PROBLEMS #9

(1) We already know that the actual path is a straight line within one medium. Therefore the segments from P_1Q and QP_2 are straight and the corresponding distances are

$$P_1Q = \sqrt{x^2+y^2+z^2}$$

$$P_2Q = \sqrt{(x-x_1)^2+y_2^2+z^2}.$$

Therefore the total time for the journey P_1QP_2 is

$$T = \frac{1}{c}\left(\sqrt{x_1^2+y_1^2+z^2} + \sqrt{(x-x_1)^2+y_2^2+z^2}\right)$$

To find the position of $Q=(x,0,z)$ for

which this is a minimum we must differentiate with respect to z and x and set the derivatives equal to zero

$$\frac{\partial T}{\partial z} = \frac{z}{c\sqrt{}} + \frac{z}{c\sqrt{}} = 0 \Rightarrow z = 0$$

which says that Q must lie in the same vertical plane as P_1 and P_2

and

$$\frac{\partial T}{\partial x} = \frac{x}{c\sqrt{}} + \frac{x - x_1}{c\sqrt{}} = 0$$

$\Rightarrow$

$$\sin\theta_1 = \sin\theta_2$$

or

$$\theta_1 = \theta_2$$

(2) The lengths of the paths P_1Q and QP_2 are

$$P_1Q = \sqrt{x^2 + h_1^2 + z^2}$$

and

$$QP_2 = \sqrt{(x_2 - x)^2 + h_2^2 + z^2}$$

The time for light to traverse each path is the path length divided by the speed of light $v = c/n$. Thus the total time is

$$T = \frac{1}{c}\left(n_1 \sqrt{x^2 + h_1^2 + z^2} + n_2 \sqrt{(x_2 - x)^2 + h_2^2 + z^2} \right)$$

To find where this is a minimum we must set $\frac{\partial T}{\partial z}$ and $\frac{\partial T}{\partial x}$ equal to zero

$$\frac{\partial T}{\partial z} = \frac{1}{c}\left(\frac{\mu_1 z}{\sqrt{x^2+h_1^2+z^2}} + \frac{\mu_2 z}{\sqrt{(x_2-x)^2+h_2^2+z^2}} \right),$$

which is zero if and only if $z=0$.

That is Fermat's principle requires that Q lie in the plane containing P_1 and P_2 and normal to the inter-face.

$$\frac{\partial T}{\partial x} = \frac{1}{c}\left(\frac{\mu_1 x}{\sqrt{x^2+h_1^2+z^2}} - \frac{\mu_2 (x_2-x)}{\sqrt{(x_2-x)^2+h_2^2+z^2}} \right)$$

$$= \frac{1}{c}\left(\mu_1 \sin\theta_1 - \mu_2 \sin\theta_2 \right)$$

which is zero if and only if

$\mu_1 \sin\theta_1 = \mu_2 \sin\theta_2$, and this is Snell's law

(3)

Consider an infinitesimal section of path on the sphere, in which θ increases by $d\theta$ and ϕ by $d\phi$. This carries us a distance to the south $R\,d\theta$, and $R\sin\theta\,d\phi$ to the east. The distance ds along the path is therefore

$$ds = \sqrt{(R\,d\theta)^2 + (R\sin\theta\,d\phi)^2} = R\sqrt{1+\sin^2\theta\left[\phi'(\theta)\right]^2}\,d\theta$$

Therefore the total path length is

$$R\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\left[\phi'(\theta)\right]^2}\,d\theta$$

Since $\frac{\partial f}{\partial \phi} = 0$, the Euler-Lagrange equation reduces to

$$\frac{\partial f}{\partial \phi'} = \frac{\sin^2\theta\, \phi'}{\sqrt{1+\sin^2\theta\, \phi'^2}} = c$$

Where we have taken

$$f = f(\phi, \phi', \theta) = \sqrt{1+\sin^2\theta\, \phi'^2}$$

and the path length L is then

$$L = \int f\, d\theta .$$

If we choose our polar axis to go through the point 1, then $\theta_1 = 0$ and the constant c has to be zero. Thus the Euler-Lagrange equation implies that $\phi' = 0$ and hence that $\phi = ct$. The curves of constant ϕ are the lines of longitude and are great circles.

Therefore the geodesic are great circles

(4) The integrand is $f(y,y',x) = \sqrt{x}\sqrt{1+y'^2}$.

Since this is independent of y, $\frac{\partial f}{\partial y} = 0$

and the Euler-Lagrange equation

implies simply that $\partial f / \partial y' = $ cte.

$$\frac{\sqrt{x}\, y'}{\sqrt{1+y'^2}} = k$$ This can be solved for

y' to find $y' = \frac{k}{\sqrt{x-k^2}}$, which

integrates to give

$$y = 2k\sqrt{x-k^2} + D$$, where D is

a constant of integration. Rearranging

terms we find that

$$x = k^2 + \frac{(y-D)^2}{4k^2}$$, which is a parabola

with its axis along the line $y = D$

(5)

The integrand is

$$f(y, y', x) = x\sqrt{1-y'^2}.$$

Since this is independent of y,

$\frac{\partial f}{\partial y} = 0$ and the EL equation

implies simply that $\frac{\partial f}{\partial y'}$ is a constant;

$$\frac{xy'}{\sqrt{1-y'^2}} = k$$

This can be solved for y' to give

$$y' = \frac{k}{\sqrt{x^2+k^2}}$$, which integrates

to give $y = k \operatorname{Arcsh}(x/k) + c$,

where c is a constant of integration.

(Making the substitution $\frac{x}{K} = \operatorname{sh} u$)

Rearranging one finds that

$$x = K \sinh\left[(y-c)\right]/K$$

(c) If we write the path as

$\phi = \phi(r)$, the distance from O to P

is $\int_O^P ds = \int_0^R f\, dr$,

where $f = \frac{2}{(1-r^2)} \sqrt{1+r^2\phi'^2}$

Since $\frac{\partial f}{\partial \phi} = 0$; the Euler-Lagrange

equation implies simply that $\frac{\partial f}{\partial \phi'} = $ cte

$$\frac{r}{(1-r^2)}\,\frac{r^2\phi'}{\sqrt{1+r^2\phi'^2}} = K$$

Because the path passes through the origin, $r=0$, the constant K must in fact be zero, and we find $\phi' = 0$. This defines a straight line through the origin

(7) The area between the string and the x axis is $A = \int y\,dx$. The length of a small element of string satisfies $ds^2 = dx^2 + dy^2$,

so $dx = \sqrt{ds^2 - dy^2} = \sqrt{1-y'^2}\,ds$,

if we regard y as a function of s and $y' = dy/ds$.

Therefore, the area A can be written

as
$$A = \int_0^{\ell} f \, ds$$

where $f(y, y', s) = y\sqrt{1 - y'^2}$

Note that f does not depend on s

i.e., $f = f(y, y')$. Then by the standard

result of two variable calculus

$$df = \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial y'} dy'$$

Dividing both sides by dx we find

$$\frac{df}{dx} = \frac{\partial f}{\partial y} y' + \frac{\partial f}{\partial y'} y'' =$$

$$\left(\frac{d}{dx} \frac{\partial f}{\partial y'}\right) y' + \frac{\partial f}{\partial y'} y'' = \frac{d}{dx}\left(y' \frac{\partial f}{\partial y'}\right)$$

where for the second equality

we have used Euler-Lagrange

equation and in the last the product rule. Moving the right side across to the left, we see that $f - y'\frac{\partial f}{\partial y'}$ is constant.

$$f - y'\frac{\partial f}{\partial y'} = y\sqrt{1-y'^2} + y'\frac{yy'}{\sqrt{1-y'^2}}$$

$$= \frac{y}{\sqrt{1-y'^2}} = R$$

where R is some constant.

This last equation implies that

$$y' = \sqrt{1 - y^2/R^2}$$

or equivalently

$$\frac{dy}{\sqrt{1-y^2/R^2}} = ds$$

Integrating both sides we conclude that

$$\arcsin(y/R) = s/R$$

(The constant of integration is zero because $y=0$ when $s=0$) Therefore

$y = R \sin(s/R)$

Since $y=0$ when $s=\ell$, we see that $\ell/R = \pi$. (It is fairly easy to see that the other solutions, $\ell/R = 2\pi, 3\pi, \ldots$ yield smaller ones.) Finally we

Saw that $dx = \sqrt{1-y'^2}\, ds$, so

$$x = \int \sqrt{1-y'^2}\, ds = R - R\cos(S/R)$$

Combining these results for x and y

we see $(x-R)^2 + y^2 = R^2$, so

the string must lie on the semicircle

with radius R centered on the

point $(R,0)$

(8) The element of path length is

$$ds = \sqrt{dx^2+dy^2+dz^2} = \sqrt{x'^2+y'^2+z'^2}\, du$$

Thus the total path length is

$$L = \int f\, du \quad \text{where } f = \sqrt{x'^2+y'^2+z'^2}.$$

There are three Euler-Lagrange equations, which involve the following six derivatives

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

and

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2+y'^2+z'^2}}$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2+y'^2+z'^2}}$$

$$\frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}}$$

Since the first three derivatives are zero, the Euler Lagrange equations imply simply that each of the last 3 is a constant. This means that the ratios $x' : y' : z'$ are constant, which implies in turn that as we move along the curve the ratios $dx : dy : dz$ are constant. In other words, the curve is a straight line