Chapter 15 Collision Theory

15.1 Introduction 1
15.2 Reference Frames Relative and Velocities 1
15.2.1 Center of Mass Reference Frame 2
15.2.2 Relative Velocities 3
15.3 Characterizing Collisions 5
15.4 One-Dimensional Elastic Collision Between Two Objects 5
15.4.1 One-Dimensional Collision Between Two Objects – Center of Mass Reference Frame 8
15.5 Worked Examples 9
Example 15.1 Elastic One-Dimensional Collision Between Two Objects 9
Example 15.2 The Dissipation of Kinetic Energy in a Completely Inelastic Collision Between Two Objects 10
Example 15.3 Elastic Two-Dimensional Collision 11
Example 15.4 Equal Mass Particles in a Two-Dimensional Elastic Collision Emerge at Right Angles 14
Example 15.5 Bouncing Superballs 15

Chapter 15 Collision Theory

> Despite my resistance to hyperbole, the LHC [Large Hadron Collider] belongs to a world that can only be described with superlatives. It is not merely large: the LHC is the biggest machine ever built. It is not merely cold: the 1.9 kelvin (1.9 degrees Celsius above absolute zero) temperature necessary for the LHC's supercomputing magnets to operate is the coldest extended region that we know of in the universe—even colder than outer space. The magnetic field is not merely big: the superconducting dipole magnets generating a magnetic field more than 100,000 times stronger than the Earth's are the strongest magnets in industrial production ever made.
>
> And the extremes don't end there. The vacuum inside the proton-containing tubes, a 10 trillionth of an atmosphere, is the most complete vacuum over the largest region ever produced. The energy of the collisions are the highest ever generated on Earth, allowing us to study the interactions that occurred in the early universe the furthest back in time.[1]
>
> *Lisa Randall*

15.1 Introduction

When discussing conservation of momentum, we considered examples in which two objects collide and stick together, and either there are no external forces acting in some direction (or the collision was nearly instantaneous) so the component of the momentum of the system along that direction is constant. We shall now study collisions between objects in more detail. In particular we shall consider cases in which the objects do not stick together. The momentum along a certain direction may still be constant but the mechanical energy of the system may change. We will begin our analysis by considering two-particle collision. We introduce the concept of the relative velocity between two particles and show that it is independent of the choice of reference frame. We then show that the change in kinetic energy only depends on the change of the square of the relative velocity and therefore is also independent of the choice of reference frame. We will then study one- and two-dimensional collisions with zero change in potential energy. In particular we will characterize the types of collisions by the change in kinetic energy and analyze the possible outcomes of the collisions.

15.2 Reference Frames Relative and Velocities

We shall recall our definition of relative inertial reference frames. Let $\vec{\mathbf{R}}$ be the vector from the origin of frame S to the origin of reference frame S'. Denote the position vector of particle i with respect to the origin of reference frame S by $\vec{\mathbf{r}}_i$ and

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[1] Randall, Lisa, *Knocking on Heaven's Door: How Physics and Scientific Thinking Illuminate the Universe and the Modern World*, Ecco, 2011.

similarly, denote the position vector of particle i with respect to the origin of reference frame S' by $\vec{\mathbf{r}}'_i$ (Figure 15.1).

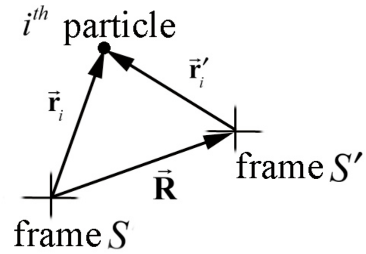

Figure 15.1 Position vector of i^{th} particle in two reference frames.

The position vectors are related by

$$\vec{\mathbf{r}}_i = \vec{\mathbf{r}}'_i + \vec{\mathbf{R}} \ . \tag{0.3.1}$$

The relative velocity (call this the *boost velocity*) between the two reference frames is given by

$$\vec{\mathbf{V}} = \frac{d\vec{\mathbf{R}}}{dt} . \tag{0.3.2}$$

Assume the boost velocity between the two reference frames is constant. Then, the relative acceleration between the two reference frames is zero,

$$\vec{\mathbf{A}} = \frac{d\vec{\mathbf{V}}}{dt} = \vec{\mathbf{0}} . \tag{0.3.3}$$

When Eq. (0.3.3) is satisfied, the reference frames S and S' are called *relatively inertial reference frames.*

Suppose the i^{th} particle in Figure 15.1 is moving; then observers in different reference frames will measure different velocities. Denote the velocity of i^{th} particle in frame S by $\vec{\mathbf{v}}_i = d\vec{\mathbf{r}}_i / dt$, and the velocity of the same particle in frame S' by $\vec{\mathbf{v}}'_i = d\vec{\mathbf{r}}'_i / dt$. Taking derivative, the velocities of the particles in two different reference frames are related according to

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}'_i + \vec{\mathbf{V}} . \tag{0.3.4}$$

15.2.1 Center of Mass Reference Frame

Let $\vec{\mathbf{R}}_{cm}$ be the vector from the origin of frame S to the center of mass of the system of particles, a point that we will choose as the origin of reference frame S_{cm}, called the ***center of mass reference frame***. Denote the position vector of particle i with respect to origin of reference frame S by $\vec{\mathbf{r}}_i$ and similarly, denote the position vector of particle i with respect to origin of reference frame S_{cm} by $\vec{\mathbf{r}}_{cm,i}$ (Figure 15.2).

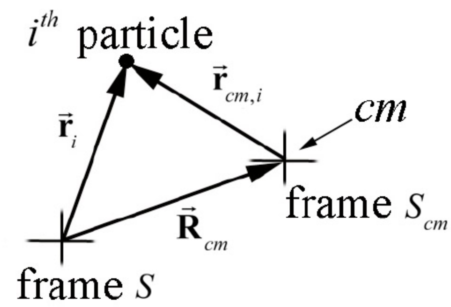

Figure 15.2 Position vector of i^{th} particle in the center of mass reference frame.

The position vector of particle i in the center of mass frame is then given by

$$\vec{\mathbf{r}}_{cm,i} = \vec{\mathbf{r}}_i - \vec{\mathbf{R}}_{cm} . \tag{0.3.5}$$

The velocity of particle i in the center of mass reference frame is then given by

$$\vec{\mathbf{v}}_{cm,i} = \vec{\mathbf{v}}_i - \vec{\mathbf{V}}_{cm} . \tag{0.3.6}$$

There are many collision problems in which the center of mass reference frame is the most convenient reference frame to analyze the collision.

15.2.2 Relative Velocities

Consider two particles of masses m_1 and m_2 interacting via some force (Figure 15.3).

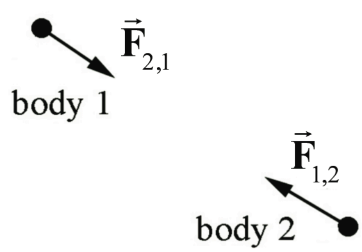

Figure 15.3 Two interacting particles

Choose a coordinate system (Figure 15.4) in which the position vector of body 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of body 2 is given by $\vec{\mathbf{r}}_2$. The *relative position* of body 1 with respect to body 2 is given by $\vec{\mathbf{r}}_{1,2} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$.

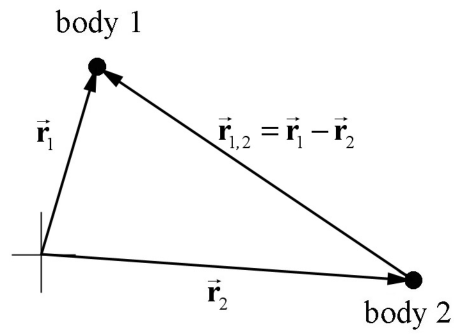

Figure 15.4 Coordinate system for two bodies.

During the course of the interaction, body 1 is displaced by $d\vec{\mathbf{r}}_1$ and body 2 is displaced by $d\vec{\mathbf{r}}_2$, so the ***relative displacement*** of the two bodies during the interaction is given by $d\vec{\mathbf{r}}_{1,2} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$. The ***relative velocity*** between the particles is

$$\vec{\mathbf{v}}_{1,2} = \frac{d\vec{\mathbf{r}}_{1,2}}{dt} = \frac{d\vec{\mathbf{r}}_1}{dt} - \frac{d\vec{\mathbf{r}}_2}{dt} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 \,. \tag{0.3.7}$$

We shall now show that the relative velocity between the two particles is independent of the choice of reference frame providing that the reference frames are relatively inertial. The relative velocity $\vec{\mathbf{v}}'_{12}$ in reference frame S' can be determined from using Eq. (0.3.4) to express Eq. (0.3.7) in terms of the velocities in the reference frame S',

$$\vec{\mathbf{v}}_{1,2} = \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = (\vec{\mathbf{v}}'_1 + \vec{\mathbf{V}}) - (\vec{\mathbf{v}}'_2 + \vec{\mathbf{V}}) = \vec{\mathbf{v}}'_1 - \vec{\mathbf{v}}'_2 = \vec{\mathbf{v}}'_{1,2} \tag{0.3.8}$$

and is equal to the relative velocity in frame S.

> *For a two-particle interaction, the relative velocity between the two vectors is independent of the choice of relatively inertial reference frames.*

We showed in Appendix 13A that when two particles of masses m_1 and m_2 interact, the change of kinetic energy between the final state B and the initial state A due to the interaction force is equal to

$$\Delta K = \frac{1}{2}\mu(v_B^2 - v_A^2) \,, \tag{0.3.9}$$

where $v_B \equiv \left|(\vec{\mathbf{v}}_{1,2})_B\right| = \left|(\vec{\mathbf{v}}_1)_B - (\vec{\mathbf{v}}_2)_B\right|$ denotes the relative speed in the state B and $v_A \equiv \left|(\vec{\mathbf{v}}_{1,2})_A\right| = \left|(\vec{\mathbf{v}}_1)_A - (\vec{\mathbf{v}}_2)_A\right|$ denotes the relative speed in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$ is the reduced mass. (If the relative reference frames were non-inertial then Eq. (0.3.9) would not be valid in all frames.

Although kinetic energy is a reference dependent quantity, by expressing the change of kinetic energy in terms of the relative velocity, then

> *the change in kinetic energy is independent of the choice of relatively inertial reference frames.*

15.3 Characterizing Collisions

In a collision, the ratio of the magnitudes of the initial and final relative velocities is called the coefficient of restitution and denoted by the symbol e,

$$e = \frac{v_B}{v_A}. \tag{0.4.1}$$

If the magnitude of the relative velocity does not change during a collision, $e = 1$, then the change in kinetic energy is zero, (Eq. (0.3.9)). Collisions in which there is no change in kinetic energy are called ***elastic collisions***,

$$\Delta K = 0, \quad \textit{elastic collision}. \tag{0.4.2}$$

If the magnitude of the final relative velocity is less than the magnitude of the initial relative velocity, $e < 1$, then the change in kinetic energy is negative. Collisions in which the kinetic energy decreases are called ***inelastic collisions***,

$$\Delta K < 0, \quad \textit{inelastic collision}. \tag{0.4.3}$$

If the two objects stick together after the collision, then the relative final velocity is zero, $e = 0$. Such collisions are called ***totally inelastic***. The change in kinetic energy can be found from Eq. (0.3.9),

$$\Delta K = -\frac{1}{2}\mu v_A^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_A^2, \quad \textit{totally inelastic collision}. \tag{0.4.4}$$

If the magnitude of the final relative velocity is greater than the magnitude of the initial relative velocity, $e > 1$, then the change in kinetic energy is positive. Collisions in which the kinetic energy increases are called ***superelastic collisions***,

$$\Delta K > 0, \quad \textit{superelastic collision}. \tag{0.4.5}$$

15.4 One-Dimensional Elastic Collision Between Two Objects

Consider a one-dimensional elastic collision between two objects moving in the x-direction. One object, with mass m_1 and initial x-component of the velocity $v_{1x,i}$, collides with an object of mass m_2 and initial x-component of the velocity $v_{2x,i}$. The scalar components $v_{1x,i}$ and $v_{1x,i}$ can be positive, negative or zero. No forces other than the interaction force between the objects act during the collision. After the collision, the final x-component of the velocities are $v_{1x,f}$ and $v_{2x,f}$. We call this reference frame the "laboratory reference frame".

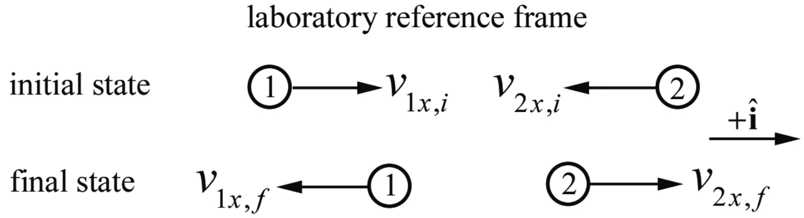

Figure 15.5 One-dimensional elastic collision, laboratory reference frame

For the collision depicted in Figure 15.5, $v_{1x,i} > 0$, $v_{2x,i} < 0$, $v_{1x,f} < 0$, and $v_{2x,f} > 0$. Because there are no external forces in the x-direction, momentum is constant in the x-direction. Equating the momentum components before and after the collision gives the relation

$$m_1 v_{1x,i} + m_2 v_{2x,0} = m_1 v_{1x,f} + m_2 v_{2x,f}\,. \tag{0.4.6}$$

Because the collision is elastic, kinetic energy is constant. Equating the kinetic energy before and after the collision gives the relation

$$\frac{1}{2} m_1 v_{1x,i}^2 + \frac{1}{2} m_2 v_{2x,i}^2 = \frac{1}{2} m_1 v_{1x,f}^2 + \frac{1}{2} m_2 v_{2x,f}^2 \tag{0.4.7}$$

Rewrite these Eqs. (0.4.6) and (0.4.7) as

$$m_1(v_{1x,i} - v_{1x,f}) = m_2(v_{2x,f} - v_{2x,i}) \tag{0.4.8}$$

$$m_1(v_{1x,i}^2 - v_{1x,f}^2) = m_2(v_{2x,f}^2 - v_{2x,i}^2)\,. \tag{0.4.9}$$

Eq. (0.4.9) can be written as

$$m_1(v_{1x,i} - v_{1x,f})(v_{1x,i} + v_{1x,f}) = m_2(v_{2x,f} - v_{2x,i})(v_{2x,f} + v_{2x,i})\,. \tag{0.4.10}$$

Divide Eq. (0.4.9) by Eq. (0.4.8), yielding

$$v_{1x,i} + v_{1x,f} = v_{2x,i} + v_{2x,f}\,. \tag{0.4.11}$$

Eq. (0.4.11) may be rewritten as

$$v_{1x,i} - v_{2x,i} = v_{2x,f} - v_{1x,f}\,. \tag{0.4.12}$$

Recall that the relative velocity between the two objects in state A is defined to be

$$\vec{\mathbf{v}}_A^{\,\text{rel}} = \vec{\mathbf{v}}_{1,A} - \vec{\mathbf{v}}_{2,A}\,. \tag{0.4.13}$$

where we used the superscript "rel" to remind ourselves that the velocity is a relative velocity. Thus $v_{x,i}^{\text{rel}} = v_{1x,i} - v_{2x,i}$ is the initial x-component of the relative velocity, and $v_{x,f}^{\text{rel}} = v_{1x,f} - v_{2x,f}$ is the final x-component of the relative velocity. Therefore Eq. (0.4.12) states that during the interaction the initial x-component of the relative velocity is equal to the negative of the final x-component of the relative velocity

$$v_{x,i}^{\text{rel}} = -v_{x,f}^{\text{rel}}\,. \tag{0.4.14}$$

Consequently the initial and final relative speeds are equal. We can now solve for the final x-component of the velocities, $v_{1x,f}$ and $v_{2x,f}$, as follows. Eq. (0.4.12) may be rewritten as

$$v_{2x,f} = v_{1x,f} + v_{1x,i} - v_{2x,i}\,. \tag{0.4.15}$$

Now substitute Eq. (0.4.15) into Eq. (0.4.6) yielding

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 v_{1x,f} + m_2 (v_{1x,f} + v_{1x,i} - v_{2x,i})\,. \tag{0.4.16}$$

Solving Eq. (0.4.16) for $v_{1x,f}$ involves some algebra and yields

$$v_{1x,f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1x,i} + \frac{2\,m_2}{m_1 + m_2} v_{2x,i}\,. \tag{0.4.17}$$

To find $v_{2x,f}$, rewrite Eq. (0.4.12) as

$$v_{1x,f} = v_{2x,f} - v_{1x,i} + v_{2x,i}\,. \tag{0.4.18}$$

Now substitute Eq. (0.4.18) into Eq. (0.4.6) yielding

$$m_1 v_{1x,i} + m_2 v_{2x,i} = m_1 (v_{2x,f} - v_{1x,i} + v_{2x,i}) v_{1x,f} + m_2 v_{2x,f}\,. \tag{0.4.19}$$

We can solve Eq. (0.4.19) for $v_{2x,f}$ and determine that

$$v_{2x,f} = v_{2x,i}\frac{m_2 - m_1}{m_2 + m_1} + v_{1x,i}\frac{2\,m_1}{m_2 + m_1}\,. \tag{0.4.20}$$

Consider what happens in the limits $m_1 >> m_2$ in Eq. (0.4.17). Then

$$v_{1x,f} \to v_{1x,i} + \frac{2}{m_1} m_2 v_{2x,i}\,; \tag{0.4.21}$$

the more massive object's velocity component is only slightly changed by an amount proportional to the less massive object's x-component of momentum. Similarly, the less massive object's final velocity approaches

$$v_{2x,f} \to -v_{2x,i} + 2v_{1x,i} = v_{1x,i} + v_{1x,i} - v_{2x,i}\,. \tag{0.4.22}$$

We can rewrite this as

$$v_{2x,f} - v_{1x,i} = v_{1x,i} - v_{2x,i} = v_{x,i}^{\text{rel}}. \tag{0.4.23}$$

i.e. the less massive object "rebounds" with the same speed relative to the more massive object which barely changed its speed.

If the objects are identical, or have the same mass, Eqs. (0.4.17) and (0.4.20) become

$$v_{1x,f} = v_{2x,i}, \quad v_{2x,f} = v_{1x,i}\,; \tag{0.4.24}$$

the objects have exchanged x-components of velocities, and unless we could somehow distinguish the objects, we might not be able to tell if there was a collision at all.

15.4.1 One-Dimensional Collision Between Two Objects – Center of Mass Reference Frame

We analyzed the one-dimensional elastic collision (Figure 15.5) in Section 15.4 in the laboratory reference frame. Now let's view the collision from the center of mass (CM) frame. The x-component of velocity of the center of mass is

$$v_{x,\text{cm}} = \frac{m_1\,v_{1x,i} + m_2\,v_{2x,i}}{m_1 + m_2}\,. \tag{0.4.25}$$

With respect to the center of mass, the x-components of the velocities of the objects are

$$
\begin{aligned}
v'_{1x,i} &= v_{1x,i} - v_{x,\mathrm{cm}} = (v_{1x,i} - v_{2x,i})\frac{m_2}{m_1 + m_2} \\
v'_{2x,i} &= v_{2x,i} - v_{x,\mathrm{cm}} = (v_{2x,i} - v_{1x,i})\frac{m_1}{m_1 + m_2}.
\end{aligned}
\tag{0.4.26}
$$

In the CM frame the momentum of the system is zero before the collision and hence the momentum of the system is zero after the collision. For an elastic collision, the only way for both momentum and kinetic energy to be the same before and after the collision is either the objects have the same velocity (a miss) or to reverse the direction of the velocities as shown in Figure 15.6.

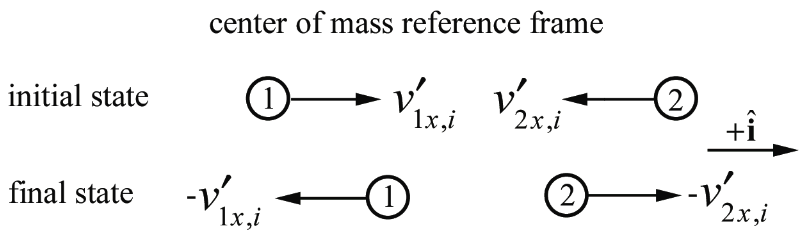

Figure 15.6 One-dimensional elastic collision in center of mass reference frame

In the CM frame, the final x-components of the velocities are

$$
\begin{aligned}
v'_{1x,f} &= -v'_{1x,i} = (v_{2x,i} - v_{1x,i})\frac{m_2}{m_1 + m_2} \\
v'_{2x,f} &= -v'_{2x,i} = (v_{2x,i} - v_{1x,i})\frac{m_1}{m_1 + m_2}.
\end{aligned}
\tag{0.4.27}
$$

The final x-components of the velocities in the "laboratory frame" are then given by

$$
\begin{aligned}
v_{1x,f} &= v'_{1x,f} + v_{x,\mathrm{cm}} \\
&= (v_{2x,i} - v_{1x,i})\frac{m_2}{m_1 + m_2} + \frac{m_1\, v_{1x,i} + m_2\, v_{2x,i}}{m_1 + m_2} \\
&= v_{1x,i}\frac{m_1 - m_2}{m_1 + m_2} + v_{2x,i}\frac{2\,m_2}{m_1 + m_2}
\end{aligned}
\tag{0.4.28}
$$

as in Eq. (0.4.17) and a similar calculation reproduces Eq. (0.4.20).

15.5 Worked Examples

Example 15.1 Elastic One-Dimensional Collision Between Two Objects

Consider the elastic collision of two carts along a track; the incident cart 1 has mass m_1 and moves with initial speed $v_{1,i}$. The target cart has mass $m_2 = 2m_1$ and is initially at rest, $v_{2,i} = 0$. Immediately after the collision, the incident cart has final speed $v_{1,f}$ and the target cart has final speed $v_{2,f}$. Calculate the final x-component of the velocities of the carts as a function of the initial speed $v_{1,i}$.

Solution Draw a "momentum flow" diagram for the objects before (initial state) and after (final state) the collision (Figure 15.7).

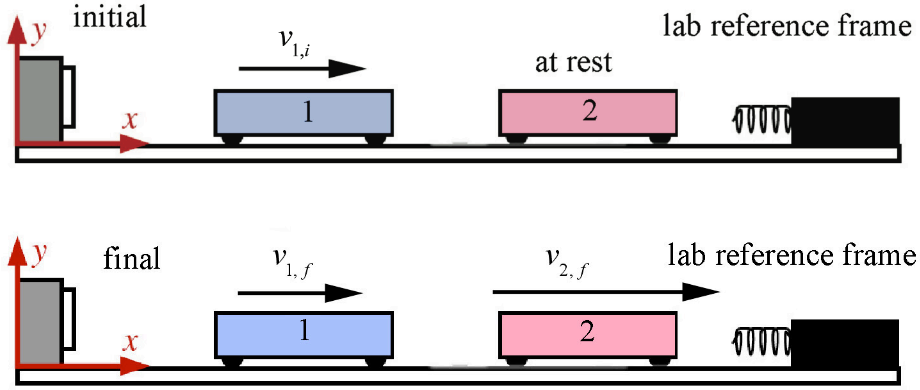

Figure 15.7 Momentum flow diagram for elastic one-dimensional collision

We can immediately use our results above with $m_2 = 2m_1$ and $v_{2,i} = 0$. The final x-component of velocity of cart 1 is given by Eq. (0.4.17), where we use $v_{1x,i} = v_{1,i}$

$$v_{1x,f} = -\frac{1}{3}v_{1,i}. \tag{0.4.29}$$

The final x-component of velocity of cart 2 is given by Eq. (0.4.20)

$$v_{2x,f} = \frac{2}{3}v_{1,i}. \tag{0.4.30}$$

Example 15.2 The Dissipation of Kinetic Energy in a Completely Inelastic Collision Between Two Objects

An incident object of mass m_1 and initial speed $v_{1,i}$ collides completely inelastically with an object of mass m_2 that is initially at rest. There are no external forces acting on the objects in the direction of the collision. Find $\Delta K / K_{\text{initial}} = (K_{\text{final}} - K_{\text{initial}}) / K_{\text{initial}}$.

Solution: In the absence of any net force on the system consisting of the two objects, the momentum after the collision will be the same as before the collision. After the collision the objects will move in the direction of the initial velocity of the incident object with a common speed v_f found from applying the momentum condition

$$\begin{aligned} m_1 v_{1,i} &= (m_1 + m_2) v_f \Rightarrow \\ v_f &= \frac{m_1}{m_1 + m_2} v_{1,i}. \end{aligned} \tag{0.4.31}$$

The initial relative speed is $v_i^{\text{rel}} = v_{1,i}$. The final relative velocity is zero because the objects stick together so using Eq. (0.3.9), the change in kinetic energy is

$$\Delta K = -\frac{1}{2}\mu (v_i^{rel})^2 = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_{1,i}^2. \tag{0.4.32}$$

The ratio of the change in kinetic energy to the initial kinetic energy is then

$$\Delta K / K_{\text{initial}} = -\frac{m_2}{m_1 + m_2}. \tag{0.4.33}$$

As a check, we can calculate the change in kinetic energy via

$$\begin{aligned} \Delta K = (K_f - K_i) &= \frac{1}{2}(m_1 + m_2) v_f^2 - \frac{1}{2} v_{1,i}^2 \\ &= \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2}\right)^2 v_{1,i}^2 - \frac{1}{2} v_{1,i}^2 \\ &= \left(\frac{m_1}{m_1 + m_2} - 1\right)\left(\frac{1}{2} m_1 v_{1,i}^2\right) = -\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} v_{1,i}^2. \end{aligned} \tag{0.4.34}$$

in agreement with Eq. (0.4.32).

Example 15.3 Elastic Two-Dimensional Collision

Object 1 with mass m_1 is initially moving with a speed $v_{1,i} = 3.0\,\text{m}\cdot\text{s}^{-1}$ and collides elastically with object 2 that has the same mass, $m_2 = m_1$, and is initially at rest. After the collision, object 1 moves with an unknown speed $v_{1,f}$ at an angle $\theta_{1,f} = 30^\circ$ with respect to its initial direction of motion and object 2 moves with an unknown speed $v_{2,f}$, at an unknown angle $\theta_{2,f}$ (as shown in the Figure 15.8). Find the final speeds of each of the objects and the angle $\theta_{2,f}$.

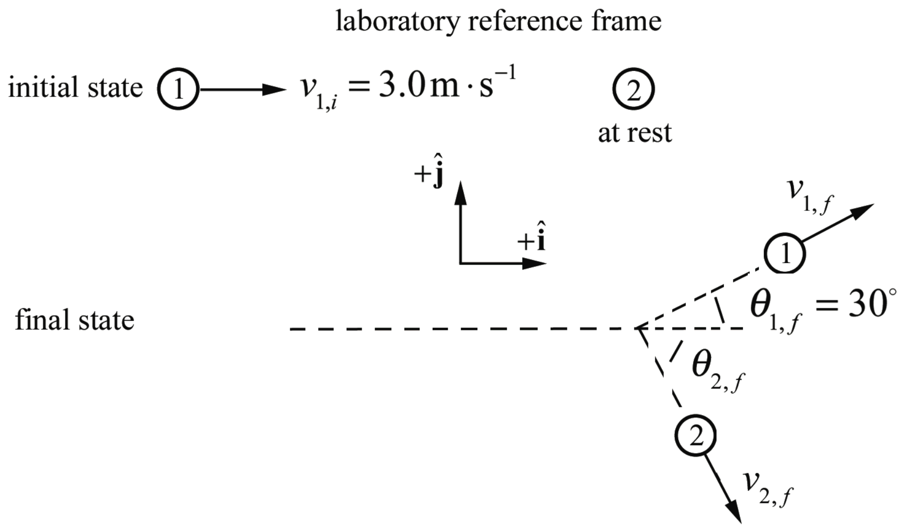

Figure 15.8 Momentum flow diagram for two-dimensional elastic collision

Solution: The components of the total momentum $\vec{\mathbf{p}}_i^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,i} + m_2\vec{\mathbf{v}}_{2,i}$ in the initial state are given by

$$\begin{aligned} p_{x,i}^{\text{sys}} &= m_1 v_{1,i} \\ p_{y,i}^{\text{sys}} &= 0. \end{aligned} \tag{0.4.35}$$

The components of the momentum $\vec{\mathbf{p}}_f^{\text{sys}} = m_1\vec{\mathbf{v}}_{1,f} + m_2\vec{\mathbf{v}}_{2,f}$ in the final state are given by

$$\begin{aligned} p_{x,f}^{\text{sys}} &= m_1\, v_{1,f} \cos\theta_{1,f} + m_1\, v_{2,f} \cos\theta_{2,f} \\ p_{y,f}^{\text{sys}} &= m_1\, v_{1,f} \sin\theta_{1,f} - m_1\, v_{2,f} \sin\theta_{2,f}. \end{aligned} \tag{0.4.36}$$

There are no any external forces acting on the system, so each component of the total momentum remains constant during the collision,

$$p_{x,i}^{\text{sys}} = p_{x,f}^{\text{sys}} \tag{0.4.37}$$

$$p_{y,i}^{\text{sys}} = p_{y,f}^{\text{sys}}. \tag{0.4.38}$$

Eqs. (0.4.37) and (0.4.37) become

$$\begin{aligned} m_1\, v_{1,i} &= m_1\, v_{1,f} \cos\theta_{1,f} + m_1\, v_{2,f} \cos\theta_{2,f} \\ 0 &= m_1\, v_{1,f} \sin\theta_{1,f} - m_1\, v_{2,f} \sin\theta_{2,f}. \end{aligned} \tag{0.4.39}$$

The collision is elastic and therefore the system kinetic energy of is constant

$$K_i^{\text{sys}} = K_f^{\text{sys}}. \tag{0.4.40}$$

Using the given information, Eq. (0.4.40) becomes

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_1 v_{2,f}^2 . \tag{0.4.41}$$

We have three equations: two momentum equations and one energy equation, with three unknown quantities, $v_{1,f}$, $v_{2,f}$ and $\theta_{2,f}$ because we are given $v_{1,i} = 3.0\ \mathrm{m \cdot s^{-1}}$ and $\theta_{1,f} = 30^o$. We first rewrite the expressions in Eq. (0.4.39), canceling the factor of m_1, as

$$\begin{aligned} v_{2,f}\cos\theta_{2,f} &= v_{1,i} - v_{1,f}\cos\theta_{1,f} \\ v_{2,f}\sin\theta_{2,f} &= v_{1,f}\sin\theta_{1,f}. \end{aligned} \tag{0.4.42}$$

We square each expressions in Eq. (0.4.42), yielding

$$v_{2,f}^2(\cos\theta_{2,f})^2 = v_{1,f}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2(\cos\theta_{1,f})^2 \tag{0.4.43}$$

$$v_{2,f}^2(\sin\theta_{2,f})^2 = v_{1,f}^2(\sin\theta_{1,f})^2 \tag{0.4.44}$$

We now add together Eqs. (0.4.43) and (0.4.44) yielding

$$v_{2,f}^2(\cos^2\theta_{2,f} + \sin^2\theta_{2,f}) = v_{1,f}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2(\cos\theta_{1,f} + \sin^2\theta_{1,f}) . \tag{0.4.45}$$

We can use identity $\cos^2\theta + \sin^2\theta = 1$ to simplify Eq. (0.4.45), yielding

$$v_{2,f}^2 = v_{1,i}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} + v_{1,f}^2 . \tag{0.4.46}$$

Substituting Eq. (0.4.46) into Eq. (0.4.41) yields

$$\frac{1}{2}m_1 v_{1,i}^2 = \frac{1}{2}m_1 v_{1,f}^2 + \frac{1}{2}m_1(v_{1,i}^2 - 2v_{1,i}\,v_{1,f}\cos\theta_{1,f} + v_{1,f}^2) . \tag{0.4.47}$$

Eq. (0.4.47) simplifies to

$$0 = 2v_{1,f}^2 - 2v_{1,i}v_{1,f}\cos\theta_{1,f} , \tag{0.4.48}$$

which may be solved for the final speed of object 1,

$$v_{1,f} = v_{1,i} \cos\theta_{1,f} = (3.0\ \mathrm{m \cdot s^{-1}})\cos 30^\circ = 2.6\ \mathrm{m \cdot s^{-1}}. \qquad (0.4.49)$$

Divide the expressions in Eq. (0.4.42), yielding

$$\frac{v_{2,f} \sin\theta_{2,f}}{v_{2,f} \cos\theta_{2,f}} = \frac{v_{1,f} \sin\theta_{1,f}}{v_{1,i} - v_{1,f} \cos\theta_{1,f}}. \qquad (0.4.50)$$

Eq. (0.4.50) simplifies to

$$\tan\theta_{2,f} = \frac{v_{1,f} \sin\theta_{1,f}}{v_{1,i} - v_{1,f} \cos\theta_{1,f}}. \qquad (0.4.51)$$

Thus object 2 moves at an angle

$$\begin{aligned} \theta_{2,f} &= \tan^{-1}\left(\frac{v_{1,f} \sin\theta_{1,f}}{v_{1,i} - v_{1,f} \cos\theta_{1,f}}\right) \\ \theta_{2,f} &= \tan^{-1}\left(\frac{(2.6\ \mathrm{m \cdot s^{-1}})\sin 30^\circ}{3.0\ \mathrm{m \cdot s^{-1}} - (2.6\ \mathrm{m \cdot s^{-1}})\cos 30^\circ}\right) \\ &= 60^\circ. \end{aligned} \qquad (0.4.52)$$

The above results for $v_{1,f}$ and $\theta_{2,f}$ may be substituted into either of the expressions in Eq. (0.4.42), or Eq. (0.4.41), to find $v_{2,f} = 1.5\mathrm{m \cdot s^{-1}}$.

Before going on, the fact that $\theta_{1,f} + \theta_{2,f} = 90^\circ$, that is, the objects move away from the collision point at right angles, is not a coincidence. A vector derivation is presented below. We can see this result algebraically from the above result. Using the result of Eq. (0.4.49), $v_{1,f} = v_{1,i} \cos\theta_{1,f}$, in Eq. (0.4.51) yields

$$\tan\theta_{2,f} = \frac{\cos\theta_{1,f} \sin\theta_{1,f}}{1 - \cos\theta_{1,f}^2} = \cot\theta_{1,f}; \qquad (0.4.53)$$

the angles $\theta_{1,f}$ and $\theta_{2,f}$ are complements. Eq. (0.4.48) also has the solution $v_{2,f} = 0$, which would correspond to the incident particle missing the target completely.

Example 15.4 Equal Mass Particles in a Two-Dimensional Elastic Collision Emerge at Right Angles

Show that the equal mass particles emerge from the collision at right angles by making explicit use of the fact that momentum is a vector quantity.

Solution: There are no external forces acting on the two objects during the collision (the collision forces are all internal), therefore momentum is constant

$$\vec{\mathbf{p}}_i^{\text{sys}} = \vec{\mathbf{p}}_f^{\text{sys}}, \tag{0.4.54}$$

which becomes

$$m_1\vec{\mathbf{v}}_{1,i} = m_1\vec{\mathbf{v}}_{1,f} + m_1\vec{\mathbf{v}}_{2,f} \ . \tag{0.4.55}$$

Eq. (0.4.55) simplifies to

$$\vec{\mathbf{v}}_{1,i} = \vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f} \ . \tag{0.4.56}$$

Recall the vector identity that the square of the speed is given by the dot product $\vec{\mathbf{v}} \cdot \vec{\mathbf{v}} = v^2$. With this identity in mind, we take the dot product of each side of Eq. (0.4.56) with itself,

$$\begin{aligned} \vec{\mathbf{v}}_{1,i} \cdot \vec{\mathbf{v}}_{1,i} &= (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \cdot (\vec{\mathbf{v}}_{1,f} + \vec{\mathbf{v}}_{2,f}) \\ &= \vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{1,f} + 2\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} + \vec{\mathbf{v}}_{2,f} \cdot \vec{\mathbf{v}}_{2,f}. \end{aligned} \tag{0.4.57}$$

This becomes

$$v_{1,i}^2 = v_{1,f}^2 + 2\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} + v_{2,f}^2 \ . \tag{0.4.58}$$

Recall that kinetic energy is the same before and after an elastic collision, and the masses of the two objects are equal, so Eq. (0.4.41) simplifies to

$$v_{1,i}^2 = v_{1,f}^2 + v_{2,f}^2 \ . \tag{0.4.59}$$

Comparing Eq. (0.4.58) with Eq. (0.4.59), we see that

$$\vec{\mathbf{v}}_{1,f} \cdot \vec{\mathbf{v}}_{2,f} = 0 \ . \tag{0.4.60}$$

The dot product of two nonzero vectors is zero when the two vectors are at right angles to each other justifying our claim that the collision particles emerge at right angles to each other.

Example 15.5 Bouncing Superballs

Two superballs are dropped from a height h_i above the ground, one on top of the other. Ball 1 is on top and has mass m_1, and ball 2 is underneath and has mass m_2 with $m_2 >> m_1$. Assume that there is no loss of kinetic energy during all collisions. Ball 2 first collides with the ground and rebounds. Then, as ball 2 starts to move upward, it collides with the ball 1 which is still moving downwards. How high will ball 1 rebound in the air? Hint: consider this collision as seen by an observer moving upward with the same speed as the ball 2 has after it collides with ground (Figure 15.9). What speed does ball 1 have in this reference frame after it collides with the ball 2?

Solution: The system consists of the two balls and the earth. There are five special states for this motion shown in the figure above.

Initial state: the balls are released from rest at a height h_i above the ground.

State A: the balls just reach the ground with speed $v_{1,A} = v_{2,A} = v_A = \sqrt{2gh_i}$.

State B: ball 2 has collided with the ground and reversed direction with the same speed, $v_{2,B} = v_A$, and ball 1 is still moving down with speed $v_{1,B} = v_A$.

State C: because we are assuming that $m_2 >> m_1$, ball 2 does not change speed as a result of the collision so it is still moving upward with speed $v_{2,C} = v_A$. As a result of the collision, ball 1 moves upward with speed $v_{1,C}$.

Final State: ball 1 reaches a maximum height $h_f = v_b^2 / 2g$ above the ground.

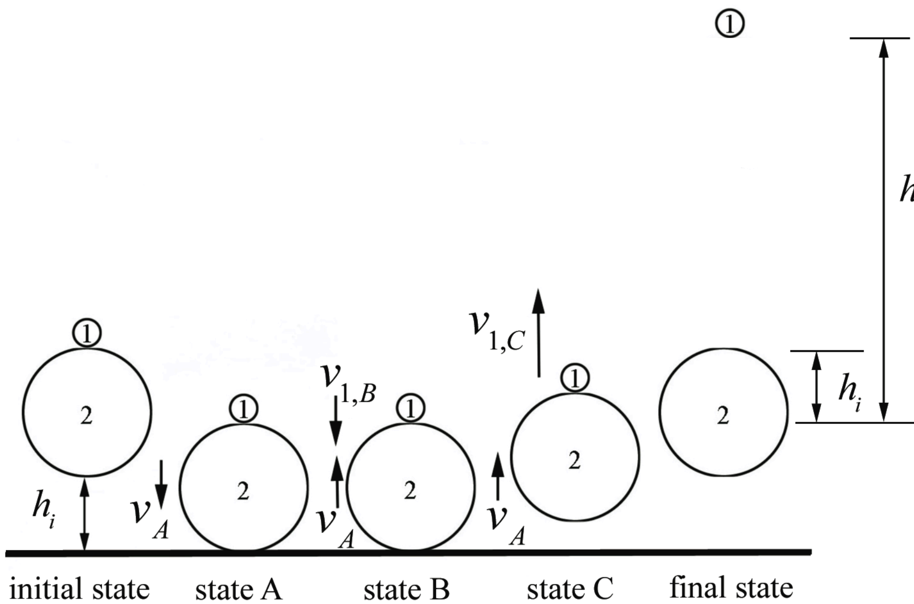

Figure 15.9 States in superball collisions

Choice of Reference Frame: For states B and C, the collision is best analyzed from the reference frame of an observer moving upward with speed v_A, the speed of ball 2 just after it rebounded from the ground. In this frame ball 1 is moving downward with a speed $v'_{1,B}$ that is twice the speed seen by an observer at rest on the ground (lab reference frame).

$$v'_{1,B} = 2v_A \qquad (0.4.61)$$

The mass of ball 2 is much larger than the mass of ball 1, $m_2 >> m_1$. This enables us to consider the collision (between states B and C) to be equivalent to ball 1 bouncing off a hard wall, while ball 2 experiences virtually no recoil. Hence ball 2 remains at rest, $v'_{2,C} = 0$, in the reference frame moving upwards with speed v_A with respect to observer at rest on ground. Before the collision, ball 1 has speed $v'_{1,B} = 2v_a$. Because we assumed the collision is elastic there is no loss of kinetic energy during the collision, therefore ball 1 changes direction but maintains the same speed,

$$v'_{1,C} = 2v_A . \tag{0.4.62}$$

However, according to an observer at rest on the ground, after the collision ball 1 is moving upwards with speed

$$v_{1,C} = 2v_A + v_A = 3v_A . \tag{0.4.63}$$

While rebounding, the mechanical energy of the smaller superball is constant hence between state C and final state,

$$\Delta K + \Delta U = 0 . \tag{0.4.64}$$

The change in kinetic energy is

$$\Delta K = -\frac{1}{2}m_1(3v_A)^2 . \tag{0.4.65}$$

The change in potential energy is

$$\Delta U = m_1 g h_f . \tag{0.4.66}$$

So the condition that mechanical energy is constant (Eq. (0.4.64)) is now

$$-\frac{1}{2}m_1(3v_A)^2 + m_1 g h_f = 0 . \tag{0.4.67}$$

Recall that we can also use the fact that the mechanical energy doesn't change between the initial state and state A. Therefore

$$m_1 g h_i = \frac{1}{2}m_1(v_A)^2 . \tag{0.4.68}$$

Substitute the expression for the kinetic energy in Eq. (0.4.68) into Eq. (0.4.67) yielding

$$m_1 g h_f = 9 m_1 g h_i . \tag{0.4.69}$$

Thus ball 1 reaches a maximum height

$$h_f = 9h_i . \tag{0.4.70}$$