Chapter 26 Elastic Properties of Materials

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26.1 Introduction

In our study of rotational and translational motion of a rigid body, we assumed that the rigid body did not undergo any deformations due to the applied forces. Real objects deform when forces are applied. They can stretch, compress, twist, or break. For example when a force is applied to the ends of a wire and the wire stretches, the length of the wire increases. More generally, when a force per unit area, referred to as *stress,* is applied to an object, the particles in the object may undergo a relative displacement compared to their unstressed arrangement. *Strain* is a normalized measure of this deformation. For example, the *tensile strain* in the stretched wire is fractional change in length of a stressed wire. The stress may not only induce a change in length, but it may result in a volume change as occurs when an object is immersed in a fluid, and the fluid exerts a force per unit area that is perpendicular to the surface of the object resulting in a *volume strain* which is the fractional change the in volume of the object. Another type of stress, known as a *shear stress* occurs when forces are applied tangential to the surface of the object, resulting in a deformation of the object. For example, when scissors cut a thin material, the blades of the scissors exert shearing stresses on the material causing one side of the material to move down and the other side of the material to move up as shown in Figure 26.1, resulting in a *shear strain*. The material deforms until it ultimately breaks.

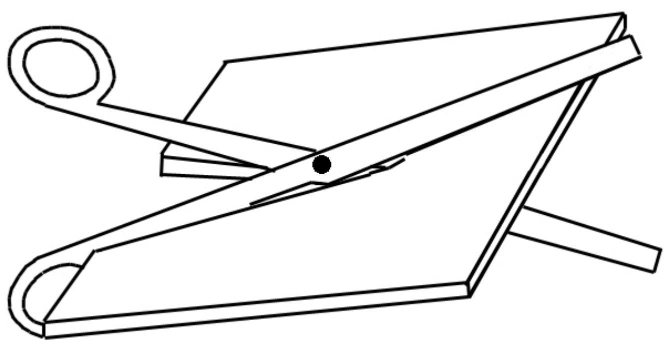

Figure 26.1: Scissors cutting a thin material[1]

In many materials, when the stress is small, the stress and strains are linearly proportional to one another. The material is then said to obey Hooke's Law. The ratio of stress to strain is called the *elastic modulus.* Hooke's Law only holds for a range of stresses, a range referred to as the *elastic region.* An *elastic body* is one in which Hooke's Law applies and when the applied stress is removed, the body returns to its initial shape. Our idealized spring is an example of an elastic body. Outside of the elastic region, the stress-strain relationship is non-linear until the object breaks.

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[1] Mohsen Mahvash, et al, IEEE Trans Biomed Eng. 2008, March; 55(3); 848-856.

26.2 Stress and Strain in Tension and Compression

Consider a rod with cross sectional area A and length l_0. Two forces of the same magnitude $F_\perp$ are applied perpendicularly at the two ends of the section stretching the rod to a length l (Figure 26.2), where the beam has been stretched by a positive amount $\delta l = l - l_0$.

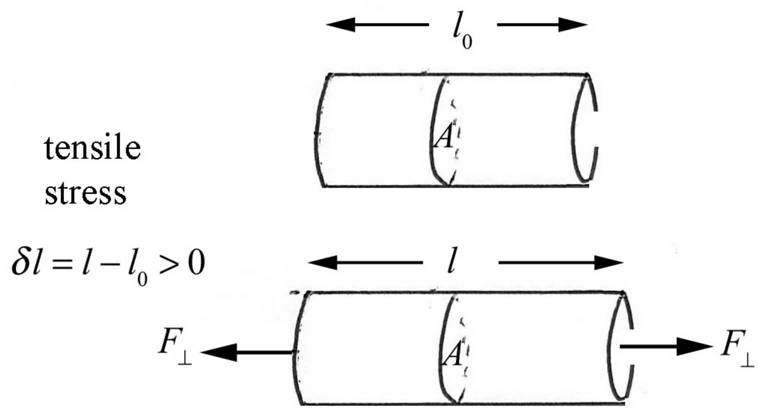

Figure 26.2: Tensile stress on a rod

The ratio of the applied perpendicular force to the cross-sectional area is called the *tensile stress*,

$$\sigma_T = \frac{F_\perp}{A}. \tag{26.2.1}$$

The ratio of the amount the section has stretched to the original length is called the *tensile strain*,

$$\varepsilon_T = \frac{\delta l}{l_0}. \tag{26.2.2}$$

Experimentally, for sufficiently small stresses, for many materials the stress and strain are linearly proportional,

$$\frac{F_\perp}{A} = Y\frac{\delta l}{l_0} \quad \text{(Hooke's Law)}. \tag{26.2.3}$$

where the constant of proportionality Y is called **Young's modulus**. The SI unit for Young's Modulus is the **pascal** where $1\,\text{Pa} \equiv 1\,\text{N}\cdot\text{m}^{-2}$. Note the following conversion factors between SI and English units: $1\,\text{bar} \equiv 10^5\,\text{Pa}$, $1\,\text{psi} \equiv 6.9\times10^{-2}\,\text{bar}$, and $1\,\text{bar} = 14.5\,\text{psi}$. In Table 26.1, Young's Modulus is tabulated for various materials. Figure 26.3 shows a plot of the stress-strain relationship for various human bones. For stresses greater than approximately $70\,\mathrm{N}\cdot\mathrm{mm}^{-2}$, the material is no longer elastic. At a certain point for each bone, the stress-strain relationship stops, representing the fracture point.

Material	Young's Modulus, Y (Pa)
Iron	21×10^{10}
Nickel	21×10^{10}
Steel	20×10^{10}
Copper	11×10^{10}
Brass	9.0×10^{10}
Aluminum	7.0×10^{10}
Crown Glass	6.0×10^{10}
Cortical Bone	$7\times10^{9}-30\times10^{9}$
Lead	1.6×10^{10}
Tendon	2×10^{7}
Rubber	$7\times10^{5}-40\times10^{5}$
Blood vessels	2×10^{5}

Table 26.1: Young's Modulus for various materials

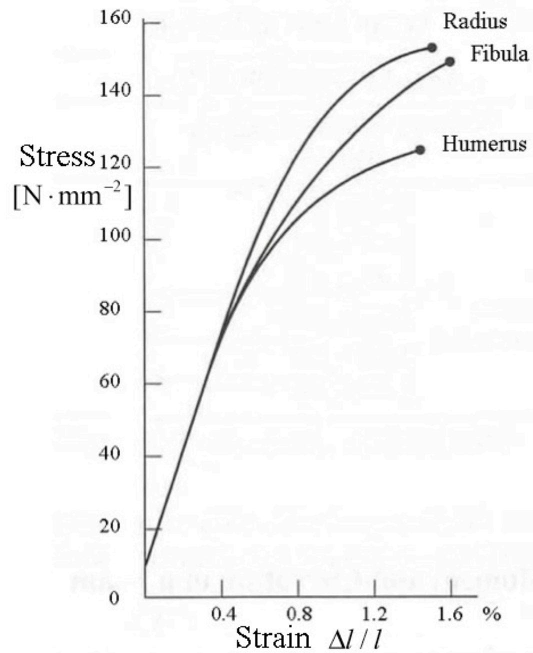

Figure 26.3: Stress-strain relation for various human bones (figure from H. Yamada, Strength of Biological Materials)

When the material is under compression, the forces on the ends are directed towards each other producing a *compressive stress* resulting in a *compressive strain* (Figure 26.4). For compressive strains, if we define $\delta l = l_0 - l > 0$ then Eq. (26.2.3) holds for compressive stresses provided the compressive stress is not too large. For many materials, Young's Modulus is the same when the material is under tension and compression. There are some important exceptions. Concrete and stone can undergo compressive stresses but fail when the same tensile stress is applied. When building with these materials, it is important to design the structure so that the stone or concrete is never under tensile stresses. Arches are used as an architectural structural element primarily for this reason.

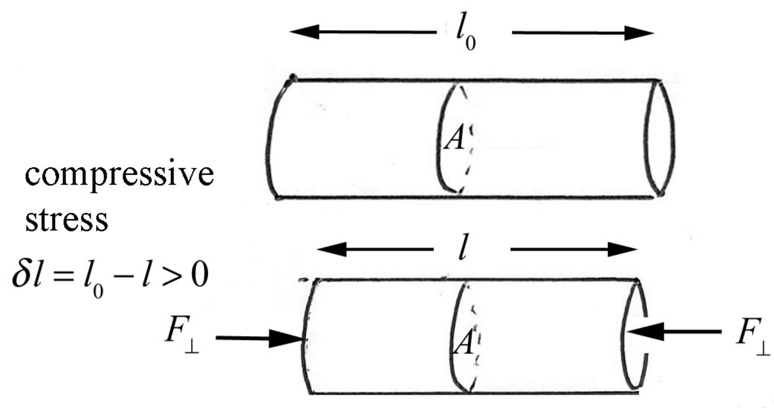

Figure 26.4: Compressive Stress

26.3 Shear Stress and Strain

The surface of material may also be subjected to tangential forces producing a shearing action. Consider a block of height h and area A, in which a tangential force, $\vec{\mathbf{F}}_{\tan}$, is applied to the upper surface. The lower surface is held fixed. The upper surface will shear by an angle α corresponding to a horizontal displacement δx. The geometry of the shearing action is shown in Figure 26.5.

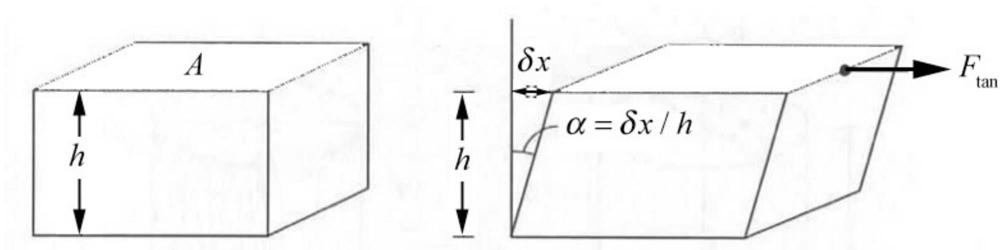

Figure 26.5: Shearing forces

The *shear stress* is defined to be the ratio of the tangential force to the cross sectional area of the surface upon which it acts,

$$\sigma_S = \frac{F_{\tan}}{A}. \qquad (26.3.1)$$

The *shear strain* is defined to be the ratio of the horizontal displacement to the height of the block,

$$\alpha = \frac{\delta x}{h}. \qquad (26.3.2)$$

For many materials, when the shear stress is sufficiently small, experiment shows that a Hooke's Law relationship holds in that the shear stress is proportional to shear strain,

$$\frac{F_{\tan}}{A} = S\frac{\delta x}{h} \quad \text{(Hooke's Law)}. \tag{26.3.3}$$

where the constant of proportional, S, is called the **shear modulus**. When the deformation angle is small, $\delta x / h = \tan\alpha \simeq \sin\alpha \simeq \alpha$, and Eq. (26.3.3) becomes

$$\frac{F_{\tan}}{A} \simeq S\alpha \quad \text{(Hooke's Law)}. \tag{26.3.4}$$

In Table 26.2, the shear modulus is tabulated for various materials.

Table 26.2: Shear Modulus for Various Materials

Material	Shear Modulus, S (Pa)
Nickel	7.8×10^{10}
Iron	7.7×10^{10}
Steel	7.5×10^{10}
Copper	4.4×10^{10}
Brass	3.5×10^{10}
Aluminum	2.5×10^{10}
Crown Glass	2.5×10^{10}
Lead	0.6×10^{10}
Rubber	$2\times10^{5}-10\times10^{5}$

Example 26.1: Stretched wire

An object of mass $1.5\times10^{1}\ \text{kg}$ is hanging from one end of a steel wire. The wire without the mass has an unstretched length of $0.50\ \text{m}$. What is the resulting strain and elongation of the wire? The cross-sectional area of the wire is $1.4\times10^{-2}\ \text{cm}^2$.

Solution: When the hanging object is attached to the wire, the force at the end of the wire acting on the object exactly balances the gravitational force. Therefore by Newton's Third Law, the tensile force stressing the wire is

$$F_{\perp} = mg. \tag{26.3.5}$$

We can calculate the strain on the wire from Hooke's Law (Eq. (26.2.3)) and the value of Young's modulus for steel $20\times10^{10}\ \text{Pa}$ (Table 26.1);

$$\frac{\delta l}{l_0} = \frac{F_\perp}{YA} = \frac{mg}{YA} = \frac{(1.5\times 10^{1}\ \text{kg})(9.8\ \text{m}\cdot\text{s}^{-2})}{(2.0\times 10^{11}\ \text{Pa})(1.4\times 10^{-6}\ \text{m}^2)} = 5.3\times 10^{-4}\,. \tag{26.3.6}$$

The elongation δl of the wire is then

$$\delta l = \frac{mg}{YA} l_0 = (5.3\times 10^{-4})(0.50\ \text{m}) = 2.6\times 10^{-4}\ \text{m}\,. \tag{26.3.7}$$

26.4 Elastic and Plastic Deformation

Consider a single sheet of paper. If we bend the paper gently, and then release the constraining forces, the sheet will return to its initial state. This process of gently bending is reversible as the paper displays *elastic behavior*. The internal forces responsible for the deformation are conservative. Although we do not have a simple mathematical model for the potential energy, we know that mechanical energy is constant during the bending. We can take the same sheet of paper and crumple it. When we release the paper it will no longer return to its original sheet but will have a permanent deformation. The internal forces now include non-conservative forces and the mechanical energy is decreased. This *plastic behavior* is irreversible.

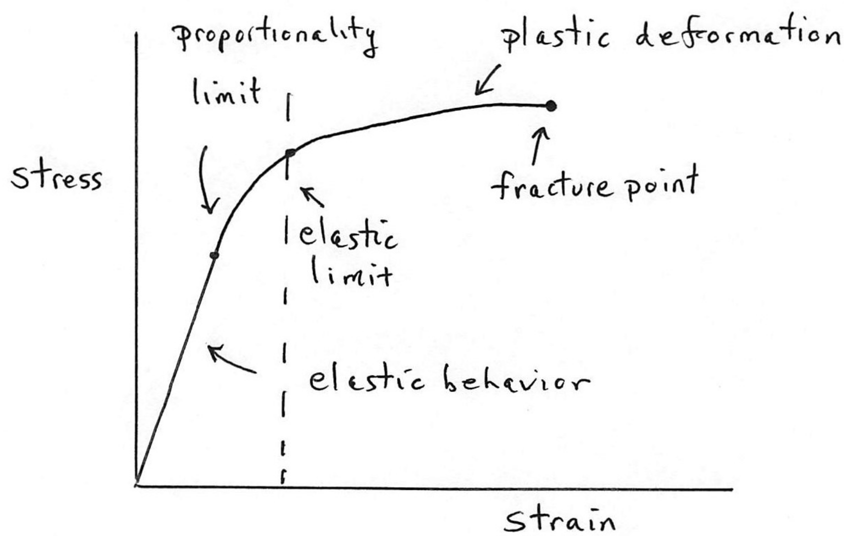

Figure 26.5: Stress-strain relationship

When the stress on a material is linearly proportional to the strain, the material behaves according to Hooke's Law. The *proportionality limit* is the maximum value of stress at which the material still satisfies Hooke's Law. If the stress is increased above the proportionality limit, the stress is no longer linearly proportional to the strain. However, if the stress is slowly removed then the material will still return to its original state; the material behaves elastically. If the stress is above the proportionality limit, but less then the elastic limit, then the stress is no longer linearly proportional to the strain. Even in this non-linear region, if the stress is slowly removed then the material will return to its original state. The maximum value of stress in which the material will still remain elastic is called the *elastic limit*. For stresses above the elastic limit, when the stress is removed the material will not return to its original state and some permanent deformation sets in, a state referred to as a *permanent set*. This behavior is referred to as *plastic deformation*. For a sufficiently large stress, the material will *fracture*. Figure 26.5 illustrates a typical stress-strain relationship for a material. The value of the stress that fractures a material is referred to as the *ultimate tensile strength.* The ultimate tensile strengths for various materials are listed in Table 26.3. The tensile strengths for wet human bones are for a person whose age is between 20 and 40 years old.

Table 26.3: Ultimate Tensile Strength for Various Materials

Material	Shear Modulus, *S* (Pa)
Femur	1.21×10^{8}
Humerus	1.22×10^{8}
Tibia	1.40×10^{8}
Fibula	1.46×10^{8}
Ulna	1.48×10^{8}
Radius	1.49×10^{8}
Aluminum	2.2×10^{8}
Iron	3.0×10^{8}
Brass	4.7×10^{8}
Steel	$5-20\times10^{8}$

Example 26.2: Ultimate Tensile Strength of Bones

The ultimate tensile strength of the wet human tibia (for a person of age between 20 and 40 years) is $1.40\times10^{8}\ \mathrm{Pa}$. If a greater compressive force per area is applied to the tibia then the bone will break. The smallest cross sectional area of the tibia, about $3.2\ \mathrm{cm}^2$, is slightly above the ankle. Suppose a person of mass $60\ \mathrm{kg}$ jumps to the ground from a height $2.0\ \mathrm{m}$ and absorbs the shock of hitting the ground by bending the knees. Assume that there is constant deceleration during the collision. During the collision, the person lowers her center of mass by an amount $d = 1.0\ \mathrm{cm}$. (a) What is the collision time Δt_{col}? (b) Find the average force of the ground on the person during the collision. (c) Can we effectively ignore the gravitational force during the collision? (d) Will the person break her ankle? (e) What is the minimum distance Δd_{min} that she would need to lower her center of mass so she does not break her ankle? What is the ratio $h_0 / \Delta d_{\mathrm{min}}$? What factors does this ratio depend on?

Solution: Choose a coordinate system with the positive y-direction pointing up, and the origin at the ground. While the person is falling to the ground, mechanical energy is constant (we are neglecting any non-conservative work due to air resistance). Choose the contact point with the ground as the zero potential energy reference point. Then the initial mechanical energy is

$$E_0 = U_0 = mgh_0 . \tag{26.3.8}$$

The mechanical energy of the person just before contact with the ground is

$$E_b = K_1 = \frac{1}{2} m v_b{}^2 . \tag{26.3.9}$$

The constancy of mechanical energy implies that

$$mgh_0 = \frac{1}{2} m v_b{}^2 . \tag{26.3.10}$$

The speed of the person the instant contact is made with the ground is then

$$v_b = \sqrt{2gh_0} . \tag{26.3.11}$$

If we treat the person as the system then there are two external forces acting on the person, the gravitational force $\vec{\mathbf{F}}^g = -mg\hat{\mathbf{j}}$ and a normal force between the ground and the person $\vec{\mathbf{F}}^N = N\hat{\mathbf{j}}$. This force varies with time but we shall consider the time average $\vec{\mathbf{F}}^N_{\text{ave}} = N_{\text{ave}}\hat{\mathbf{j}}$. Then using Newton's Second Law,

$$N_{\text{ave}} - mg = ma_{y,\text{ave}} . \tag{26.3.12}$$

The y-component of the average acceleration is equal to

$$a_{y,\text{ave}} = \frac{N_{\text{ave}}}{m} - g . \tag{26.3.13}$$

Set $t = 0$ for the instant the person reaches the ground; then $v_{y,0} = -v_b$. The displacement of the person while in contact with the ground for the time interval Δt_{col} is given by

$$\Delta y = -v_b \Delta t_{\text{col}} + \frac{1}{2} a_{y,\text{ave}} \Delta t_{\text{col}}^2 . \tag{26.3.14}$$

The y-component of the velocity is zero at $t = \Delta t_{\text{col}}$ when the person's displacement is $\Delta y = -d$,

$$0 = -v_b + a_{y,\text{ave}} \Delta t_{\text{col}} . \tag{26.3.15}$$

Solving Eq. (26.3.15) for the collision time yields

$$\Delta t_{\text{col}} = v_b / a_{y,\text{ave}} . \tag{26.3.16}$$

We can now substitute $\Delta y = -d$, Eq. (26.3.16), and Eq. (26.3.11) into Eq. (26.3.14) and solve for the y-component of the acceleration, yielding

$$a_{y,\text{ave}} = \frac{g h_0}{d} . \tag{26.3.17}$$

We can solve for the collision time by substituting Eqs. (26.3.17) and Eq. (26.3.11) into Eq. (26.3.16) and using the given values in the problem statement, yielding

$$\Delta t_{\text{col}} = \frac{2d}{\sqrt{2 g h_0}} = \frac{2(1.0 \times 10^{-2}\,\text{m})}{\sqrt{2(9.8\,\text{m}\cdot\text{s}^{2})(2.0\,\text{m})}} = 3.2 \times 10^{-3}\,\text{s} . \tag{26.3.18}$$

Now substitute Eq. (26.3.17) for the y-component of the acceleration into Eq. (26.3.13) and solve for the average normal force

$$N_{\text{ave}} = mg\left(1 + \frac{h_0}{d}\right) = (60\,\text{kg})(9.8\,\text{m}\cdot\text{s}^{-2})\left(1 + \frac{(2.0\,\text{m})}{(1.0 \times 10^{-2}\,\text{m})}\right) = 1.2 \times 10^{5}\,\text{N} . \tag{26.3.19}$$

Notice that the factor $1 + h_0 / d \simeq h_0 / d$ so *during the collision* we can effectively ignore the external gravitational force. The average compressional force per area on the person's ankle is the average normal force divided by the cross sectional area

$$P = \frac{N_{\text{ave}}}{A} \simeq \frac{mg}{A}\left(\frac{h_0}{d}\right) = \frac{1.2 \times 10^{5}\,\text{N}}{3.2 \times 10^{-4}\,\text{m}^2} = 3.7 \times 10^{8}\,\text{Pa} . \tag{26.3.20}$$

From Table 26.3, the tensile strength of the tibia is $1.4 \times 10^{8}\,\text{Pa}$, so this fall is enough to break the tibia.

In order to find the minimum displacement that the center of mass must fall in order to avoid breaking the tibia bone, we set the force per area in Eq. (26.3.20) equal to $P = 1.4 \times 10^{8}\,\text{Pa}$. Because at this value of tensile strength,

$$\frac{PA}{mg} = \frac{(1.4\times10^{8}\ \mathrm{Pa})((3.2\times10^{-4}\ \mathrm{m}^2)}{(60\ \mathrm{kg})(9.8\ \mathrm{m\cdot s^{-2}})} = 80 \tag{26.3.21}$$

and so $PA >> mg$. We can solve Eq. (26.3.20) for the minimum displacement

$$d_{\min} = \frac{h_0}{\left(\frac{PA}{mg} - 1\right)} \simeq \frac{mgh_0}{PA} = \frac{(60\ \mathrm{kg})(9.8\ \mathrm{m\cdot s^{-2}})(2.0\ \mathrm{m})}{(1.4\times10^{8}\ \mathrm{Pa})(3.2\times10^{-4}\ \mathrm{m}^2)} = 2.6\ \mathrm{cm}, \tag{26.3.22}$$

where we used the fact that

$$\frac{PA}{mg} = \frac{(1.4\times10^{8}\ \mathrm{Pa})((3.2\times10^{-4}\ \mathrm{m}^2)}{(60\ \mathrm{kg})(9.8\ \mathrm{m\cdot s^{-2}})} = 76 \tag{26.3.23}$$

and so $PA >> mg$. The ratio

$$h_0 / d_{\min} \simeq PA / mg = 76\,. \tag{26.3.24}$$

This ratio depends on the compressive strength of the bone, the cross sectional area, and inversely on the weight of the person. The maximum normal force is anywhere from two to ten times the average normal force. A safe distance to lower the center of mass would be about 20 cm.