PROBLEMS # 3

(1) Expand the potential about the equilibrium point

$$U(x) = \sum_{i=n+1}^{\infty} \frac{1}{i!} \left[\frac{d^i U}{dx^i}\right]_0 x^i$$

The leading term in the force is then

$$F(x) = -\frac{dU(x)}{dx} = \frac{-1}{n!} \left[\frac{d^{n+1} U}{dx^{n+1}}\right]_0 x^n$$

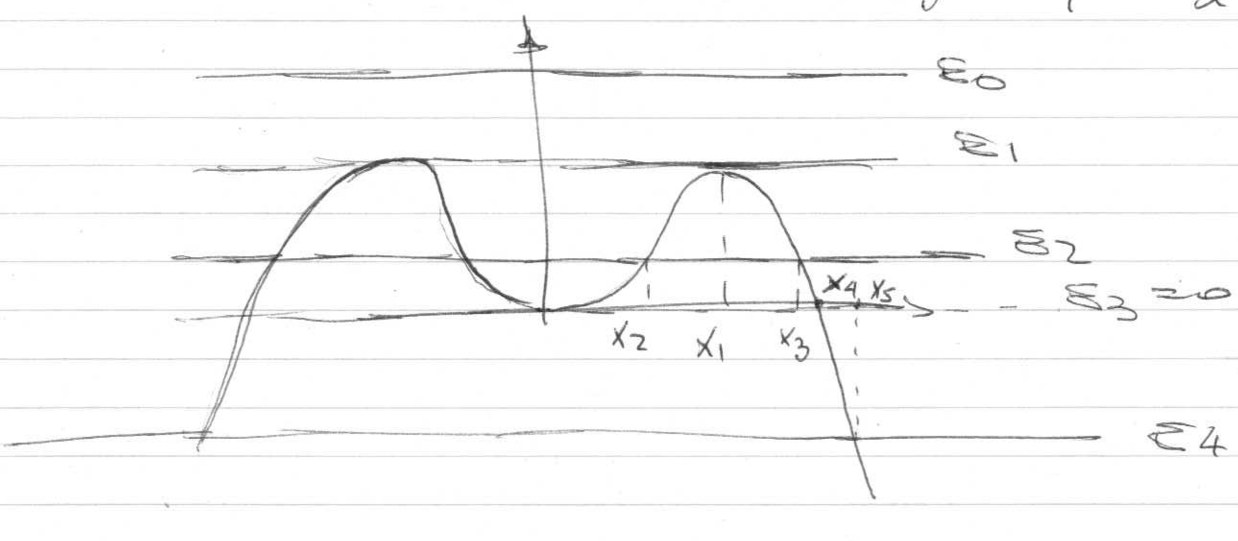
The force is restoring for a stable point, so we need $F(x>0) < 0$ and $F(x<0) > 0$. This is never true when n is even (e.g. $U = kx^3$) and only true for n odd when

$$\left(\frac{d^{n+1} U}{dx}\right)_0 < 0$$

(2) · $F = -kx + \frac{kx^3}{\alpha^2}$

$$U(x) = -\int F\,dx = \frac{1}{2}kx^2 - \frac{1}{4}k\frac{x^4}{\alpha^2}$$

To sketch $U(x)$, we note that for small x $U(x)$ behaves like the parabola $\frac{1}{2}kx^2$. For large x the behavior is determined by $-\frac{1}{4}k\frac{x^4}{\alpha^2}$



$$E = \frac{1}{2}mv^2 + U(x)$$

For $E = E_0$, the motion is unbounded, the particle may be anywhere. For $E = E_1$ (at the maximum in $U(x)$) the particle is at a point of unstable equilibrium. It may remain at rest where it is, but if perturbed slightly, it will move away from the equilibrium.

What is the value of E_1? We find the x values by setting $\frac{dU}{dx} = 0$

$$0 = kx - kx^3/\alpha^2$$

$x = 0, \pm\alpha$ are the equilibrium points

$$U(\pm\alpha) = E_1 = \frac{1}{2}k\alpha^2 - \frac{1}{4}k\alpha^2 = \frac{1}{4}k\alpha^2$$

For $E = E_2$, the particle is either bounded and oscillates between $-x_2$ and x_2; or the particle comes in from $\pm\infty$ to $\pm x_3$ and returns to $\pm\infty$

For $E_3 = 0$, the particle is either at the stable equilibrium point $x=0$, or beyond $x = \pm x_4$.

For E_4, the particle comes in from $\pm\infty$ to $\pm x_5$ and returns.

③ We are given $U(x) = U_0 (a/x + x/a)$ for $x > 0$

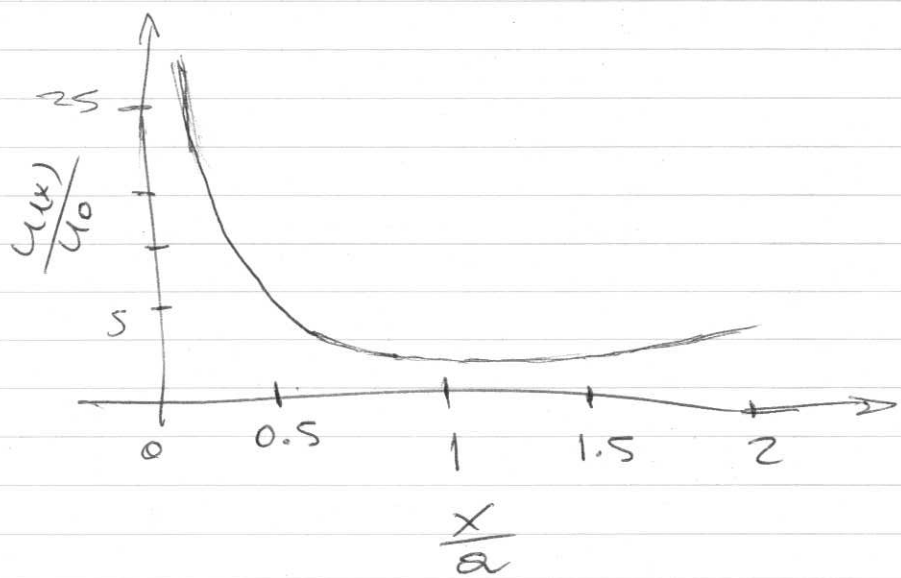
Equilibrium points are defined by $dU/dx = 0$, with stability determined by d^2U/dx^2 at those points. Here we have

$$\frac{dU}{dx} = U_0 \left[-\frac{a}{x^2} + \frac{1}{a} \right]$$

which vanishes at $x = a$. Now evaluate

$$\left.\frac{d^2U}{dx^2}\right|_a = \frac{2U_0}{a^3} > 0$$

indicating that the equilibrium point is stable



(4)

A.– Since the rate of change of mass of the droplet is proportional to its cross-sectional area, we have

$$\frac{dm}{dt} = k\pi r^2 \qquad (1)$$

If the density of the droplet is ρ

$$m = \frac{4}{3}\pi\rho r^3 \qquad (2)$$

So that

$$\frac{dm}{dt} = \frac{dm}{dr}\frac{dr}{dt} = 4\pi\rho r^2 \frac{dr}{dt} = \pi k r^2 \qquad (3)$$

Therefore

$$\frac{dr}{dt} = \frac{k}{4\rho} \qquad (4)$$

or

$$r = r_0 + \frac{k}{4\rho}t \qquad (5)$$

as required

B– The mass changes with time, so the equation of motion is

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt} = mg$$

(6)

Using (1) and (2) this becomes

$$\frac{4}{3}\pi \rho r^3 \frac{dv}{dt} + \pi k r^2 v = \frac{4\pi}{3} \rho r^3 g \qquad (7)$$

or

$$\frac{dv}{dt} + \frac{3k}{4\rho r} v = g \qquad (8)$$

Using (5) this becomes

$$\frac{dv}{dt} + \frac{3k}{4\rho} \frac{v}{r_0 + \frac{k}{4\rho} t} = g \qquad (9)$$

If we set $A = \frac{3k}{4\rho}$ and $B = \frac{k}{4\rho}$, this equation becomes

$$\frac{dv}{dt} + \frac{A}{r_0 + Bt} v = g \qquad (10)$$

and we recognize a standard form for a first order differential equation

$$\frac{dv}{dt} + P(t)\, v = Q(t) \qquad (11)$$

in which we identify

$$P(t) = \frac{A}{r_0 + Bt}, \quad Q(t) = g \qquad (12)$$

The solution of Eq (11) is

$$v(t) = e^{-\int P(t)\,dt}\left[\int e^{\int P(t)\,dt}\,Q\,dt + \text{constant}\right] \tag{13}$$

Now
$$\int P(t)\,dt = \int \frac{A}{r_0 + Bt}\,dt = \frac{A}{B}\ln(r_0 + Bt)$$

$$= \ln(r_0 + Bt)^3 \tag{14}$$

since $A/B = 3$. Therefore

$$e^{\int P(t)\,dt} = (r_0 + Bt)^3 \tag{15}$$

Thus,
$$v(t) = (r_0 + Bt)^{-3}\left[\int (r_0 + Bt)^3 g\,dt + \text{constant}\right]$$

$$= (r_0 + Bt)^{-3}\left[\frac{g}{4B}(r_0 + Bt)^4 + C\right] \tag{16}$$

The constant C can be evaluated by setting $v(t=0) = v_0$

$$v_0 = \frac{1}{r_0^3}\left[\frac{f}{4B} r_0^4 + C\right] \tag{17}$$

so that

$$C = v_0 r_0^3 - \frac{f}{4B} r_0^4 \tag{18}$$

We then have

$$v(t) = \frac{1}{(r_0 + Bt)^3}\left[\frac{f}{4B}(r_0 + Bt)^4 + v_0 r_0^3 - \frac{f}{4B} r_0^4\right]$$

or

$$v(t) = \frac{1}{(Bt)^3}\left[\frac{f}{4B}(Bt)^4 + O\left(r_0^3\right)\right] \tag{19}$$

where $O(r_0^3)$ means terms of order r_0^3 and higher. If r_0 is sufficiently small so that we can neglect these terms, we have

$$v(t) \propto t \tag{20}$$

as required.

(5) Start with our definition of work

$$W = \int F \cdot dx = \int \frac{dp}{dt}\, dx = \int \frac{dx}{dt}\, dp$$

We know that for constant acceleration we must have $v = at$ (zero initial velocity)

$$v = v_0 + u \ln\left(\frac{m_0}{m}\right)$$ (Equation of rocket motion in free space)

$$\Rightarrow \quad m = m_0\, e^{-at/u}$$

We can then compute dp

$$dp = d(mv) = d(mat) = ma\, dt + at\, dm$$

$$= m_0\, a\, e^{-at/u}\left[1 - \frac{at}{u}\right] dt$$

This makes our expression for the work done on the rocket

$$W_r = \frac{m_0 a}{u}\int_0^t (at)(u - at)\, e^{-at/u}\, dt$$

the work done on the exhaust, on the other hand, is given with $v \to (v-u)$ and $dp \to dm_{exhaust}\,(v-u)$, so that

$$W_e = \frac{m_0 a}{u}\int_0^t (at-u)^2\, e^{-at/u}\, dt$$

The upper limit on the integrals is the burnout time, which I take to be the final velocity divided by the acceleration. The total work done by the rocket engine is the sum of this two quantities, so that

$$W = \frac{m_0 a}{u}\int_0^{V/a} (u^2 - uat)\, e^{-at/u}\, dt =$$

$$m_0 u^2 \int_0^{V/u} (1-x)\, e^{-x}\, dx$$

where $x = at/u$. Evaluating this integral we find

$$W = m_0 u V e^{-V/u} = m u V$$

where m is the mass of the rocket after its engines have turned off and v is its final velocity.

(6) To hover above the surface requires the thrust to counteract the gravitational force of the Moon

$$-u \frac{dm}{dt} = \frac{1}{6} m g$$

$$-\frac{6u}{g} \frac{dm}{m} = dt$$

we integrate from $m = m_0$ to $0.8 m_0$ and $t = 0$ to T

$$T = -\frac{6u}{g} \ln 0.8 = -\frac{6 \,(2000 \text{ m/s}) \ln 0.8}{9.8 \text{ m/s}^2}$$

$$T = 273 \text{ sec.}$$

(7) The rocket will lift off when the thrust just exceeds the weight of the rocket

$$\text{Thrust} = -u \frac{dm}{dt} = u\alpha$$

$$\text{Weight} = mg = (m_0 - \alpha t) g$$

Set the thrust = weight and solve for t

$$u\alpha = (m_0 - \alpha t) g \qquad t = \frac{m_0}{\alpha} - \frac{u}{g}$$

with $m_0 = 7000$ kg, $\alpha = 250$ kg/s, $u = 2500$ m/s and $g = 9.8$ m/s^2

$$t = 25 \text{ s}$$

The design problem is that there is too much fuel on board. The rocket sits on the ground burning off fuel until the thrust equals the weight. A real rocket will lift off as soon as the engines reach full thrust. The time the rocket sits on the ground with the engines on is spent building up to full thrust, not burning off excess fuel.

(8) (a) There is only constant acceleration due to gravity to worry about, so the problem can be solved analytically. The rocket height at burnout is

$$y_b = u t_b - \frac{1}{2} g t_b^2 - \frac{m u}{\alpha} \ln\left[\frac{m_0}{m_b}\right]$$

where m_b is the mass of the rocket at burnout and $\alpha = (m_0 - m_b)/t_b$. Substitution of the given values leads to $y_b \simeq 250$ km. After burnout, the rocket travels an additional $v_b^2/2g$, where v_b is the rocket velocity at burnout. The final height of the rocket ends up being 3700 km, after everything is taken into account.

(b) The situation and hence the differential equation becomes more complicated when air resistance is added.

$$F_{ext} = -mg - c_w \rho A v^2/2$$

where $\rho = 1.3\ \mathrm{kg/m^3}$

$$\frac{dv}{dt} = \frac{u\alpha}{m} - g - \frac{c_w \rho A v^2}{2m}$$

We must remember that the mass m is also a function of time, and we must therefore include also in the system of equations. To be specific, the system of equations we must use to do this by computer is

$$\begin{pmatrix} \dot{y} \\ \dot{v} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} v \\ \frac{u\alpha}{m} - g - c_w \rho A v^2 \\ -\alpha \end{pmatrix}$$

These must be integrated from the beginning to the burnout time.

Firstly we get the velocity and height at burnout to be $v_b = 7000$ m/s and $y_b = 230$ km.

We can numerically integrate to get the final part of the journey. The total height to which the rocket rises is ~ 890 km, in total flight time of 410 s

(e) The variation in the acceleration of gravity is taken into account by substituting $GM_\oplus/(R_\oplus + y)^2 = g\left[R_\oplus/(R_\oplus + y)\right]^2$ for g in the differential equation.

This gives $v_b = 6900$ m/s, $y_b \approx 230$ km, with total height ~ 950 km and time of flight 460 s (d). Now one simply substitutes the given expression for the air density, $\rho(y)$ for ρ, into the differential equation. This gives $v_b = 8200$ m/s, $y_b \sim 250$ km, and total height 8900 km with time of flight $\cong 2900$ s.