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Problems set # 12 **Physics 303** November 25 and December 2, 2014

1. A spherical black body of radius r at absolute temperature T is surrounded by a thin spherical and concentric shell of radius R, black on both sides. Show that the factor by which this radiation shield reduces the rate of cooling of the body (consider space between spheres evacuated, with no thermal conduction losses) is given by the following expression $aR^2/(R^2 + br^2)$, and find the numerical coefficients a and b.

Solution: Let the surrounding temperature be T_0. The rate of energy loss of the black body before being surrounded by the spherical shell is $Q = 4\pi r^2 \sigma(T^4 - T_0^4)$. The energy loss per unit time by the black body after being surrounded by the shell is $Q' = 4\pi r^2 \sigma(T^4 - T_1^4)$, where T_1 is the temperature of the shell. The energy loss per unit time by the shell is $Q'' = 4\pi R^2 \sigma(T_1^4 - T_0^4)$. Since $Q'' = Q'$, we obtain $T_1^4 = (r^2 T^4 + R^2 T_0^4)/(R^2 + r^2)$. Hence, $Q'/Q = R^2/(R^2 + r^2)$, *i.e.* $a = 1$ and $b = 1$.

2. The solar constant (radiant flux at the surface of the earth) is about 0.1 W/cm^2. Find the temperature of the sun assuming that it is a black body.

Solution: The radiant flux density of the sun is $J = \sigma T^4$, where $\sigma = 5.7 \times 10^{-8}$ W/m^{2}K^4. Thus, $\sigma T^4 (R_\odot/d_{\oplus-\odot})^2 = 0.1$, where $R_\odot = 7.0 \times 10^5$ km is the radius of the sun and $d_{\oplus-\odot} = 1.5 \times 10^8$ km is the distance between the earth and the sun. Hence, $T = \left[\frac{0.1}{\sigma}\left(\frac{d_{\oplus-\odot}}{R_\odot}\right)^2\right]^{1/4} \approx 6 \times 10^3$ K.

3. Estimate the temperature of the sun's surface given that the sun subtends an angle θ as seen from the earth's and the earth's surface temperature is T_0. (Assume the earth's surface temperature is uniform, and that the earth reflects a fraction, ε of the solar radiation incident upon it.) Use your result to obtain a rough estimate of the sun's surface temperature by putting in "reasonable" values for all parameters.

Solution: The earth radiates heat while it is absorbing heat from the solar radiation. Assume that the sun can be taken as a black body. Because of reflection, th earth is a greybody of emissivity $1 - \epsilon$. The equilibrium condition is $(1 - \epsilon) J_\odot 4\pi R_\odot^2 \pi R_\oplus^2/(4\pi d_{\oplus-\odot}^2) = J_\oplus 4\pi R_\oplus^2$, where $J_\odot$ and $J_\oplus$ are the radiated energy flux densities on the surfaces of the sun and the earth respectively, $R_\odot$, $R_\oplus$, and $d_{\oplus-\odot}$ are the radius of the sun, the radius of the earth, and the distance between the earth and the sun, respectively. Obviously $R_\odot/d_{\oplus-\odot} = \tan(\theta/2)$. From the Stefan-Boltzmann law, we have: *(i)* for the sun $J_\odot = \sigma T_\odot^4$ and for the earth $J_\oplus = (1 - \epsilon)\sigma T_\oplus^4$. Therefore $T_\odot = T_\oplus \left(\frac{2d_{\oplus-\odot}}{R_\odot}\right)^{1/2} \approx 300 \text{ K} \left(2\frac{1.5\times 10^8 \text{ km}}{7\times 10^5 \text{ km}}\right)^{1/2} \approx 6000$ K.

4. Consider an idealized sun and earth, both black bodies, in otherwise empty space. The sun is at a temperature of $T_\oplus = 6000$ K and heat transfer by oceans and atmosphere on the earth is so effective as to keep the earth surface uniform. The radius of the earth is $R_\oplus = 6 \times 10^8$ cm, the

radius of the sun is $R_\odot = 7 \times 10^{10}$ cm, and the earth radius distance is $d_{\oplus-\odot} = 1.5 \times 10^{13}$ cm. *(i)* Find the temperature of the earth. *(ii)* Find the radiation force on the earth. *(iii)* Compare these results with those for an interplanetary "chondrule" in the form of a spherical, perfectly conducting black-body with a radius of $R = 0.1$ cm, moving in a circular orbit around the sun with a radius equal to the earth-sun distance $d_{\oplus-\odot}$.

Solution: *(i)* The radiation received per second by the earth from the sun is approximately $q_{\oplus\odot} = 4\pi R_\odot^2(\sigma T_\odot^4)\frac{\pi R_\oplus^2}{4\pi d_{\oplus-\odot}^2}$. The radiation per second from the earth itself is $q_\oplus = 4\pi R_\oplus^2 \sigma T_\oplus^4$. Neglecting the earth's own heat sources, energy conservation leads to the relation $q_\oplus = q_{\oplus\odot}$, so that $T_\oplus^4 = \frac{R_\oplus^2}{4d_{\oplus-\odot}^2}T_\odot^4$, that is $T_\oplus = \sqrt{R_\odot/(2d_{\oplus-\odot})}T_\odot = 290$ K $= 17°$ C. *(ii)* The angles subtended by the earth in respect of the sun and by the sun in respect to the earth are very small, so the radiation force is $F_\oplus = \frac{q_\oplus}{c} = \frac{1}{c}\frac{R_\odot^2}{d_{\oplus-\odot}^2}\pi R_\oplus^2 \sigma T_\odot^4 = 6 \times 10^8$ N. *(iii)* As $R_\oplus \to R$, $T = T_\oplus = 17°$ C, $F = (R/R_\oplus)^2 F_\oplus = 1.7 \times 10^{-11}$ N.

5. Making reasonable assumptions, estimate the surface temperature of Neptune. Neglect any possible internal source of heat. What assumptions have you made about the planet's surface and/or atmosphere? [*Hint:* Astronomical data which may be helpful: radius of sun $= 7 \times 10^5$ km; radius of Neptune $= 2.2 \times 10^4$ km; mean sun-earth distance $= 1.5 \times 10^8$ km; mean sun-Neptune distance $= 4.5 \times 10^9$ km; temperature of the sun $= 6000$ K; rate at which sun's radiation reaches earth $= 1.4$ kW/m^2; Stefan-Boltzmann constant $= 5.7 \times 10^{-8}$ W/m^{2}K^4.]

Solution: We assume that the surface of Neptune and the thermodynamics of its atmosphere are similar to those of the earth. The radiation flux on the earth's surface is $J_\oplus = 4\pi R_\oplus^2 \sigma T_\oplus^4/(4\pi d_{\oplus-\odot}^2)$. The equilibrium condition on Neptune's surface gives $\frac{4\pi R_\odot^2 \sigma T_\odot^4 \pi R_\mathrm{N}^2}{4\pi d_{\mathrm{N}\odot}^2} = \sigma T_\mathrm{N}^4 4\pi R_\mathrm{N}^2$. Hence $d_\oplus^2 J_\oplus / d_{\mathrm{N}\odot}^2 = 4\sigma T_\mathrm{N}^4$, and we have $T_\mathrm{N} = \left(\frac{d_{\oplus\odot}^2 J_\oplus}{4\sigma d_{\mathrm{N}\odot}^2}\right)^{1/4} \approx 52$ K.

6. Consider a photon gas enclosed in a volume V and in equilibrium at temperature T. The photon is a massless particle, so that $\varepsilon = pc$. *(i)* What is the chemical potential of the gas? *(ii)* Determine how the number of photons in the volume depends upon the temperature. *(iii)* One may write the energy density in the form

$$\frac{\overline{U}}{V} = \int_0^\infty \rho(\omega)d\omega.$$

Determine the form of $\rho(\omega)$, the spectral density of the energy. *(iv)* What is the temperature dependence of the energy $\overline{U}$?

Solution: *(i)* The chemical potential of the photon gas is zero. Since the number of photons is not conserved at a given temperature and volume, the average photon number is determined by the expression $\left(\frac{\partial F}{\partial \overline{N}}\right)_{T,V} = 0$, then $\mu = \left(\frac{\partial F}{\partial \overline{N}}\right)_{T,V} = 0$. *(ii)* The density of states is $8\pi V P dP/h^3$, or $V\omega^2 d\omega/\pi^2 c^3$. Then the number of photons is $\overline{N} = \int \frac{V}{\pi^2 c^3}\omega^2 \frac{1}{e^{\hbar\omega/kT}-1}d\omega = \frac{V}{\pi^2 c^3}\left(\frac{kT}{\hbar}\right)^3 \int_0^\infty \frac{\alpha^2\, d\alpha}{e^\alpha - 1} \propto T^3$. *(iii)*, *(iv)* $\frac{\overline{U}}{V} = \int \frac{\omega^2}{\pi^2 c^3}\frac{\hbar\omega}{e^{\hbar\omega/kT}-1}d\omega = \frac{(kT)^4}{\pi^2 c^3 \hbar^3}\int \frac{\xi^3 d\xi}{e^\xi - 1}$. Hence, $\rho(\omega) = \frac{\hbar}{\pi^2 c^3}\frac{\omega^3}{e^{\hbar\omega/kT}-1}$, and $\overline{U} \propto T^4$.

7. Consider a gas of non-interacting, non-relativistic, identical bosons. Explain whether and why the Bose-Einstein condensation effect that applies to a three-dimensional gas applies also to a two -dimensional gas and to a one-dimensional gas.

Solution: Roughly speaking, the Bose-Einstein condensation occurs when $\mu = 0$. For a two dimensional gas it follows that $N = \frac{2\pi m A}{h^2} \int_0^\infty \frac{d\varepsilon}{e^{(\varepsilon-\mu)/kT}-1} = \frac{2\pi m A}{h^2} \int_0^\infty \left(\sum_{l=1}^\infty e^{-l(\varepsilon-\mu)/kT} \right) d\varepsilon = \frac{2\pi m A}{h^2} kT \sum_{l=1}^\infty \frac{1}{l} e^{l\mu/kT}$. If $\mu = 0$, the above expression diverges. Hence $\mu \neq 0$ and Bose-Einstein condensation does not occur. For a one-dimensional gas we have $N = \frac{\sqrt{2mL}}{2h} \int_0^\infty \frac{d\varepsilon}{\sqrt{\varepsilon}(e^{(\varepsilon-\mu)/kT}-1)}$. If $\mu = 0$, the integral diverges. Again, Bose-Einstein condenstaion does not occur.

8. The universe is pervaded by 3K black body radiation. In a simple view, this radiation arose from the adiabatic expansion of a much hotter photon cloud which was produced during the big bang. *(i)* Why is the recent expansion adiabatic rather than, for example, isothermal? *(ii)* Write down an integral which determines how many photons per cubic centimeter are contained in this cloud of radiation. Estimate the result within an order of magnitude. *(iii)* Show that a freely expanding blackbody radiation remains described by the Planck formula, but with a temperature that drops in proportion to the scale expansion. *(iv)* If in the next 10^{10} yr the volume of the universe increases by a factor of 2, what then will be the temperature of the blackbody radiation.

Solution: *(i)* The photon cloud is an isolated system, so its expansion is adiabatic. *(ii)* In 1964, Arno Penzias and Robert Wilson were experiencing difficulty with what they assumed to be background noise, or "static," in their radio telescope. Eventually, they became convinced that it was real and that it was coming from outside the Galaxy. They made precise measurements at wavelength $\lambda = 7.35$ cm, in the microwave region of the electromagnetic spectrum. The intensity of this radiation was found initially not to vary by day or night or time of the year, nor to depend on the direction. It came from all directions in the universe with equal intensity, to a precision of better than 1%. It could only be concluded that this radiation came from the universe as a whole. The intensity of this cosmic microwave background (CMB) as measured at $\lambda = 7.35$ cm corresponds to a blackbody radiation at a temperature of about 3 K. When radiation at other wavelengths was measured, the intensities were found to fall on a blackbody curve, corresponding to a temperature of 2.725 K. The CMB provides strong evidence in support of the Big Bang, and gives us information about conditions in the very early universe. To understand why, let us look at what a Big Bang might have been like. The temperature must have been extremely high at the start, so high that there could not have been any atoms in the very early stages of the universe. Instead the universe would have consisted solely of radiation (photons) and a plasma of charged electrons and other elementary particles. The universe would have been opaque - the photons in a sense "trapped," travelling very short distances before being scattered again, primarily by electrons. Indeed, the details of the CMB provide strong evidence that matter and radiation were once in thermal equilibrium at very high temperature. As the universe expanded, the energy spread out over an increasingly larger volume and the temperature dropped. Only when the temperature had fallen to about 3,000 K was the universe cool enough to allow the combination of nuclei and electrons into atoms. (In the astrophysical literature this is usually called "recombination," a singularly inappropriate term, for at the time we were considering the nuclei and electrons had never in the previous history of the universe been combined into atoms!) The sudden disappearance of electrons broke the thermal contact between radiation and matter, and the radiation continued thereafter to expand freely. At the moment this happened, the energy in the radiation field at various wavelengths was governed by the conditions of the thermal equilibrium, and was therefore given by the Planck blackbody formula with a temperature equal to that of the matter $\sim 3,000$ K. In particular, the typical photon wavelength would have been about one micron, and the average distance between photons would have been roughly equal to this typical wavelength. What has happened to the photons since then? Individual photons would not be created or destroyed, so the average distance between photons would simply increase in proportion to the size of the universe, i.e., in proportion to the average distance between typical galaxies. The Planck distribution that gives the energy du of a blackbody radiation per unit volume, in a narrow range of wavelengths from λ to $\lambda + d\lambda$, is $du = \frac{8\pi hc}{\lambda^5}\, d\lambda \,\frac{1}{e^{hc/\lambda kT}-1}$. For long wavelengths, the denominator in the Planck distribution may be approximated by $e^{hc/\lambda kT} - 1 \simeq hc/\lambda kT$. Hence, in this wavelength region, $du = \frac{8\pi kT}{\lambda^4}\, d\lambda$. This is the Rayleigh-Jeans formula. If this formula held down to arbitrarily small wavelengths, $du/d\lambda$ would become infinite for $\lambda \to 0$, and the total energy density in the blackbody radiation would be infinite. Fortunately, as we saw before, the Planck formula for du reaches a maximum at a wavelength $\lambda = 0.2014052\, hc/kT$ and then falls steeply off for decreasing wavelengths. The total energy density in the blackbody radiation is $u = \int_0^\infty \frac{8\pi hc}{\lambda^5}\, d\lambda \,\frac{1}{e^{hc/\lambda kT}-1}$. Integrals of this sort can be looked up in standard tables of definite integrals; the result gives the Stefan-Boltzmann law $u = \frac{8\pi^5 (kT)^4}{15(hc)^3} = 7.56464 \times 10^{-15}\,(T/\mathrm{K})^4 \mathrm{erg/cm^3}$. (Recall that 1 J $\equiv 10^7$ erg $= 6.24 \times 10^{18}$ eV.) We can easily interpret the Planck distribution in terms of quanta of light or photons. Each photon has an energy $E = hc/\lambda$. Hence the number dn of photons per unit volume in blackbody radiation in a narrow range of wavelengths from λ to $\lambda + d\lambda$ is $dn = \frac{du}{hc/\lambda} = \frac{8\pi}{\lambda^4}\, d\lambda \,\frac{1}{e^{hc/\lambda kT}-1}$. Then the total number of photons per unit volume is $n = \int_0^\infty dn = 8\pi \left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2\, dx}{e^x - 1}$, where $x = hc/(\lambda kT)$. The integral cannot be expressed in terms of elementary functions, but it can be expressed as an infinite series $\int_0^\infty \frac{x^2\, dx}{e^x-1} = 2\sum_{j=1}^\infty \frac{1}{j^3} \approx 2.4$. Therefore, the number photon density is $n = 60.42198 \left(\frac{kT}{hc}\right)^3 = 20.28 \left(\frac{T}{\mathrm{K}}\right)^3$ photons $\mathrm{cm^{-3}} \approx 400$ photons $\mathrm{cm^{-3}}$, and the average photon energy is $\langle E_\gamma \rangle = u/n = 3.73 \times 10^{-16}\,(T/\mathrm{K})$ erg . *(iii)* Now, let's consider what happens to blackbody radiation in an expanding universe. Suppose the size of the universe changes by a factor f, for example, if it doubles in size, then $f = 2$. As predicted by the Doppler effect, the wavelengths will change in proportion to the size of the universe to a new value $\lambda' = f\lambda$. After the expansion, the energy density du' in the new wavelength range λ' to $\lambda' + d\lambda'$ is less than the original energy density du in the old wavelength range $\lambda + d\lambda$, for two different reasons: (a) Since the volume of the universe has increased by a factor of f^3, as long as no photons have been created or destroyed, the numbers of photons per unit volume has decreased by a factor of $1/f^3$. (b) The energy of each photon is inversely proportional to its wavelength, and therefore is decreased by a factor of $1/f$. It follows that the energy density is decreased by an overall factor $1/f^3 \times 1/f = 1/f^4$: $du' = \frac{1}{f^4}\, du = \frac{8\pi hc}{\lambda^5 f^4}\, d\lambda \,\frac{1}{e^{hc/\lambda kT}-1}$. If we rewrite the previous equation in terms of the new wavelengths λ', it becomes $du' = \frac{8\pi hc}{\lambda'^5}\, d\lambda' \,\frac{1}{e^{hcf/\lambda' kT}-1}$, which is exactly the same as the old formula for du in terms of λ and $d\lambda$, except that T has been replaced by a new temperature $T' = T/f$. Therefore, we conclude that freely expanding blackbody radiation remains described by the Planck formula, but with a temperature that drops in inverse proportion to the scale of expansion. *(iv)* The energy density of black body radiation is $u = aT^4$, so that the total energy $U \propto VT^4$. From the first law $TdS = dU + PdV$, we have $T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V \propto VT^3$. Hence, $S = VT^3\times$ constant. For a reversible adiabatic expansion, the entropy S remains unchanged. Thus, when V doubles T will decrease by a factor $(2)^{-1/3}$. So after another 10^{10} yr, the temperature of black body radiation will become $T = 3$ K$/2^{1/3}$.