

# Lecture 21: The Doppler effect

## 1 Moving sources

We'd like to understand what happens when waves are produced from a moving source. Let's say we have a source emitting sound with the frequency  $\nu$ . In this case, the maxima of the amplitude of the wave produced occur at intervals of the period  $T = \frac{1}{\nu}$ . If the source is at rest, an observer would receive these maxima spaced by  $T$ . If we draw the waves, the maxima are separated by a wavelength  $\lambda = Tc_s$ , with  $c_s$  the speed of sound.

Now, say the source is moving at velocity  $v_s$ . After the source emits one maximum, it moves a distance  $v_s T$  towards the observer before it emits the next maximum. Thus the two successive maxima will be closer than  $\lambda$  apart. In fact, they will be  $\lambda_{\text{ahead}} = (c_s - v_s)T$  apart. The second maximum will arrive in less than  $T$  from the first blip. It will arrive with period

$$T_{\text{ahead}} = \frac{\lambda_{\text{ahead}}}{c_s} = \left( \frac{c_s - v_s}{c_s} \right) T \quad (1)$$

The frequency of the blips/maxima directly ahead of the siren is thus

$$\nu_{\text{ahead}} = \frac{1}{T_{\text{ahead}}} = \left( \frac{c_s}{c_s - v_s} \right) \frac{1}{T} = \left( \frac{c_s}{c_s - v_s} \right) \nu. \quad (2)$$

In other words, if the source is traveling directly towards us, the frequency we hear is shifted upwards by a factor of  $\frac{c_s}{c_s - v_s}$ .

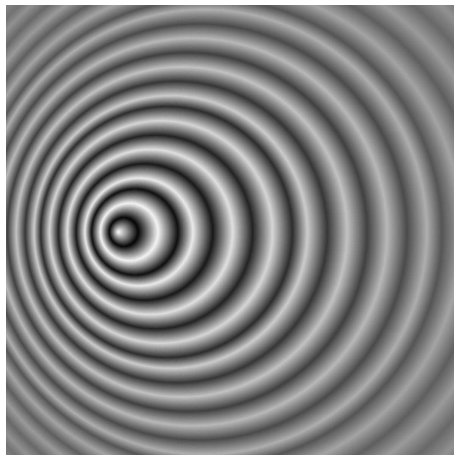
We can do a similar calculation for the case in which the source is traveling directly away from us with velocity  $v$ . In this case, in between pulses, the source travels a distance  $T$  and the old pulse travels outwards by a distance  $c_s T$ . The physical spacing between maxima is therefore  $\lambda_{\text{behind}} = (c_s + v)T$ . The frequency as perceived by an observer behind the siren is thus

$$\nu_{\text{behind}} = \left( \frac{c_s}{c_s + v_s} \right) \nu \quad (3)$$

i.e. lower than for a stationary source.

In other words, the frequency goes up when the source is approaching us, and goes down when it is traveling away from us.

The sound waves produced by the moving source are depicted in Fig. 1.



**Figure 1.** Sound waves emitted by a source moving to the left. The peaks in pressure ahead of the source are more closely spaced than the peaks behind the source.

We can summarize these two results by saying that for a stationary observer directly ahead or behind the moving source,

$$\nu' = \left( \frac{c_s}{c_s + v_s} \right) \nu \quad (4)$$

where  $v_s$  is positive if the source is moving away from the observer, and negative if the source is moving towards the observer.

It is not hard to include also the case when the observer is in motion. Say the source is moving away from the observer. Then the spacing between adjacent peaks are spread out as before  $\lambda_{\text{behind}} = (c_s + v_s)T = \frac{c_s + v_s}{\nu}$ . If the observer is moving towards the source with velocity  $v_r$ , then she passes these peaks faster, at the rate

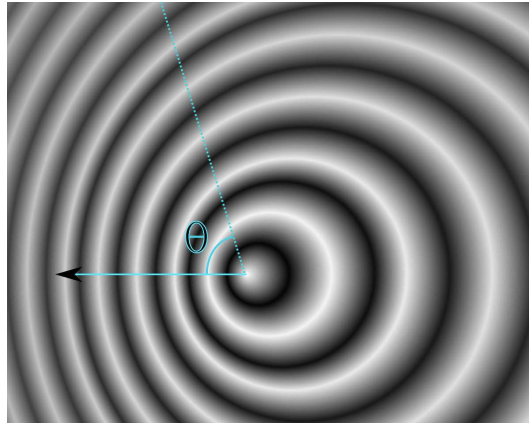
$$\nu' = \frac{c_s + v_r}{\lambda_{\text{ahead}}} = \left( \frac{c_s + v_r}{c_s + v_s} \right) \nu \quad (5)$$

In the cases where the direction of either the source or observer is flipped, the sign of  $v_s$  or  $v_r$  flips, but this equation still holds. Note that if the observer and source are moving at the same speed in the same direction, no frequency change is detected.

This type of change in frequency due to motion is called the **Doppler effect**.

## 2 Motion at an angle

What happens if the source is not moving directly towards or away from the receiver? Say the source is moving at an angle  $\theta$  with respect to a stationary receiver, as shown in Fig. 2.



**Figure 2.** Source moving at angle  $\theta$  relative to the axis connecting the source and the receiver. In this picture, the observer is along the angled blue line, say on the top of the image.

It's easiest to see what the observed frequency is in this case by looking at the picture. Now the maxima are spaced  $\lambda_\theta = (c_s - v_s \cos \theta)$ . Check that for  $\theta = 0$ , this reduces to the ahead case, for  $\theta = \pi$  it reduces to the behind case and for  $\theta = \frac{\pi}{2}$ , where the observer is orthogonal to the direction, there is no change. Following the same logic as before, the frequency of the sound the receiver hears is given by

$$\nu' = \left( \frac{c_s}{c_s - v_s \cos \theta} \right) \nu \quad (6)$$

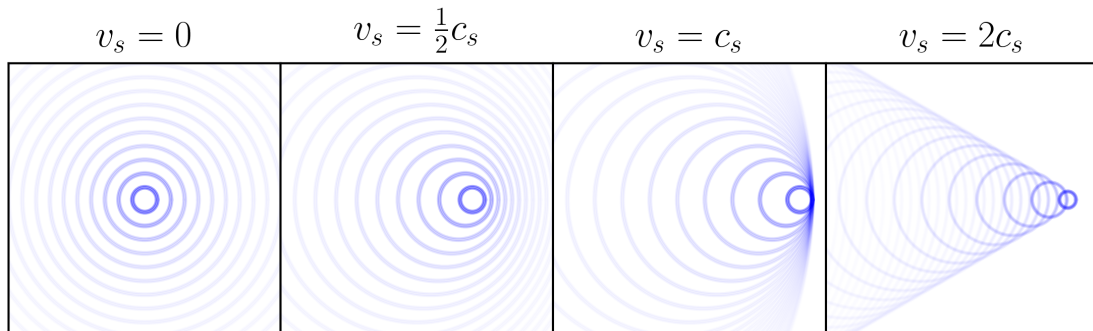
This angular dependence explains why the siren of a police car or ambulance sounds the way it does when it passes you. While approaching at a distance, the car is basically going towards you and the frequency is increased. When it's going away, the frequency is lowered. As the car passes us, the angle transitions pretty quickly, and the sound transitions from high to low as the car goes through the intermediate angles.

Note that the angle  $\theta$  denotes the angle to the velocity vector of the police car when the source was emitted, when the sound is received. To see this, imagine that we are very far away from the source, and it took a couple of days for the sound to get to us. Clearly what the source did during those days, such as the position it ended up at, is irrelevant to the frequency we hear.

Thus the only possibly relevant piece of information is where the source was and how it was moving when it emitted the sound. So the fastest frequency change is not when the car is moving perpendicular to your line of sight, but slightly before the car gets to that point.

### 3 Source moving at or faster than sound

What happens if the source approaches the speed of sound, or surpasses it? Again, the result is easiest to understand with pictures. Here are some wavefronts as the speed of sound is approached and surpassed.



**Figure 3.** Pulses given off by a source moving to the right at various speeds. The older pulses are depicted as dimmer. All the circles are growing with time.

The first two panels we've already discussed: stationary source and a source moving **subsonically** (less than the speed of sound).

As the source approaches the speed of sound, you can see from the picture that the wavefronts in the forward direction start bunching up. When the speed of sound is hit, they all come together. Remember for sound these are wavefronts of pressure. The large increase in pressure from the accumulation of maxima is followed by a large decrease in pressure from the minima. This rapid change in pressure is very loud. It is called a **sonic boom**. You heard it as the cracking of a whip.

When the source goes **supersonically** (faster than the speed of sound), the sonic boom goes from a straight wavefront perpendicular to the motion of the source to being bent backwards in a cone. The sonic boom is now at an angle to the motion of the source, but it is still there. So, to be clear, the sonic boom is a sign that the speed of sound has been surpassed; it is not something that happens only exactly when  $v = c_s$ .

### 4 Redshift

The picture with the wavefronts works just as well for light as for sound, at least for the case  $v_s \ll c$ . Of course, the source can never go faster than the speed of light, due to special relativity, so these equations need some modification.

The nice thing about relativity is that you can pick whatever inertial reference frame you want. So let's work in the rest frame of the source, and call  $v$  the velocity of the observer. Since the source is at rest, the wave crests are spaced  $\lambda = \frac{c}{\nu}$  apart. If the moving observer is heading away from the source, she passes the crests at a rate  $\nu_{\text{move}} = \frac{c-v}{\lambda} = \frac{c-v}{c}\nu$ , as in Eq. (5). However, since the observer is moving very fast there is also a time dilation effect. Time is slower for her, so she really sees successive crests at the lower frequency

$$\nu' = \frac{\nu_{\text{move}}}{\sqrt{1 - \frac{v^2}{c^2}}} = \sqrt{\frac{1 - \frac{v}{c}}{1 + \frac{v}{c}}}\nu \quad (7)$$

More generally,  $v$  is the relative velocity of the source and the observer:  $v = v_s - v_r$  in the notation from before. If  $v > 0$  the two are moving away from each other and  $v < 0$  if they are moving towards each other.

Note that for small  $v \ll c$ , we can Taylor expand Eq. (7) giving  $\nu' = \nu(1 + \frac{v_r - v_s}{c} + \dots)$ . Taylor expanding Eq. (5) gives  $\nu' = \nu(1 + \frac{v_r - v_s}{c_s} + \dots)$ . So in the small velocity limit, this relativistic analysis reduces to our previous results.

In astronomy, it is useful to define the **redshift** of a signal as

$$z \equiv \frac{\Delta \lambda}{\lambda} = -\frac{\Delta \nu}{\nu} = \frac{\nu - \nu'}{\nu} = 1 - \frac{\nu'}{\nu}. \quad (8)$$

Negative redshift is referred to as **blueshift**. These names are used because a signal which is redshifted is shifted to longer wavelengths, and red is longest-wavelength light visible to humans. Blueshifted signals are shifted to shorter wavelengths, and blue is the shortest-wavelength light visible to humans. Astronomers commonly use “red” as a synonym for long-wavelength, and “blue” as a synonym for short-wavelength. For objects at low velocity compared to the speed of light, plugging Eq. (8) into the above formula for redshift yields

$$z \approx \frac{v}{c} \quad (9)$$

where  $v$  is positive if the source is moving away from the receiver. Sources receding from the observer thus appear redshifted, while sources moving towards the observer appear blueshifted. This is an incredibly useful fact in astrophysics, as it allows us to measure the velocity with which distant sources of light are receding from or approaching us.

So what do we find? Everywhere we look, the objects are redshifted. We can sometimes measure the distance to an object by how bright it is, or using parallax. When we do this we find an essentially linear relationship between distance and redshift.

$$z = \frac{1}{c} H_0 r \quad (10)$$

where  $H_0$  is a constant, called **Hubble’s constant**, named after Edwin Hubble who first observed this relation. The value is  $H_0 = 20 \frac{\text{km}}{\text{s}} / 10^6 \text{ light years} = (13.8 \text{ billion years})^{-1}$ . This means that stars, galaxies, and everything else is moving away from us, and the farther stuff is moving faster. That’s exactly what would happen in a big explosion: the stuff farther away is moving faster (that’s how it got to be farther away), and distance is proportional to velocity. So the proportionality of  $z$  to distance gives direct evidence that there was once a big bang. Extrapolating the velocities back to when they all met, it seems that the big bang was around 13.8 billion years ago. That’s not really very long, considering the earth is only 5 billion years old. It’s also pretty neat that when we look farther away in the universe (at larger redshift, we are looking back in time).

## 5 Cosmic microwave background

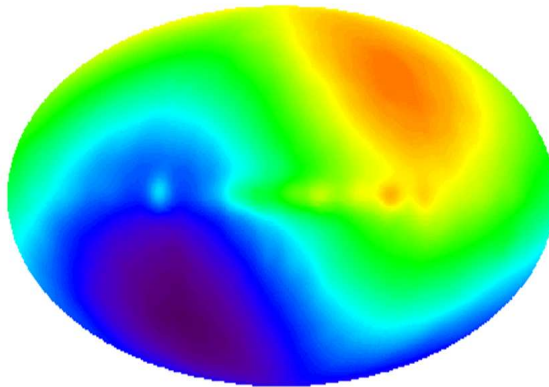
When you go closer to the big bang, you have to fit all the energy of the universe in a smaller volume, so it has to be hotter. Working out the relevant physics, the result is that the temperature  $T$  in Kelvin is related to time  $t$  in seconds by

$$T \approx \frac{10^{10}}{\sqrt{t}} \quad (11)$$

At an early enough time, the universe was so hot that the thermal energy was enough to ionize atoms. At this time, the universe was essentially a plasma. In a plasma, photons cannot get very far without hitting a free electron or proton, so the universe is opaque. The ionization energy of atoms is around the Rydberg constant 13 eV, which corresponds to a temperature of  $T = 160,000 \text{ K}$ . By the time the universe has cooled to around 3000 K, the rate for electrons to be captured by protons exceeds the rate for photons to knock them out. This temperature is called the **reionization temperature**. Using Eq. (11) this happened at a time  $t = 380,000 \text{ yrs}$ . This time corresponds to a redshift of  $z = 1100$ , which is called the **surface of last scattering**. We can’t see anything farther than this, since the universe was opaque.

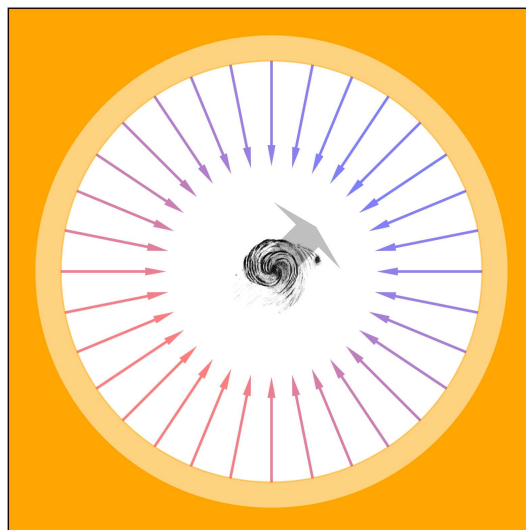
What does the surface of last scattering look like? Well, the last photons that scattered had a blackbody spectrum at  $T \approx 3000K$ . But these photons all get redshifted since they are very far away. The effect is that the spectrum looks like a blackbody at  $\frac{T}{z} \approx 3K$ . People sometimes say that the photons have cooled as the universe expanded, but I think it's more accurate to say that they redshifted. The net result is that there is a thermal bath of photons at  $3K$ . This bath was first observed by 1964 by Penzias and Wilson (somewhat by accident – they thought it was reducible noise in another experiment they were doing, but couldn't get rid of it!). The peak photon wavelength at  $3K$  is in the microwave region, and this is called the **cosmic microwave background (CMB)**.

In 1995, the Cobe satellite found that the CMB wasn't perfectly thermal. It has temperature fluctuations around the blackbody spectrum at around one part in  $10^5$ . Here is a picture of their observations



**Figure 4.** Temperature anisotropies in the Cosmic Microwave Background, as seen by NASA's COBE mission. This oval represents all the directions in the universe, much like a map of the earth can be projected on a similar ellipse. Notice that one end of the sky is hotter than the opposite end.

One thing we see from this is that the universe is slightly hotter in one direction than the other. This means that we are moving with respect to the CMB, as explained in this figure



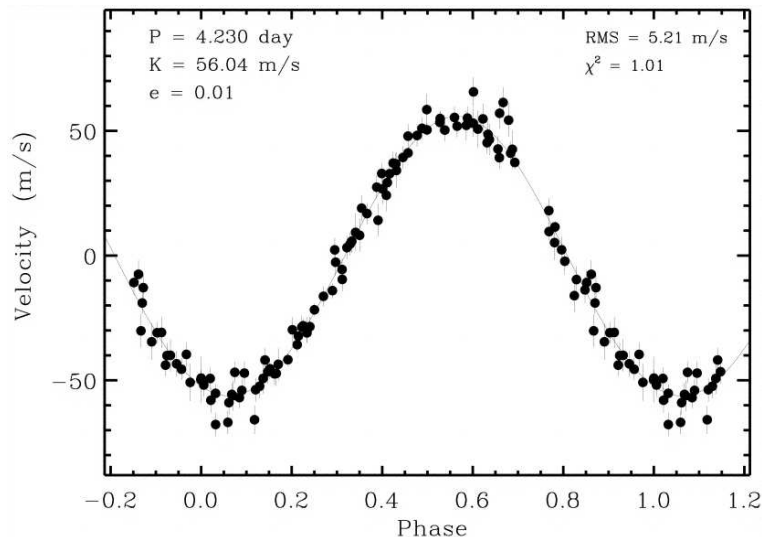
**Figure 5.** A schematic representation of the dipole anisotropy in the Cosmic Microwave Background. CMB photons emitted from the surface of last scattering approach the Milky Way. The relative velocity of the Milky Way to the surface of last scattering determines the redshift of the photons an observer in the Galaxy perceives. Photons traveling inwards along the Milky Way's velocity vector appear blueshifted, while photons traveling inwards from the opposite point on the sky appear redshifted.

More precision studies of the CMB have told us (and continue to tell us) a lot about our universe: the amount of dark matter, dark energy, the total number of particles.

## 6 Doppler spectroscopy

Gasses in stars emit and absorb light at specific frequencies, corresponding to transition energies between electron orbitals. The spectrum of a star therefore contains many narrow peaks and troughs, called spectral lines. If the star is moving relative to our telescope, then its entire spectrum is either redshifted or blueshifted. The redshift of a star can therefore be determined very precisely by measuring the offset of certain spectral lines from their standard frequencies. This method, called **Doppler spectroscopy**, allows a precise measurement of the velocity of a star along the line of sight. This velocity is called the radial velocity of the star.

Here is an example of the motion of a star (51 Pegasi) determined from Doppler spectroscopy:



**Figure 6.** Radial velocity of the star 51 Pegasi, inferred by Doppler shifting of emission and absorption lines in the star's spectrum. Detected in 1995, 51 Pegasi b was the first planet found orbiting another star. This plot is from Marcy *et al.* (1997).

We see that there is a periodic oscillation of the velocity. This can be explained if there is a planet orbiting the star. For example, imagine some alien is looking at the earth from far away. The average velocity of the earth is the average velocity of our solar system. But when the earth is on one side of the sun, it moves away from the alien and on the other side, it moves towards the alien. Thus the earth's velocity as viewed from far away has wiggles due to the annual orbit. If the earth were much bigger, the sun would also have detectable wiggles. So what we are seeing in the 51 Pegasi spectrum above are wiggles due to a very large planet in a tight orbit with that star.

Doppler spectroscopy can be extremely powerful when the spectral features one is interacting with are very narrow (or high  $Q$ ). One of the most high  $Q$  spectroscopic systems are Mossbauer transitions in radioactive nuclei ( $^{56}\text{Fe}$  for example). A Mossbauer transition has a  $Q > 10^{11}$ . Mossbauer transitions are such high  $Q$  due to an unusual effect where nuclei locked in a crystal must all recoil together in response to the emission of a photon. Additionally, nuclear energy levels are highly protected from perturbing electric fields, mostly because they are well shielded by all of the orbiting electrons. Thus, every nucleus in a solid block of iron will have a nearly identical resonant frequency.

Because the  $Q$  of a Mossbauer transition is so high, they can be used for extremely high precision Doppler spectroscopy measurements. In general, these experiments work as follows. One Mossbauer emitter is placed at rest and a second absorbing piece of Mossbauer material on a moving stage. By measuring the absorption of photons emitted by the first stationary piece in the second moving piece, one can measure velocities as low as  $10^{-6} \frac{m}{s}$ .

One of the most clever experiments ever performed using this effect was the measurement of the gravitational redshift. A gravitational redshift happens from a combination of quantum mechanics and relativity. Photons have energies  $E = h\nu$ . Since  $E = mc^2$ , this energy acts just like mass from the point of view of gravity. Thus if we shine a photon towards to the center of the earth, it must gain energy, just like dropped mass would. This energy can only be stored in the photon's frequency. So this frequency must change. In 1959 Pound and Rebka placed a iron 57 Mossbauer emitter in the basement of Jefferson labs and another on the fourth floor. Working out the numbers, the expected energy shift is just  $\frac{dE}{E} < 10^{-14}$  of the photon. For the 14 keV photon, this corresponds to a Doppler shift of the second absorber moving at  $10^{-6} \frac{m}{s}$ . They moved the second iron piece at the speed required to have perfect absorption (on resonance) of the emitted photons. With this carefully designed experiment, they were eventually able to observe the gravitational redshift to within 1% of the predicted value.

In gases, the  $Q$  values for spectral lines are not nearly as high as in Mossbauer materials. The reason that emission peaks, say from Hydrogen, are not infinitely narrow is due to **Doppler broadening**. If the gas is at a temperature  $T$ , the probability  $M(v)dv$  of finding an atom with velocity between  $v$  and  $v + dv$  is given by the Maxwell-Boltzmann distribution

$$M(v) = \sqrt{\frac{m}{2\pi k_B T}} e^{-\frac{mv^2}{2k_B T}} \quad (12)$$

where  $m$  is the mass of the Hydrogen molecules  $k_B$  is the Boltzmann constant. At rest, a Hydrogen atom would have an emission at  $\lambda_0 = 656\text{nm}$  or equivalently at a frequency of  $\nu_0 = 4.5 \times 10^{17}\text{Hz}$ . If the molecules are moving towards the observer, the frequency will go up due to the Doppler effect, and if they are moving away, it will go down. The probability of finding the emission at  $\nu$  is then  $P(\nu)d\nu$  where

$$P(\nu) = \frac{c}{\nu_0} M\left(c\left(\frac{\nu}{\nu_0} - 1\right)\right) = \sqrt{\frac{mc^2}{2\pi k_B T \nu_0^2}} \exp\left[-\frac{mc^2(\nu - \nu_0)^2}{2k_B T \nu_0^2}\right] \quad (13)$$

This is a Gaussian distribution with width

$$\frac{\Delta\nu}{\nu_0} = \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{8k_B T}{mc^2}} = \sqrt{\frac{8}{\gamma}} \frac{c_s}{c} \quad (14)$$

where  $c_s = \sqrt{\gamma \frac{k_B T}{m}}$  is the speed of sound. For example, at room temperature, where  $c_s = 343 \frac{m}{s}$  and  $\gamma = \frac{7}{5}$  for Hydrogen gas,  $\frac{\Delta\lambda}{\lambda} = 2 \times 10^{-6}$  and the emission line at  $\lambda = 656\text{nm}$  will have a width of  $\Delta\lambda = 0.002\text{nm}$ .