Group Problems #20 - Solutions

Friday, October 17

Problem 1 *Solid Angle*

The Moon has a surface area of $\sim 3.8 \times 10^7$ km^2 and has a mean orbital radius about the Earth of $\sim 3.8 \times 10^5$ km.

(a) What is the solid angle subtended by the Moon in the sky?

The solid angle, $\Omega = A/d^2$, is the 2D angle subtended by a cross-sectional area A at a distance d from the point of observation. The problem gives the surface area of the moon, $4\pi r^2$, where r is the radius of the moon. The cross-sectional area of the moon is then $A = \pi r^2 = (3.8/4) \times 10^7 = 9.5 \times 10^6$ km^2. Thus, $\Omega = (9.5 \times 10^6)/(3.8 \times 10^5)^2 = 6.6 \times 10^{-5}$ sr (steradians).

(b) What fraction of the total sky does this solid angle represent?

The full 2D solid angle of the sky (or of any complete sphere) is 4π sr. Thus, the moon represents $(6.6 \times 10^{-5})/(4\pi) = 5.2 \times 10^{-6}$, or about 5 millionths of the sky. So, about 20 million moons can fit into the sky (at the moon's orbital radius). Equivalently, you could calculate the total surface area of the sphere with radius equal to the orbital radius of the Moon about the Earth, and then divide the cross-sectional area of the Moon by this total surface area. This must yield the same answer!

(c) Pluto has a mean diameter of 2,300 km and a very eccentric orbit around the sun, with a perihelion distance of 4.4×10^9 km and an aphelion distance of 7.3×10^9 km. What is the variation in the solid angle subtended by Pluto from Earth's perspective during one of Pluto's orbits around the sun? Assume that Earth's distance from the sun can be neglected.

The radius of Pluto is $r = 2300/2 = 1150$ km, so its cross-sectional area is $A = \pi(1150 \text{ km})^2 = 4.15 \times 10^6$ km^2. At perihelion, we have $d_p^2 = (4.4 \times 10^9 \text{ km})^2 = 1.94 \times 10^{19}$ km^2. At Aphelion, we have $d_a^2 = (7.3 \times 10^9 \text{ km})^2 = 5.33 \times 10^{19}$ km^2. Using these values, we can calculate the solid angles subtended by pluto at perihelion and aphelion:

$$\Omega_p = \frac{A}{d_p^2} = \frac{4.15 \times 10^6}{1.94 \times 10^{19}} = 2.14 \times 10^{-13} \text{ sr} \quad (1)$$

$$\Omega_a = \frac{A}{d_a^2} = \frac{4.15 \times 10^6}{5.33 \times 10^{19}} = 0.78 \times 10^{-13} \text{ sr} \quad (2)$$

It's possible to write this in terms of a percent change:

$$\% \text{ change in } \Omega = \frac{\Omega_p - \Omega_a}{\frac{\Omega_p + \Omega_a}{2}} = \frac{2(2.14 - 0.78)}{2.14 + 0.78} = 0.93 = 93\%. \quad (3)$$

Problem 2 *Differential Solid Angle*

A beam of charged particles (e.g., α particles) is directed toward a target foil. The incoming particles are scattered at various angles, θ, relative to the initial direction of the beam.

(a) What is the differential solid angle for particles scattered at a mean angle $\theta = 5°$ over an angular range of $d\theta = 1°$?

The differential solid angle is defined as $d\Omega = 2\pi \sin\theta \, d\theta$. First we must convert $d\theta$ to radians: $d\theta = 1° \times \pi/180° = 0.017$ rad. We also have $\sin 5° = 0.087$, so $d\Omega = 2\pi(0.087)(0.017) = 0.009$ sr.

(b) What is the differential solid angle for particles scattered at a mean angle $\theta = 85°$ over an angular range of $d\theta = 1°$?

Now we have $\sin 85° = 0.996$, so $d\Omega = 2\pi(0.996)(0.017) = 0.106$ sr.

(c) Based on geometric arguments alone, would you expect more particles to be scattered at 5° or at 85°?

This is a bit of a subtle question. If we had no knowledge of the physical interaction between the incoming particles and the nuclei within the target, we might assume that the particles scatter equally in all directions. If this were the case, then we would expect more particles would be scattered into the differential solid angle centered at 85° since it represents a much larger fraction of the total 4π solid angle, or equivalently, it constitutes a much larger fraction of the entire surface area of a sphere centered at the scattering center (a nucleus within the target). When we discuss Rutherford scattering in the next lecture, we will find that in fact it is relatively rare for an incoming α particle to be scattered at a large angle because it requires that the incoming particle pass very close to a nucleus. This is very improbable since nuclei are so small relative to the size of atoms.