch, although I_1 is anomalously small. The latter is the current through the 14 V battery. Finally, the voltage between the points *a* and *b* in the sketch is

$$V_a-V_b=R_1I_1-R_2I_2.$$

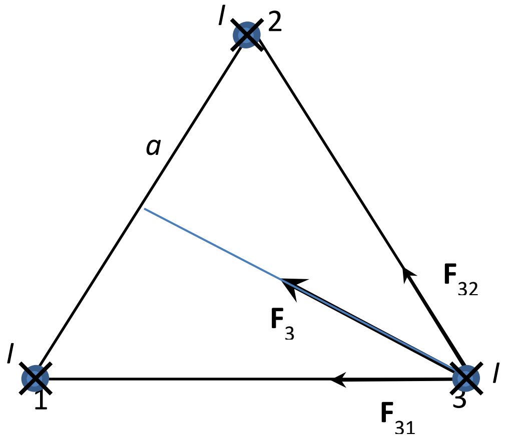
Here the minus sign arises because we move from *a* to *b* across the resistor R_2 in the direction opposite to the chosen positive direction of the current I_2. As I_1 is anomalously small, the voltage is dominated by the second term and is negative, that is, V_a is lower than V_b. Numerically,

$$V_a-V_b=10\frac{1}{15}-15\frac{10}{9}=-16\,V.$$

The solution in the Giancoli book is purely numerical: one plugs numbers already into Eq. (1).

Magnetic field created by electric currents

63. Triangle of wires with the same direction of currents



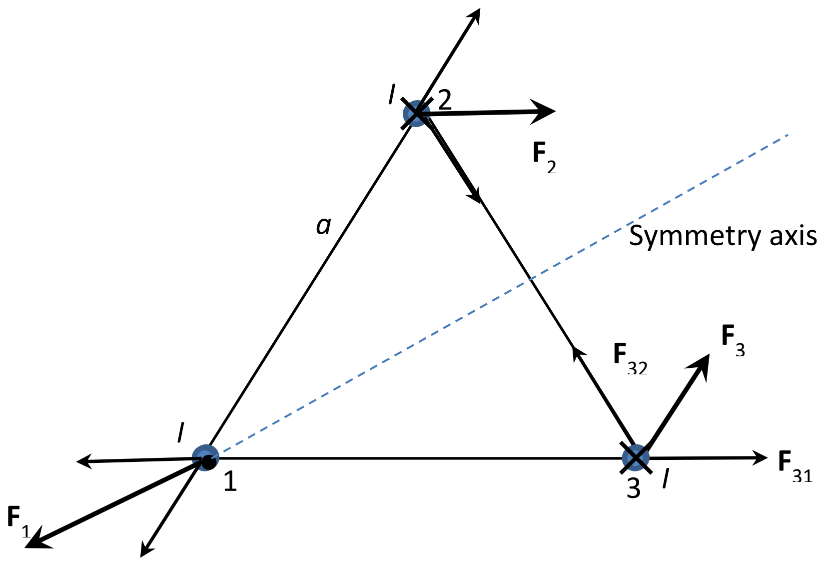
Find the forces acting on the long parallel wires forming an equilateral triangle with a side a in the cross-section. All currents flow in the same direction.

Solution. The solution of such problems resembles the solution of the problems with charges in electrostatics. The difference is that the currents flowing in the same direction attract and the currents flowing in different directions repel each other. As in the case of the Coulomb interaction, the forces between the wires are directed along the lines connecting them. Thus, the problem is to add up force vectors acting on each wire, as shown in the figure. In this case, because of the symmetry, the direction of the total forces is obvious, so that one has to project the forces acting from the individual wires on the direction of the total force. One obtains, for each wire,

$$F = \frac{\mu_0}{2\pi}\frac{I^2}{a} l \times 2\cos 30° = \frac{\mu_0}{2\pi}\frac{I^2}{a} l\sqrt{3},$$

where l is the length of the wire.

64. Triangle of wires with different directions of currents



Find the forces acting on the long parallel wires forming an equilateral triangle with a side a in the cross-section. Two currents flow in one direction and one current is flowing in the other direction.

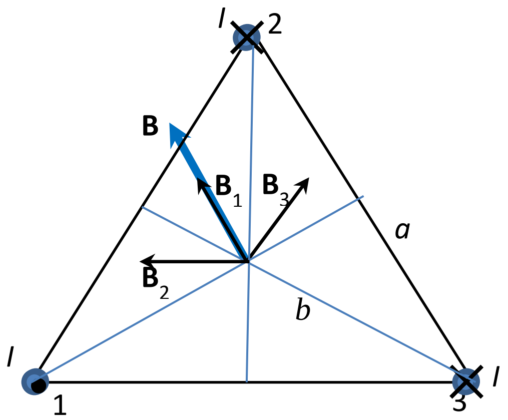
Solution. Here the currents in wires 2 and 3 flows in one direction, while the current in wire 1 flows in the other direction. Again, there is symmetry in the problem, so that the directions of all forces are obvious. The magnitude of $\mathbf{F}_2$ and $\mathbf{F}_3$ are the same:

$$F_2 = F_3 = \frac{\mu_0}{2\pi}\frac{I^2}{a} l \times 2\cos 60° = \frac{\mu_0}{2\pi}\frac{I^2}{a} l.$$

The magnitude of $\mathbf{F}_1$ is the same as in the preceding problem, only its direction is inverted:

$$F = \frac{\mu_0}{2\pi}\frac{I^2}{a} l \times 2\cos 30° = \frac{\mu_0}{2\pi}\frac{I^2}{a} l\sqrt{3}.$$

65. Magnetic field in the center of the triangle of wires



Here we find the magnetic field in the center of the equilateral triangle of wires in the case when two currents are flowing in one direction and the third current is flowing in the other direction, as in the preceding problem.

Solution. The total magnetic field is the sum of all three contributions created by each wire:

$$\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 + \mathbf{B}_3.$$

The magnetic field from wires 2 and 3 is rotating clockwise around the respective wires, whereas the magnetic field from wire 1 rotates counterclockwise. Each magnetic field is perpendicular to the line connecting the wire and the observation point. The directions of the magnetic fields shown in the sketch makes the angles 30° with the bisectrices. The direction of the total magnetic field coincides with that of $\mathbf{B}_1$, thus we project $\mathbf{B}_2$ and $\mathbf{B}_3$ onto this direction. The result is

$$B = \frac{\mu_0}{2\pi}\frac{I}{b}(1 + 2\cos 60°) = \frac{\mu_0}{2\pi}\frac{I}{b}\left(1 + 2\frac{1}{2}\right) = \frac{\mu_0}{2\pi}\frac{I}{b} \times 2,$$

where b is the distance between the corner of the triangle and its center. It can be obtained as

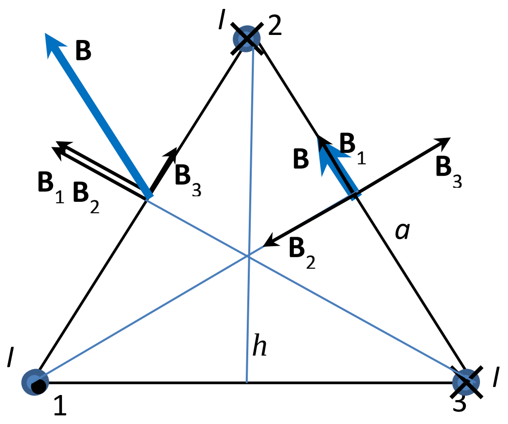
$$b=\frac{a/2}{\cos 30^\circ}=\frac{a/2}{\sqrt{3}/2}=\frac{a}{\sqrt{3}}.$$

Substituting this into the formula for B, one finally obtains

$$B=\frac{\mu_0}{2\pi}\frac{I}{a}2\sqrt{3}.$$

If the current in wire 1 flow in the same direction as the other two, then $\mathbf{B}_1$ is inverted and the three vectors pointing in different directions cancel each other, $\mathbf{B}=0$.

66. Magnetic field in the middles of the sides of the triangle of wires



Here we find the magnetic field in the middles the sides of the equilateral triangle of wires in the case when two currents are flowing in one direction and the third current is flowing in the other direction, as in the preceding problem.

Solution: In general,

$$\mathbf{B}=\mathbf{B}_1+\mathbf{B}_2+\mathbf{B}_3.$$

In the middle of the 23 side, $\mathbf{B}_2$ and $\mathbf{B}_3$ are opposite and cancel each other, so that only $\mathbf{B}_1$ remains, $\mathbf{B}=\mathbf{B}_1$. Thus, one has

$$B=\frac{\mu_0}{2\pi}\frac{I}{h},$$

where h is the height of the triangle,

$$h=a\cos 30^\circ=\frac{a\sqrt{3}}{2}.$$

Substituting this into the formula for B, one finally obtains

$$B=B_1=\frac{\mu_0}{2\pi}\frac{I}{a}\frac{2}{\sqrt{3}}$$

for the middle of the 23 side. The situation in the middle of the sides 12 and 13 is similar by symmetry. For instance, for the 12 side, $\mathbf{B}_1$ and $\mathbf{B}_2$ are the same, while $\mathbf{B}_3$ is perpendicular to them. One has

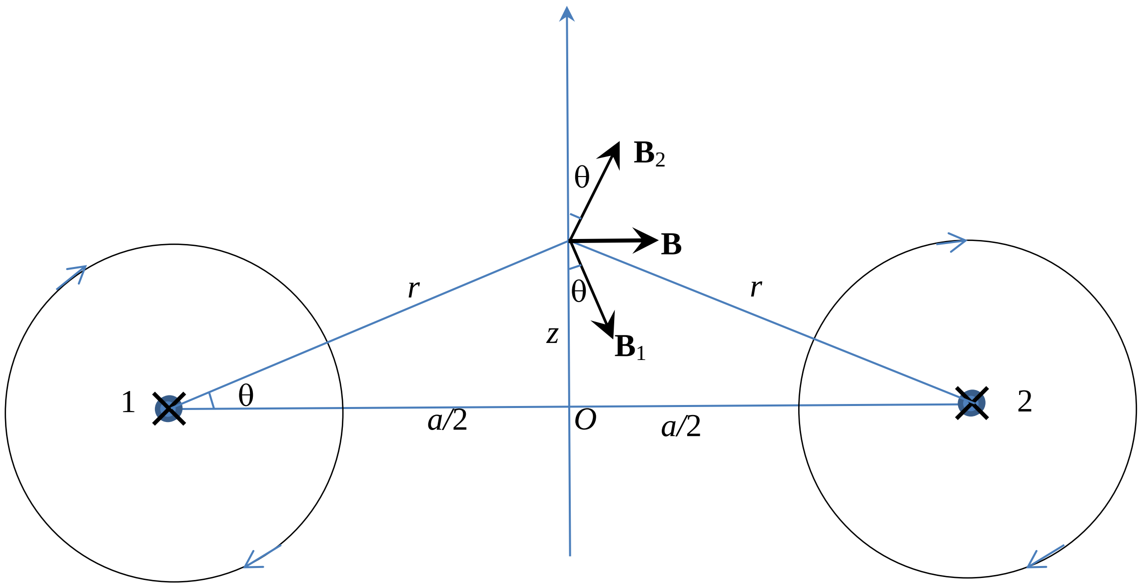
$$B_1 = B_2 = \frac{\mu_0}{2\pi}\frac{I}{a/2}, \qquad B_3 = \frac{\mu_0}{2\pi}\frac{I}{a}\frac{2}{\sqrt{3}}.$$

The magnitude of the total field is given by

$$B = \frac{\mu_0}{2\pi}\frac{I}{a}\sqrt{4^2 + \left(\frac{2}{\sqrt{3}}\right)^2} = \frac{\mu_0}{2\pi}\frac{I}{a} \times 2\sqrt{\frac{12+1}{3}} = \frac{\mu_0}{2\pi}\frac{I}{a} \times 2\sqrt{\frac{13}{3}}$$

67. Magnetic field at the center line between two long wires with the currents in the same direction

Two long straight wires are going parallel to each other at the distance *a* from each other. The wires carry the currents I that go in the same direction. Find the magnetic field created by this system at the center line between the wires (the vertical line on the drawing).



Solution. We set the origin of the coordinate system in the point *O* in the middle between the wires and the *z*-axis going up along the center line. Let the currents be directed inside the paper sheet, then, according to the screw rule, the magnetic field created by each wire is directed clockwise. The currents 1 and 2 create magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ directed as shown in the drawing. By symmetry, the resulting magnetic field $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ is directed horizontally to the right. Using the basic formula for the magnetic field created by the long wire

$$B = \frac{\mu_0}{2\pi}\frac{I}{r},$$

and projecting the vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ on the horizontal direction *x*, one obtains

$$B = B_{1,x} + B_{2,x} = \frac{\mu_0}{2\pi}\frac{I}{r} \times 2\sin\theta.$$

With

$$r = \sqrt{z^2 + (a/2)^2}, \qquad \sin\theta = \frac{z}{r}$$

one finally obtains

$$B = \frac{\mu_0 I}{2\pi} \frac{2z}{z^2 + (a/2)^2}.$$

Here, it does not make sense to cancel 2 in the numerator and denominator.

Let us analyze particular and limiting cases. First, our formula yields *B*=0 for *z*=0 (at the point *O*). This is an expected result, as in this case fields $\mathbf{B}_1$ and $\mathbf{B}_2$ are opposite and cancel each other.

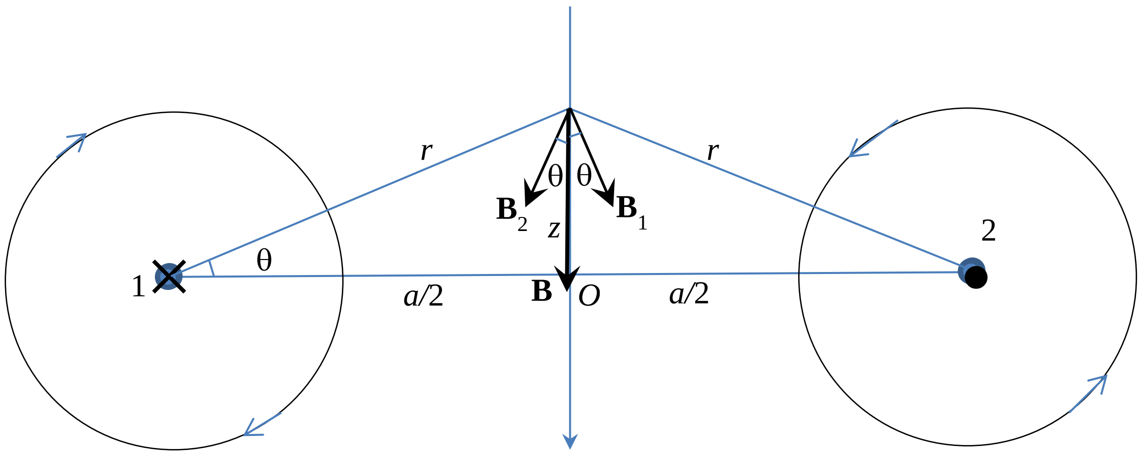
At large distances, *z*>>*a*, one can neglect $(a/2)^2$ in the denominator that yields

$$B = \frac{\mu_0}{2\pi} \frac{2I}{z}.$$

This is the field produced by one wire with the current 2*I*. Indeed, from large distances these two wires are seen as one wire with the double current. This limiting case serves as one of the checks of the general formula.

68. Magnetic field at the center line between two long wires with the currents in the opposite directions

Two long straight wires are going parallel to each other at the distance *a* from each other. The wires carry the currents I that go in the opposite directions. Find the magnetic field created by this system at the center line between the wires (the vertical line on the drawing).



Solution. We set the origin of the coordinate system in the point *O* in the middle between the wires and the *z*-axis going up along the center line. Let current 1 be directed inside the paper sheet and current 2 be directed outside the paper sheet, then, according to the screw rule, the magnetic fields created by each wire are directed clockwise and counterclockwise, respectively. The currents 1 and 2 create magnetic fields $\mathbf{B}_1$ and $\mathbf{B}_2$ directed as shown in the drawing. By symmetry, the resulting magnetic field $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$ is directed vertically down. Using the basic formula for the magnetic field created by the long wire

$$B = \frac{\mu_0}{2\pi} \frac{I}{r},$$

and projecting the vectors $\mathbf{B}_1$ and $\mathbf{B}_2$ on the vertical direction *x*, one obtains

$$B = B_{1,z} + B_{2,z} = \frac{\mu_0}{2\pi}\frac{I}{r}2\cos\theta.$$

With

$$r = \sqrt{z^2 + (a/2)^2}, \qquad \cos\theta = \frac{a/2}{r}$$

one finally obtains

$$B = \frac{\mu_0 I}{2\pi}\frac{a}{z^2 + (a/2)^2}.$$

Let us consider particular and limiting cases of this formula. At z=0 the result reads

$$B = \frac{\mu_0 I}{2\pi}\frac{4}{a}$$

that is twice the magnetic field created by a wire at the distance $a/2$, an expected result.

At large distances, $z \gg a$, one can neglect $(a/2)^2$ in the denominator that yields

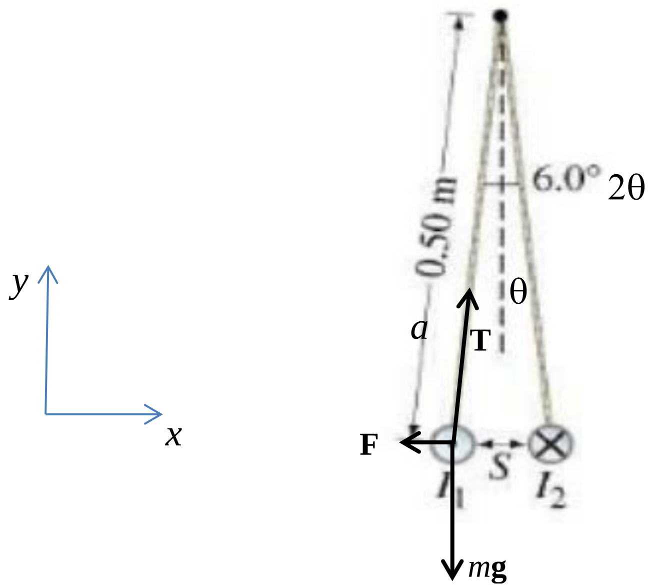
$$B = \frac{\mu_0 I}{2\pi}\frac{a}{z^2}.$$

This decreases at large distances faster than the magnetic field from one wire since $\mathbf{B}_1$ and $\mathbf{B}_2$ become almost opposite and nearly cancel each other.

Different problems on the magnetic field

69. Problem 86, end of chapter 20, Giancoli 6th edition: Suspended wires with opposite currents

86. Two long straight aluminum wires, each of diameter 0.50 mm, carry the same current but in opposite directions. They are suspended by 0.50-m-long strings as shown in Fig. 20–71. If the suspension strings make an angle of 3.0° with the vertical, what is the current in the wires?



Solution. First, we introduce missing notations. Second, we identify the forces acting on a wire, as shown in the sketch. The three forces, including the tension force $\mathbf{T}$ from the suspending cord, balance each other:

$$\mathbf{F} + m\mathbf{g} + \mathbf{T} = 0.$$

In components (with explicit signs of the force components), this vector equation becomes

$$(x):\ -F + T\sin\theta = 0, \qquad (y):\ -mg + T\cos\theta = 0.$$

The magnetic force F is given by

$$F = \frac{\mu_0}{2\pi}\frac{I^2}{S}l = \frac{\mu_0}{2\pi}\frac{I^2}{2a\sin\theta}l,$$

where l is the length of the wires.

Now, one can eliminate the tension force from the mechanical equilibrium equation to relate F to mg. Multiplying the *x*-equation by $\cos\theta$, the *y*-equation by $\sin\theta$, and subtracting them from each other, one obtains

$$-F\cos\theta + mg\sin\theta = 0$$

or

$$F = mg\tan\theta.$$

In fact, this relation could be written immediately without considering T. Equating this to the magnetic expression for F, one obtains

$$mg\tan\theta = \frac{\mu_0}{2\pi}\frac{I^2}{2a\sin\theta}l.$$

Solving this for I yields

$$I = \sqrt{mg\frac{\sin^2\theta}{\cos\theta}\frac{2\pi}{\mu_0}\frac{2a}{l}}.$$

Here, the mass of the wire is proportional to its length, so that the result does not depend on l. Expressing the wire mass as

$$m = \rho l\pi d^2/4,$$

one obtains

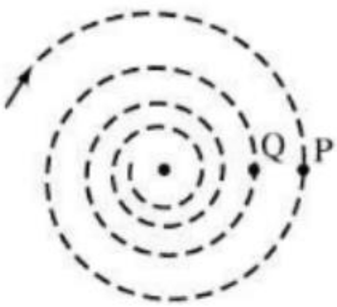
$$I = \sqrt{\frac{\rho l\pi d^2}{4}g\frac{\sin^2\theta}{\cos\theta}\frac{2\pi}{\mu_0}\frac{2a}{l}} = d\sqrt{\frac{2\pi}{\mu_0}\frac{\pi a}{2}\rho g\frac{\sin^2\theta}{\cos\theta}}$$

that is the final analytical result. Now one can substitute the given numerical values, including the aluminum density $\rho = 2700\ kg/m^3$. One obtains

$$I = 0.5\times10^{-3}\sqrt{\frac{2\pi}{4\pi\times10^{-7}}\frac{\pi\times0.5}{2}2700\times9.8}\frac{\sin 3°}{\sqrt{\cos 3°}} = 8.44\ A$$

70. Problem 84, end of chapter 20, Giancoli 6th edition: Energy loss in cyclotron motion

84. A proton follows a spiral path through a gas in a magnetic field of 0.010 T, perpendicular to the plane of the spiral, as shown in Fig. 20–70. In two successive loops, at points P and Q, the radii are 10.0 mm and 8.5 mm, respectively. Calculate the change in the kinetic energy of the proton as it travels from P to Q.



Solution. The proton circling in the magnetic field interacts with the gas molecules and gives them a part of its kinetic energy. As the gas is rarified, these interactions occur at large distances and are weak. This is why the kinetic energy of the proton changes slowly. The equation describing the cyclotron motion (the second Newton's law) has the form

$$evB = m\frac{v^2}{R},$$

where from the well-known formula for the orbit radius follows:

$$R = \frac{mv}{eB}.$$

To tailor it to the current problem, one can relate the velocity with the kinetic energy:

$$E_k = \frac{mv^2}{2}.$$

Finding v from the formula for R and substituting it into the kinetic energy, one obtains

$$E_k = \frac{m}{2}\left(\frac{ReB}{m}\right)^2 = \frac{(ReB)^2}{2m}.$$

In this formula, ReB is the linear momentum of the proton. Now the loss of the kinetic energy can be represented as

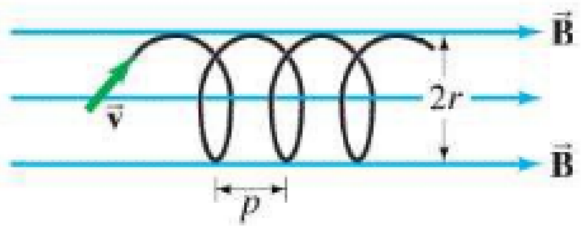
$$\Delta E_k = \frac{(eB)^2}{2m}(R_1^2 - R_2^2).$$

Substituting the numbers, one obtains

$$\Delta E_k = \frac{(1.6 \times 10^{-19} \times 0.01)^2}{2 \times 1.67 \times 10^{-27}}(10^2 - 8.5^2) \times 10^{-6} = 2.1 \times 10^{-20}\ \text{J}.$$

71. Problem 87, end of chapter 20, Giancoli 6th edition: Helical motion of the charge in the magnetic field

87. An electron enters a uniform magnetic field $B = 0.23\ \text{T}$ at a 45° angle to $\vec{\mathbf{B}}$. Determine the radius r and pitch p (distance between loops) of the electron's helical path assuming its speed is $3.0 \times 10^6\ \text{m/s}$. See Fig. 20–72.



Solution. First, we introduce the missing notation of θ as the angle between the electron's velocity and the magnetic field. The motion of the electron is described, as usual by the second Newtons' law:

$$\mathbf{F} = m\mathbf{a}.$$

Here $\mathbf{F}$ is the Lorentz force,

$$\mathbf{F} = e\mathbf{v} \times \mathbf{B}.$$

Only the velocity component perpendicular to $\mathbf{B}$ make a contribution to the force, and the force is perpendicular to $\mathbf{B}$. On the other hand, the velocity component parallel to $\mathbf{B}$ does not create and force, and there is no force in this direction. One can conclude that the motion parallel to the magnetic field if free, while the motion in the plane perpendicular to $\mathbf{B}$ should be a cyclotron motion.

Introducing the coordinate axes z along the magnetic field, x and y perpendicular to the magnetic field, and projecting the equation of motion onto these axis, one obtains

$$v_z = \mathrm{const} = v\cos\theta$$

for the motion along the magnetic field and the circular motion with the speed

$$v_\perp = \mathrm{const} = v\sin\theta$$

in the xy plane. The radius of the orbit is

$$R = \frac{mv_\perp}{eB} = \frac{mv\sin\theta}{eB}.$$

The period of the orbiting is

$$T = \frac{2\pi R}{v_\perp} = \frac{2\pi m}{eB}.$$

During this time, the electron covers the distance

$$p = v_z T = v\cos\theta\,\frac{2\pi m}{eB}.$$

Substituting the numerical values, one obtains

$$R = \frac{0.91\times10^{-30}\times3\times10^{6}\sin45^\circ}{1.6\times10^{-19}\times0.23} = 0.0000518\,m = 0.518\times10^{-4}\,m$$

and

$$p = 3\times10^{6}\cos45^\circ\frac{2\pi\times0.91\times10^{-30}}{1.6\times10^{-19}\times0.23} = 0.00032959\,m = 3.30\times10^{-4}\,m.$$

Electromagnetic induction

72. Problem 13 end of chapter 21, Giancoli 6th edition

13. (II) A circular loop in the plane of the paper lies in a 0.75-T magnetic field pointing into the paper. If the loop's diameter changes from 20.0 cm to 6.0 cm in 0.50 s, (*a*) what is the direction of the induced current, (*b*) what is the magnitude of the average induced emf, and (*c*) if the coil resistance is 2.5 Ω, what is the average induced current?

Solution. First, introduce missing notations. The magnetic field $B = 0.75\ T$, The loop diameters $D_1 = 20\ cm = 0.2\ m$, $D_2 = 6\ cm = 0.06\ m$, the time interval $\Delta t = 0.5\ s$, Resistance of the coil $R = 2.5\ \Omega$.

We use the Faraday-Lenz law

$$\mathcal{E} = -\frac{\Delta\Phi}{\Delta t}.$$

Here, we take into account only the external magnetic flux and neglect the contribution of the currents in the loop (this is valid for the resistance of the loop large enough so that the current is small)

$$\Delta\Phi = \Phi_2 - \Phi_1 = \frac{\pi D_2^2}{4}B - \frac{\pi D_1^2}{4}B = \frac{\pi}{4}B(D_2^2 - D_1^2).$$

Thus one obtains

$$\mathcal{E} = -\frac{\pi}{4}B\frac{D_2^2 - D_1^2}{\Delta t},$$

where we keep the symbolic Lenz sign that shows that the direction of the EMF is such that it induces currents whose magnetic field partially compensates the change of the external magnetic flux. Substituting the numbers, one obtains

$$\mathcal{E} = -\frac{\pi}{4}0.75\frac{0.06^2 - 0.2^2}{0.5} = 0.0439\ V = 43.9\ mV.$$

(a) The loop is contracting, thus the magnetic flux into the plane of the loop (away from us) is decreasing. Thus, the current will flow in the clockwise direction to create its own magnetic flux into the plane of the loop.

(b) The induced current is given by the Ohm's law

$$I = \frac{\mathcal{E}}{R} = \frac{\pi B}{4R}\frac{D_2^2 - D_1^2}{\Delta t}.$$

Substituting the numbers, one obtains $I = \frac{0.0439}{2.5} = 0.01756\ A = 17.56\ mA$.

73. Problem 18 end of chapter 21, Giancoli 6th edition

18. (III) A 22.0-cm-diameter coil consists of 20 turns of circular copper wire 2.6 mm in diameter. A uniform magnetic field, perpendicular to the plane of the coil, changes at a rate of 8.65×10^{-3} T/s. Determine (*a*) the current in the loop, and (*b*) the rate at which thermal energy is produced.

Solution. First, we introduce missing notations. The coil's diameter $D = 22\ cm = 0.22\ m$, the number of turns of wire $N = 20$, the diameter of the wire $d = 2.6\ mm = 2.1 \times 10^{-3}\ m$ the rate of change of the magnetic field $\frac{\Delta B}{\Delta t} = 8.65 \times 10^{-3}\ T/s$. We have to add the resistivity of the copper $\rho = 1.72 \times 10^{-8}\ \Omega\ m$.

We use the Faraday-Lenz law for a coil with N turns

$$\mathcal{E} = -N\frac{\Delta\Phi}{\Delta t}.$$

Here, we take into account only the external magnetic flux and neglect the contribution of the currents in the coil (this is valid for the resistance of the coil large enough so that the current is small). The applicability of this approach can be tested *a posteriori*.

In our case, the Faraday-Lenz law becomes

$$\mathcal{E} = -N\frac{\pi D^2}{4}\frac{\Delta B}{\Delta t}$$

and the current is given by

$$I = \frac{\mathcal{E}}{R} = -N\frac{\pi D^2}{4R}\frac{\Delta B}{\Delta t}.$$

The resistance of the wire is given by

$$R = \rho\frac{L}{S} = \rho\frac{\pi DN}{\pi d^2/4} = \rho\frac{4DN}{d^2}.$$

Substituting this into the formula for the current, one obtains

$$I = -N\frac{\pi D^2}{4}\frac{d^2}{\rho \times 4DN}\frac{\Delta B}{\Delta t} = -\frac{\pi D d^2}{16\rho}\frac{\Delta B}{\Delta t}.$$

Substituting the numerical values, one obtains

$$I = -\frac{\pi \times 0.22 \times (2.1 \times 10^{-3})^2}{16 \times 1.72 \times 10^{-8}} 8.65 \times 10^{-3} = 0.0958\ A.$$

(b) The dissipated power is

$$P = I^2 R.$$

Substituting the formulas for the current and resistance above, one obtains

$$P=\left(\frac{\pi D d^2}{16\rho}\frac{\Delta B}{\Delta t}\right)^2 \rho\frac{4DN}{d^2}=\frac{\pi^2 D^3 d^2 N}{64\rho}\left(\frac{\Delta B}{\Delta t}\right)^2.$$

Substituting the numbers yields

$$P=\frac{\pi^2 0.22^3(2.1\times 10^{-3})^2\times 20}{64\times 1.72\times 10^{-8}}(8.65\times 10^{-3})^2=0.00063\,W=0.63\,mW.$$

Let us look at what happens if we take into account the magnetic flux created by the current in the wire. The total magnetic EMF in this case will be

$$\mathcal{E}=-N\frac{\Delta\Phi}{\Delta t}-L\frac{\Delta I}{\Delta t},$$

where

$$L=\frac{\mu_0 N^2 S}{l}=\frac{\mu_0 N^2\pi D^2}{4l}$$

and l is the coil's length whose numerical value is not given. Using the Ohm's law, one obtains the equation for the current

$$RI+L\frac{\Delta I}{\Delta t}=-N\frac{\Delta\Phi}{\Delta t}.$$

This equation contains both the current and the rate of the current's change, that is, it is a differential equation that belongs to the calculus-based course. Qualitatively one can say that the current cannot increase immediately from zero to a finite value obtained above after the external magnetic flux started to change. The current will increase from zero to the value obtained above during the characteristic time

$$\tau_{LR}=\frac{L}{R}.$$

74. EMF in a rotating coil

A coil with diameter D and N turns of wire is initially oriented with its axis parallel to the magnetic field B. It begins to rotate with the angular velocity ω around an axis perpendicular to the magnetic field. What is the average EMF during the time Δt? Work out the result in the limit of short Δt. What is the exact condition for Δt to be short? What is the EMF at the initial moment of time?

Tip. Use the formula

$$\cos\theta\cong 1-\frac{\theta^2}{2},\qquad \theta\ll 1.$$

Solution. We use the Faraday law

$$\mathcal{E}=-N\frac{\Delta\Phi}{\Delta t}.$$

The magnetic flux is defined by

$$\Phi = S\mathbf{B}\cdot\mathbf{n} = SB\cos\theta,$$

where $\mathbf{n}$ is the normal to the coil's plane and θ is the angle between the vectors $\mathbf{B}$ and $\mathbf{n}$. In the initial state $\mathbf{n} \parallel \mathbf{B}$ and thus $\theta = 0$. In the initial state, $t = 0$, the magnetic flux has its maximal possible value

$$\Phi(0) = \Phi_0 \equiv SB.$$

Because of the coil's rotation, the angle changes at the linear rate:

$$\theta = \omega t.$$

At the time t the magnetic flux is given by

$$\Phi(t) = \Phi_0\cos\theta = \Phi_0\cos(\omega t).$$

Thus, the average EMF in the process is given by

$$\mathcal{E} = -N\frac{\Phi(\Delta t) - \Phi(0)}{\Delta t} = -N\Phi_0\frac{1-\cos(\omega\Delta t)}{\Delta t}.$$

If the time interval is short, that is,

$$\omega\Delta t \ll 1 \qquad \text{or} \qquad \Delta t \ll 1/\omega$$

(or the rotation angle is small), then one can use the small-argument expansion of the cosine that yields

$$\mathcal{E} = -N\Phi_0\frac{(\omega\Delta t)^2}{2\Delta t} \cong -N\Phi_0\frac{\omega^2\Delta t}{2}.$$

One can see that the average EMF is small for small Δt. In the limit $\Delta t \to 0$, that is, at the initial moment of time, the EMF is zero.

Geometrical optics

75. Mirror equation

EXAMPLE 23–2 **Image in a concave mirror.** A 1.50-cm-high diamond ring is placed 20.0 cm from a concave mirror with radius of curvature 30.0 cm. Determine (*a*) the position of the image, and (*b*) its size.

Solution: Introduce missing notations: $h_o = 1.5\ cm$, $d_o = 20\ cm$, $r = 30\ cm$; find d_i and h_i.

Basic formulas. The focal length is given by

$$f = r/2.$$

The mirror equation is

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}.$$

The magnification m is defined by

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}.$$

(a) From the mirror equation follows the solution for d_i:

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}$$

thus

$$d_i = \frac{1}{\frac{1}{f} - \frac{1}{d_o}} = \frac{d_o f}{d_o - f}.$$

From here one can see that for $d_o > f$ the image is real ($d_i > 0$), while for $d_o < f$ the image is virtual ($d_i < 0$). If $d_o \to f + 0$, then $d_i \to \infty$, while for $d_o \to f - 0$, then $d_i \to -\infty$. If the object is at the center of curvature, $d_o = r = 2f$, then also $d_i = r = 2f$.

Substituting the numbers, one obtains

$$f = \frac{30}{2} = 15\ cm$$

and

$$d_i = \frac{20 \times 15}{20 - 15} = 60\ cm.$$

(b) Substituting the formula for d_i into that for the magnification, one obtains

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o} = -\frac{f}{d_o - f} = \frac{f}{f - d_o}$$

and

$$h_i = mh_o = \frac{h_o f}{f - d_o}.$$

Substituting the numbers, one obtains

$$m = \frac{15}{15 - 20} = -3$$

and

$$h_i = \frac{1.5 \times 15}{15 - 20} = -4.5\ cm.$$

76. The depth of the pool

We wish to determine the depth of a swimming pool filled with water. We measure the width ($x = 5.50$ m) and then note that the bottom edge of the pool is just visible at an angle of 14.0° above the horizontal as shown in Fig. 23–54. Calculate the depth of the pool.

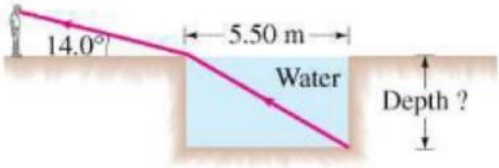


FIGURE 23–54 Problem 73.

Solution. First, introduce missing notations: $\theta'_1 = 14°$, $L = 5.5\ m$ (the length of the pool), $n = 1.33$ (refraction index of water). Find h, the depth of the basin.

The angle θ'_1 is the angle complimentary to the incidence angle θ_1, that is,

$$\theta_1 = 90° - \theta_1'.$$

We use the Snell's law

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

with $n_1 = 1$ for the air and $n_2 = n$ for the water. From here, one finds the refraction angle:

$$\sin\theta_2 = \frac{\sin\theta_1}{n} = \frac{\cos\theta_1'}{n}.$$

Now, from the triangle formed by the light ray inside the pool, its left wall, and the bottom, one finds

$$\frac{L}{h} = \tan\theta_2,$$

so that

$$h = L\cot\theta_2 = L\cot\arcsin\frac{\cos\theta_1'}{n}.$$

This is the analytical answer to the problem that is sufficient for an exam. However, it can be simplified, as tangent and cotangent can be expressed via the sine and cosine. Here,

$$\cot\theta_2 = \frac{\cos\theta_2}{\sin\theta_2} = \frac{\sqrt{1-\sin^2\theta_2}}{\sin\theta_2} = \sqrt{\frac{1}{\sin^2\theta_2} - 1}.$$

Using this, one obtains

$$h = L\cot\theta_2 = L\sqrt{\frac{n^2}{\cos^2\theta_1'} - 1}.$$

Substituting the numbers yields

$$h = 5.5\sqrt{\frac{1.33^2}{\cos^2 14°} - 1} = 5.16\, m.$$

77. Critical angle in another material

The critical angle of a certain piece of plastic in air is $\theta_C = 37.3°$. What is the critical angle of the same plastic if it is immersed in water?

Solution. The critical angle is defined by the condition that the angle of refraction is 90°, that is,

$$n_1 \sin\theta_1 \equiv n_1 \sin\theta_C = n_2 \sin 90° = 1,$$

as there is the air on the other side. For another material on the refractive side, one can write a similar formula:

$$n_1 \sin\theta'_C = n'_2,$$

where n'_2 is the refraction index of the water, so that

$$\sin\theta'_C = \frac{n'_2}{n_1}.$$

One has to find n_1. From the first equation one finds

$$n_1 = \frac{1}{\sin\theta_C}.$$

So that

$$\sin\theta'_C = n'_2 \sin\theta_C$$

and

$$\theta'_C = \arcsin(n'_2 \sin\theta_C), \qquad \sin\theta_C < \frac{1}{n'_2}.$$

78. Total internal reflection in a prism

If the apex angle of a prism is $\phi = 72°$ (see Fig. 23–56), what is the minimum incident angle for a ray if it is to emerge from the opposite side (i.e., not be totally internally reflected), given $n = 1.50$?

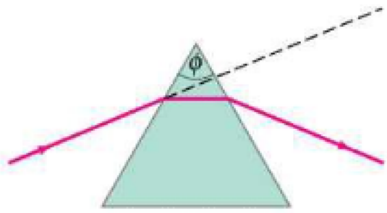
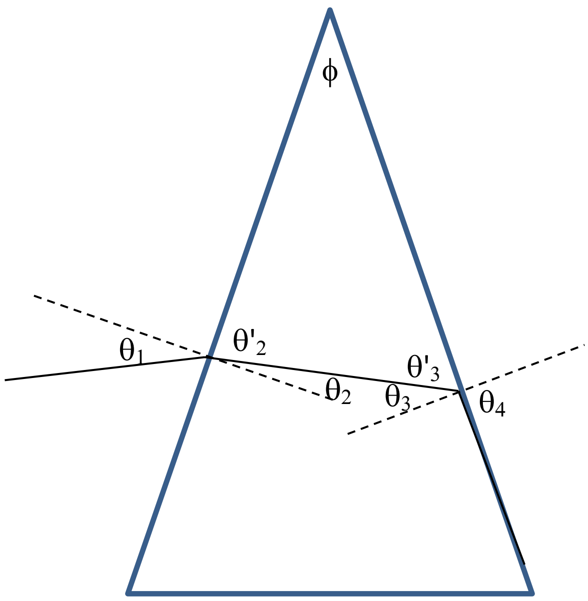


FIGURE 23–56 Problem 79.

Solution. To solve the problem, we need a more detailed sketch.



We have to apply the Snell's law two times, for both refractions,

$$\sin\theta_1 = n\sin\theta_2\,, \qquad n\ \sin\theta_3 = \sin\theta_4 = 1.$$

An additional relation for the angles can be obtained from the triangle formed by the two sides of the prism and the light ray inside the prism. One has

$$\phi + \theta'_2 + \theta'_3 = 180°$$

as well as

$$\theta'_2 = 90° - \theta_2, \qquad \theta'_3 = 90° - \theta_3.$$

Substituting this into the relation for the angles in the triangle, one obtains

$$\phi - \theta_2 - \theta_3 = 0.$$

To find the required θ_1 as

$$\theta_1 = \sin^{-1}(n\sin\theta_2),$$

one needs θ_2 that follows from the relation above:

$$\theta_2 = \phi - \theta_3.$$

Here, θ_3 follows from the second Snell's law:

$$\theta_3 = \sin^{-1}\left(\frac{1}{n}\right),$$

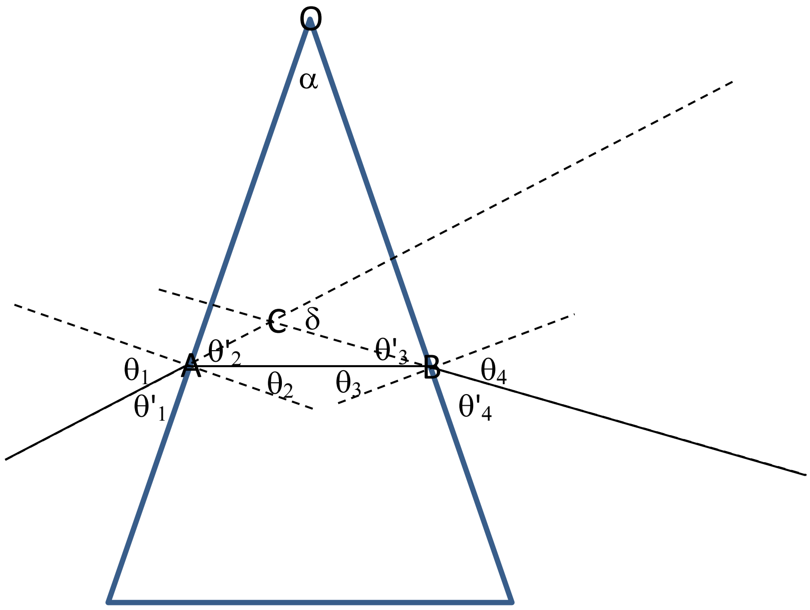
so that

$$\theta_1 = \sin^{-1}\left(n\sin\left(\phi - \sin^{-1}\left(\frac{1}{n}\right)\right)\right),$$

Substituting the numbers, one obtains $\theta_1 \cong 49°$. Note that for ϕ small enough the value of θ_1 is negative that corresponds to the incident ray (on the left of the prism) deviating clockwise from the normal (not counterclockwise, as shown in the sketch).

79. Change of the direction of light in the prism

a) Find the change-of-the-direction angle δ in a prism with the apex angle α and the refraction index n for the arbitrary incidence angle θ_1; b) Find the result in the symmetric case; c) Express the refraction index n via θ_1 in the symmetric case; d*) Obtain the limiting expression for δ for small α.



Solution.

a) The change-of-direction angle δ shown in the sketch can be found considering the triangle ABC. One has

$$\delta = 180° - \measuredangle ACB.$$

(We do not introduce a notation for the angle $\measuredangle ACB$ to avoid a mess on the sketch). On the other hand, as in any triangle the sum of all angles is $180°$, one finds

$$\measuredangle ACB = 180° - \measuredangle CAB - \measuredangle ABC.$$

Thus

$$\delta = \measuredangle CAB + \measuredangle ABC.$$

Further,

$$\measuredangle CAB = \theta_2' - \theta_1' = 90° - \theta_2 - (90° - \theta_1) = \theta_1 - \theta_2$$

and

$$\measuredangle ABC = \theta_3' - \theta_4' = 90° - \theta_3 - (90° - \theta_4) = \theta_4 - \theta_3.$$

Substituting these expressions into the formula for δ, one obtains

$$\delta = \theta_1 - \theta_2 - \theta_3 + \theta_4.$$

To find the remaining angles for a given θ_1, one uses the Snell's law for the two sides of the prism,

$$\sin\theta_1 = n\sin\theta_2\,, \qquad n\ \sin\theta_3 = \sin\theta_4$$

Another relation can be obtained from the triangle AOB:

$$\alpha + \theta'_2 + \theta'_3 = 180°,$$

as well as $\theta'_2 = 90° - \theta_2$ and $\theta'_3 = 90° - \theta_3$. Substituting this into the relation above, one obtains

$$\alpha - \theta_2 - \theta_3 = 0 \qquad (1)$$

and thus

$$\theta_3 = \alpha - \theta_2.$$

This simplifies the formula for δ to

$$\delta = \theta_1 - \alpha + \theta_4. \qquad (2)$$

Now, one has to find θ_4:

$$\theta_4 = \sin^{-1}(n\ \sin\theta_3) = \sin^{-1}(n\ \sin(\alpha - \theta_2)) = \sin^{-1}\left(n\ \sin\left(\alpha - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right)\right).$$

The final result for δ reads

$$\delta = \theta_1 - \alpha + \sin^{-1}\left(n\ \sin\left(\alpha - \sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\right)\right) \qquad (3).$$

One can see that for $\alpha = 0$ (that corresponds to a flat glass) one has $\delta = 0$. It can be shown that δ reaches its minimal value in the symmetric case.

b) In the symmetric case, one has $\theta_1 = \theta_4$ and $\theta_2 = \theta_3$. From (1) one obtains

$$\theta_2 = \frac{\alpha}{2}.$$

Now from the Snell's law follows

$$\theta_1 = \sin^{-1}(n\sin\theta_2) = \sin^{-1}\left(n\sin\frac{\alpha}{2}\right)$$

Now from (2) one obtains

$$\delta = 2\theta_1 - \alpha = 2\sin^{-1}\left(n\sin\frac{\alpha}{2}\right) - \alpha.$$

c) We now resolve the last formula for n:

$$\sin\left(\frac{\delta+\alpha}{2}\right) = n\sin\frac{\alpha}{2},$$

thus

$$n = \frac{\sin\left(\frac{\delta+\alpha}{2}\right)}{\sin\frac{\alpha}{2}}.$$

This formula is used in the lab to find the refraction index.

d*) To simplify (3) for small α, one can use the trigonometric formula

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha\,. \qquad (4)$$

Thus one obtains

$$\delta = \theta_1 - \alpha + \sin^{-1}\left[n\left(\sin\alpha\cos\sin^{-1}\left(\frac{\sin\theta_1}{n}\right) - \sin\sin^{-1}\left(\frac{\sin\theta_1}{n}\right)\cos\alpha\right)\right].$$

Here

$$\sin\sin^{-1}\left(\frac{\sin\theta_1}{n}\right) = \frac{\sin\theta_1}{n}$$

$$\cos\sin^{-1}\left(\frac{\sin\theta_1}{n}\right) = \sqrt{1-\left(\frac{\sin\theta_1}{n}\right)^2}.$$

Thus

$$\delta = \theta_1 - \alpha + \sin^{-1}\left[n\left(\sin\alpha\sqrt{1-\left(\frac{\sin\theta_1}{n}\right)^2} - \frac{\sin\theta_1}{n}\cos\alpha\right)\right]$$

or

$$\delta = \theta_1 - \alpha + \sin^{-1}\left[n\sin\alpha\sqrt{1-\left(\frac{\sin\theta_1}{n}\right)^2} - \sin\theta_1\cos\alpha\right]$$

As α is small, one can use $\sin\alpha \cong \alpha$ and $\cos\alpha \cong 1$ to simplify this to

$$\delta = \theta_1 - \alpha - \sin^{-1}\left[\sin\theta_1 - n\alpha\sqrt{1 - \left(\frac{\sin\theta_1}{n}\right)^2}\right]. \qquad (5)$$

In the argument of arcsin, the first term is regular, while the second one is small. Thus, it can be approximately simplified. Writing

$$\gamma = \sin^{-1}\left[\sin\theta_1 - n\alpha\sqrt{1 - \left(\frac{\sin\theta_1}{n}\right)^2}\right]$$

and further

$$\sin\gamma = \sin\theta_1 - n\alpha\sqrt{1 - \left(\frac{\sin\theta_1}{n}\right)^2}, \qquad (6)$$

one can see that γ is very close to θ_1. Thus we write $\gamma = \theta_1 - \varepsilon$, where ε is a small angle. Now, using (4), one can write

$$\sin\gamma = \sin(\theta_1 - \varepsilon) = \sin\theta_1\cos\varepsilon - \sin\varepsilon\cos\theta_1 \cong \sin\theta_1 - \varepsilon\cos\theta_1.$$

Substituting this into (6), one obtains

$$\varepsilon\cos\theta_1 = n\alpha\sqrt{1 - \left(\frac{\sin\theta_1}{n}\right)^2},$$

where from

$$\varepsilon = n\alpha\frac{\sqrt{1 - \left(\frac{\sin\theta_1}{n}\right)^2}}{\cos\theta_1} = \alpha\frac{\sqrt{n^2 - \sin^2\theta_1}}{\cos\theta_1}.$$

Now, (5) becomes

$$\delta = \theta_1 - \alpha - \gamma = \theta_1 - \alpha - (\theta_1 - \varepsilon) = \varepsilon - \alpha$$

and, finally,

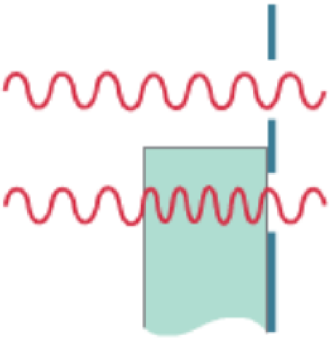
$$\delta = \alpha\left(\frac{\sqrt{n^2 - \sin^2\theta_1}}{\cos\theta_1} - 1\right).$$

One can see that if $\alpha = 0$ or $n = 1$, the deviation angle δ becomes zero, as it should be. Thus, our formula passes the available checks. If the incidence angle is small, then one can use $\cos\theta_1 \cong 1$ and $\sin\theta_1 \cong 0$ to obtain the simplified result

$$\delta = \alpha(n - 1).$$

Wave optics

80. Modified two-slit experiment



A glass plate of thickness t is placed before one of the slits in the two-slit experiment. How does it modify the two-slit interference?

Solution. As the wave length in glass is smaller than that in the air ($\lambda_n = \lambda/n < \lambda$), the wave going through the glass make an effective extra distance in comparison to that going through the air. To define this extra distance, one has to consider phases of the waves. Phase φ in radians, accrued at the distance x is given by

$$\varphi = 2\pi\frac{x}{\lambda}. \qquad (1)$$

Indeed, at the distance $x = \lambda$ the accrued phase is 2π, that is, $360°$. The phase accrued in the glass plate is

$$\varphi_n = 2\pi\frac{t}{\lambda_n} = 2\pi n\frac{t}{\lambda}.$$

On the other hand, for the wave traveling the same distance through the air the accrued phase φ is given by a similar formula without n. The phase difference between the two waves is thus

$$\Delta\varphi \equiv \varphi_n - \varphi = 2\pi(n-1)\frac{t}{\lambda}.$$

Now we can define the effective extra distance Δx by inverting the phase-distance relation (1):

$$\Delta x = \lambda\frac{\Delta\varphi}{2\pi} = (n-1)t.$$

This adds up to the extra distance in the double-slit experiment $d\sin\theta$. In particular, the condition for the interference maximum now has the form

$$(n-1)t + d\sin\theta = m\lambda, \qquad m = \text{integer}.$$

The glass plate rotates all interference maxima in the negative direction. In particular, the zero-order fringe is seen at the angle corresponding to $\sin\theta = -(n-1)t/d$.

81. One-slit diffraction

83. A laser beam passes through a slit of width 1.0 cm and is pointed at the Moon, which is approximately 380,000 km from the Earth. Assume the laser emits waves of wavelength 630 nm (the red light of a He-Ne laser). Estimate the width of the beam when it reaches the Moon.

Solution. First, introduce missing notations: $\lambda = 630\ nm = 6.3 \times 10^{-7}\ m$, $D = 1\ cm = 10^{-2}$, $L = 380{,}000\ km = 3.8 \times 10^{8} m$.

The laser beam undergoes one-slit diffraction, and its angular width is determined by the angle θ corresponding to the first diffraction minimum that is defined by

$$D \sin\theta = m\lambda, \qquad m = \pm 1.$$

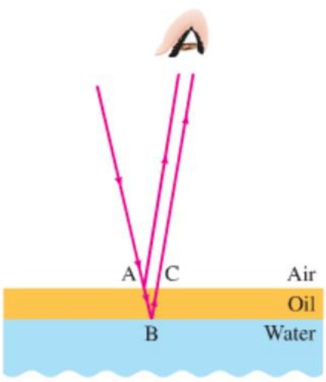
Since this angle is very small, $\sin\theta \cong \tan\theta \cong \theta$. The width of the beam on the Moon's surface is

$$\Delta x = 2L\tan\theta \cong 2L\theta = \frac{2L\lambda}{D}.$$

Substituting the numbers, one obtains

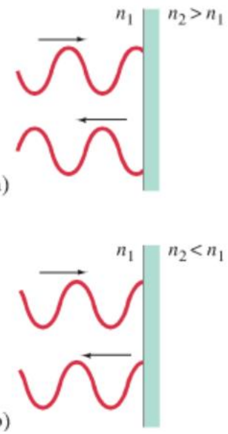
$$\Delta x = \frac{2 \times 3.8 \times 10^{8} \times 6.3 \times 10^{-7}}{10^{-2}} = 47880\ m = 48\ km.$$

82. Thin-film interference (theory)



Thin-film interference occurs between the light rays reflected from different surfaces of a thin film, as shown for the thin layer of oil on the water. To observe interference in natural light that is incoherent, the thickness of the film t has to be much smaller than the coherence length of light, $t \ll \xi$. For thicker films, the two rays will have a random phase with respect to each other so that there will be no interference. Practically, thin-film interference can be observed on films with the thickness comparable to the wave length of light λ. In the laser light that is coherent, interference can be observed on thicker films.

The condition for constructive interference is that the phase shift between the two reflected waves is a multiple of 360° that corresponds to the extra distance covered by one ray with respect to the other equal to a multiple of the wave lengths of light in the film $\lambda_n = \lambda/n$. In addition, there is a phase jump by 180° upon reflection if the refraction index of the other media is larger than that of the media hosting the incident and reflected rays (case (a) in the figure below). This corresponds to changing the extra distance by $\lambda_n/2$.



(a) (b)

Thus, the condition for the constructive interference can be written as

$$2t + [\lambda_n/2] + [\lambda_n/2] = m\lambda_n, \quad m = \text{integer}$$

Here the terms in the square brackets should be added or not depending on whether there is a phase jump or not. If there are phase jumps on both surfaces, they cancel each other and have no effect. The condition for the destructive interference has the form

$$2t + [\lambda_n/2] + [\lambda_n/2] = \lambda_n/2 + m\lambda_n, \quad m = \text{integer}$$

In this form, the interference conditions can be understood and memorized.

In the case of an oil film on the water surface, we use $n = 1.47$ for the oil that is larger than that of the water, 1.33. Thus there is a phase jump for the ray reflected at A but no jump for that reflected at B. The condition for the constructive interference becomes

$$2t + \lambda_n/2 = m\lambda_n, \quad m = \text{integer}$$

83. Thin-film interference

85. A thin film of soap ($n = 1.34$) coats a piece of flat glass ($n = 1.52$). How thick is the film if it reflects 643-nm red light most strongly when illuminated normally by white light?

Solution. In this case, there is a phase jump on both interfaces, thus phase jumps can be ignored and the condition for the constructive interference reads

$$2t = m\lambda_n, \quad m = \text{integer}.$$

With increasing the film width t and thus the number m the incoherence of the natural light leads to gradual washing out the interference. Thus the strongest interference is observed for $m = 1$ that requires

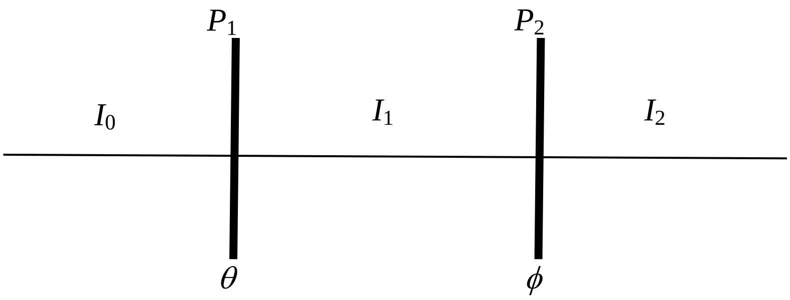
$$t = \frac{\lambda_n}{2} = \frac{\lambda}{2n}.$$

Substituting the numbers, one obtains

$$t = \frac{643}{2 \times 1.34} = 240\ nm.$$

84. Polarizers

58. (II) Two polarizers are oriented at 40° to each other and plane-polarized light is incident on them. If only 15% of the light gets through both of them, what was the initial polarization direction of the incident light?



Solution. First, we introduce missing notations: $\phi = 40°$, $\eta = 0.15$. Using the Malus law for the intensities of light, we obtain

$$I_1 = I_0 \cos^2\theta, \qquad I_2 = I_1 \cos^2\phi.$$

Combining these two formulas, one obtains

$$I_2 = I_0 \cos^2\theta \cos^2\phi.$$

According to the data in the problem,

$$\frac{I_2}{I_0} = \eta.$$

Thus, for the angle θ one obtains the equation

$$\eta = \cos^2\theta \cos^2\phi.$$

Solving it for θ, one obtains

$$\theta = \cos^{-1}\sqrt{\frac{\eta}{\cos^2\phi}} = \cos^{-1}\frac{\sqrt{\eta}}{\cos\phi}.$$

Substituting the numerical values yields

$$\theta = \cos^{-1}\frac{\sqrt{0.15}}{\cos 40°} = 59.63° \approx 60°.$$