Chapter 14 Potential Energy and Conservation of Energy

Equation Chapter 8 Section 1 Chapter 14 Potential Energy and Conservation of Energy

There is a fact, or if you wish, a law, governing all natural phenomena that are known to date. There is no exception to this law — it is exact as far as we know. The law is called the conservation of energy. It states that there is a certain quantity, which we call energy that does not change in the manifold changes which nature undergoes. That is a most abstract idea, because it is a mathematical principle; it says that there is a numerical quantity, which does not change when something happens. It is not a description of a mechanism, or anything concrete; it is just a strange fact that we can calculate some number and when we finish watching nature go through her tricks and calculate the number again, it is the same. [1]

Richard Feynman

So far we have analyzed the motion of point-like objects under the action of forces using Newton's Laws of Motion. We shall now introduce the Principle of Conservation of Energy to study the change in energy of a system between its initial and final states. In particular we shall introduce the concept of potential energy to describe the effect of conservative internal forces acting on the constituent components of a system.

14.1 Conservation of Energy

Recall from Chapter 13.1, the principle of conservation of energy. When a system and its surroundings undergo a transition from an initial state to a final state, the change in energy is zero,

$$\Delta E = \Delta E_{\text{system}} + \Delta E_{\text{surroundings}} = 0\,. \tag{14.1.1}$$

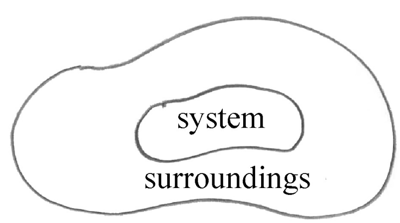

Figure 14.1 Diagram of a system and its surroundings

We shall study types of energy transformations due to interactions both inside and across the boundary of a system.

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[1] Richard P. Feynman, Robert B. Leighton, and Matthew Sands, *The Feynman Lectures on Physics,* Vol. 1, p. 4.1.

14.2 Conservative and Non-Conservative Forces

Our first type of "energy accounting" involves *mechanical energy.* There are two types of mechanical energy, *kinetic energy* and *potential energy*. Our first task is to define what we mean by the change of the potential energy of a system.

We defined the work done by a force $\vec{\mathbf{F}}$, on an object, which moves along a path from an initial position $\vec{\mathbf{r}}_i$ to a final position $\vec{\mathbf{r}}_f$, as the integral of the component of the force tangent to the path with respect to the displacement of the point of contact of the force and the object,

$$W = \int_{\text{path}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} . \quad (14.2.1)$$

Does the work done on the object by the force depend on the path taken by the object?

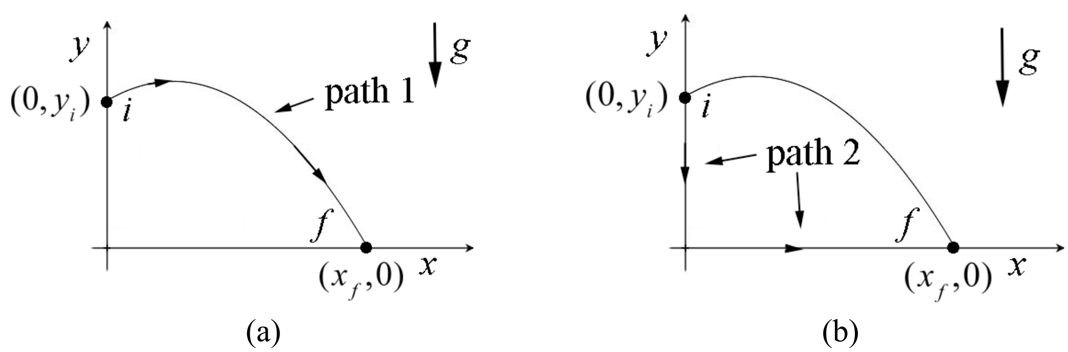

Figure 14.2 (a) and (b) Two different paths connecting the same initial and final points

First consider the motion of an object under the influence of a gravitational force near the surface of the earth. Let's consider two paths 1 and 2 shown in Figure 14.2. Both paths begin at the initial point $(x_i, y_i) = (0, y_i)$ and end at the final point $(x_f, y_f) = (x_f, 0)$. The gravitational force always points downward, so with our choice of coordinates, $\vec{\mathbf{F}} = -mg\,\hat{\mathbf{j}}$. The infinitesimal displacement along path 1 (Figure 14.2a) is given by $d\vec{\mathbf{r}}_1 = dx_1\,\hat{\mathbf{i}} + dy_1\,\hat{\mathbf{j}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_1 = -mg\,\hat{\mathbf{j}} \cdot (dx_1\,\hat{\mathbf{i}} + dy_1\,\hat{\mathbf{j}}) = -mgdy_1 . \quad (14.2.2)$$

The work done by gravity along path 1 is the integral

$$W_1 = \int_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{(0,y_i)}^{(x_f,0)} -mg\,dy_1 = -mg(0 - y_i) = mgy_i . \quad (14.2.3)$$

Path 2 consists of two legs (Figure 14.2b), leg A goes from the initial point $(0, y_i)$ to the origin $(0,0)$, and leg B goes from the origin $(0,0)$ to the final point $(x_f, 0)$. We shall calculate the work done along the two legs and then sum them up. The infinitesimal displacement along leg A is given by $d\vec{\mathbf{r}}_A = dy_A\, \hat{\mathbf{j}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_A = -mg\, \hat{\mathbf{j}} \cdot dy_A\, \hat{\mathbf{j}} = -mgdy_A \,. \tag{14.2.4}$$

The work done by gravity along leg A is the integral

$$W_A = \int_{\text{leg } A} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_{\text{A}} = \int_{(0,y_i)}^{(0,0)} -mg\, dy_A = -mg(0 - y_i) = mgy_i \,. \tag{14.2.5}$$

The infinitesimal displacement along leg B is given by $d\vec{\mathbf{r}}_B = dx_B\, \hat{\mathbf{i}}$. The scalar product is then

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}_B = -mg\, \hat{\mathbf{j}} \cdot dx_B\, \hat{\mathbf{i}} = 0 \,. \tag{14.2.6}$$

Therefore the work done by gravity along leg B is zero, $W_B = 0$, which is no surprise because leg B is perpendicular to the direction of the gravitation force. Therefore the work done along path 2 is equal to the work along path 1,

$$W_2 = W_A + W_B = mgy_i = W_1 \,. \tag{14.2.7}$$

Now consider the motion of an object on a surface with a kinetic frictional force between the object and the surface and denote the coefficient of kinetic friction by μ_{k}. Let's compare two paths from an initial point x_i to a final point x_f. The first path is a straight-line path. Along this path the work done is just

$$W^f = \int_{\text{path 1}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path 1}} F_x\, dx = -\mu_{\text{k}} N s_1 = -\mu_{\text{k}} N \Delta x < 0 \,, \tag{14.2.8}$$

where the length of the path is equal to the displacement, $s_1 = \Delta x$. Note that the fact that the kinetic frictional force is directed opposite to the displacement, which is reflected in the minus sign in Equation (14.2.8). The second path goes past x_f some distance and them comes back to x_f (Figure 14.3). Because the force of friction always opposes the motion, the work done by friction is negative,

$$W^f = \int_{\text{path 2}} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_{\text{path 2}} F_x\, dx = -\mu_{\text{k}} N s_2 < 0 \,. \tag{14.2.9}$$

The work depends on the total distance traveled s_2, and is greater than the displacement $s_2 > \Delta x$. The magnitude of the work done along the second path is greater than the magnitude of the work done along the first path.

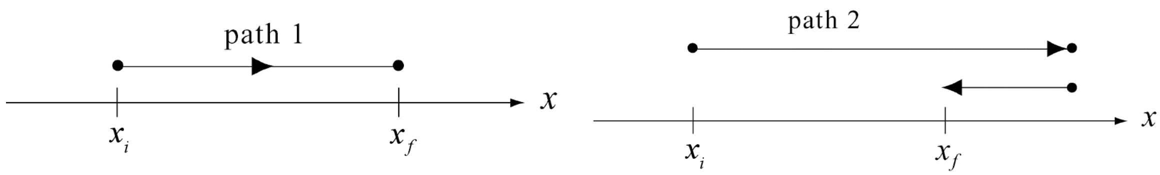

Figure 14.3 Two different paths from x_i to x_f.

These two examples typify two fundamentally different types of forces and their contribution to work. The work done by the gravitational force near the surface of the earth is independent of the path taken between the initial and final points. In the case of sliding friction, the work done depends on the path taken.

*Whenever the work done by a force in moving an object from an initial point to a final point is independent of the path, the force is called a **conservative force**.*

The work done by a conservative force $\vec{\mathbf{F}}_c$ in going around a closed path is zero. Consider the two paths shown in Figure 14.4 that form a closed path starting and ending at the point A with Cartesian coordinates $(1,0)$.

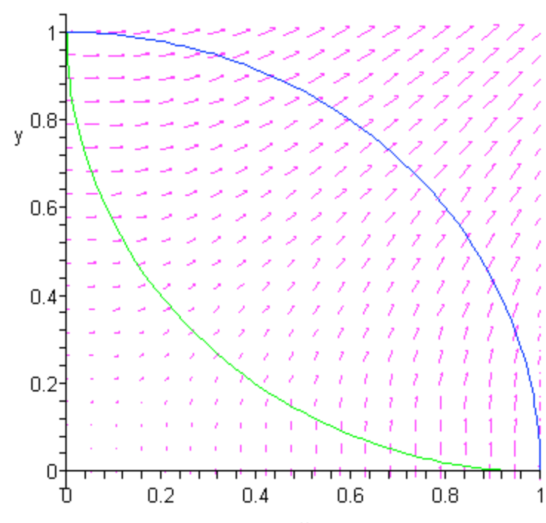

Figure 14.4 Two paths in the presence of a conservative force.

The work done along path 1 (the upper path in the figure, blue if viewed in color) from point A to point B with coordinates $(0,1)$ is given by

$$W_1 = \int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 \,. \tag{14.2.10}$$

The work done along path 2 (the lower path, green in color) from B to A is given by

$$W_2 = \int_B^A \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.11}$$

The work done around the closed path is just the sum of the work along paths 1 and 2,

$$W = W_1 + W_2 = \int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 + \int_B^A \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.12}$$

If we reverse the endpoints of path 2, then the integral changes sign,

$$W_2 = \int_B^A \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 = -\int_A^B \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.13}$$

We can then substitute Equation (14.2.13) into Equation (14.2.12) to find that the work done around the closed path is

$$W = \int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.14}$$

Since the force is conservative, the work done between the points A to B is independent of the path, so

$$\int_A^B \vec{\mathbf{F}}_c(1) \cdot d\vec{\mathbf{r}}_1 = \int_A^B \vec{\mathbf{F}}_c(2) \cdot d\vec{\mathbf{r}}_2 \,. \tag{14.2.15}$$

We now use path independence of work for a conservative force (Equation (14.2.15) in Equation (14.2.14)) to conclude that the work done by a conservative force around a closed path is zero,

$$W = \oint_{\substack{\text{closed}\\ \text{path}}} \vec{\mathbf{F}}_c \cdot d\vec{\mathbf{r}} = 0 \,. \tag{14.2.16}$$

14.3 Changes in Potential Energies of a System

Consider an object near the surface of the earth as a system that is initially given a velocity directed upwards. Once the object is released, the gravitation force, acting as an external force, does a negative amount of work on the object, and the kinetic energy decreases until the object reaches its highest point, at which its kinetic energy is zero. The gravitational force then does positive work until the object returns to its initial starting point with a velocity directed downward. If we ignore any effects of air resistance, the descending object will then have the identical kinetic energy as when it was thrown. All the kinetic energy was completely recovered.

Now consider both the earth and the object as a system and assume that there are no other external forces acting on the system. Then the gravitational force is an internal conservative force, and does work on both the object and the earth during the motion. As the object moves upward, the kinetic energy of the system decreases, primarily because the object slows down, but there is also an imperceptible increase in the kinetic energy of the earth. The change in kinetic energy of the earth must also be included because the earth is part of the system. When the object returns to its original height (vertical distance from the surface of the earth), all the kinetic energy in the system is recovered, even though a very small amount has been transferred to the Earth.

If we included the air as part of the system, and the air resistance as a non-conservative internal force, then the kinetic energy lost due to the work done by the air resistance is not recoverable. This lost kinetic energy, which we have called thermal energy, is distributed as random kinetic energy in both the air molecules and the molecules that compose the object (and, to a smaller extent, the earth).

We shall define a new quantity, the change in the internal *potential energy* of the system, which measures the amount of lost kinetic energy that can be recovered during an interaction.

When only internal conservative forces act in a closed system, the sum of the changes of the kinetic and potential energies of the system is zero.

Consider a closed system, $\Delta E_{sys} = 0$, that consists of two objects with masses m_1 and m_2 respectively. Assume that there is only one conservative force (internal force) that is the source of the interaction between two objects. We denote the force on object 1 due to the interaction with object 2 by $\vec{\mathbf{F}}_{2,1}$ and the force on object 2 due to the interaction with object 1 by $\vec{\mathbf{F}}_{1,2}$. From Newton's Third Law,

$$\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}\,. \tag{14.3.1}$$

The forces acting on the objects are shown in Figure 14.5.

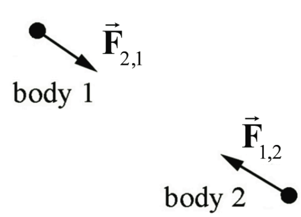

Figure 14.5 Internal forces acting on two objects

Choose a coordinate system (Figure 14.6) in which the position vector of object 1 is given by $\vec{\mathbf{r}}_1$ and the position vector of object 2 is given by $\vec{\mathbf{r}}_2$. The relative position of object 1 with respect to object 2 is given by $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. During the course of the interaction, object 1 is displaced by $d\vec{\mathbf{r}}_1$ and object 2 is displaced by $d\vec{\mathbf{r}}_2$, so the relative displacement of the two objects during the interaction is given by $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$.

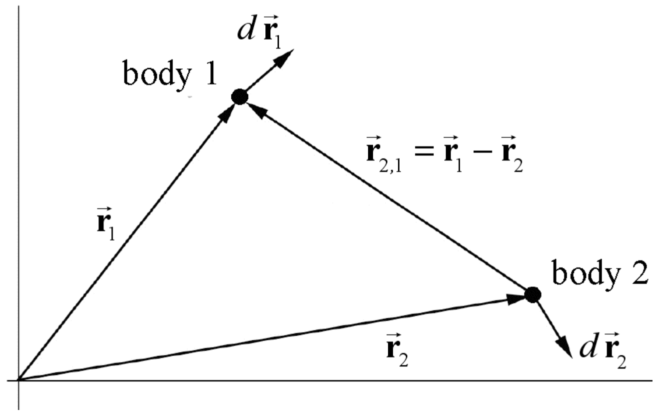

Figure 14.6 Coordinate system for two objects with relative position vector $\vec{\mathbf{r}}_{2,1} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$

Recall that the change in the kinetic energy of an object is equal to the work done by the forces in displacing the object. For two objects displaced from an initial state A to a final state B,

$$\Delta K_{\text{sys}} = \Delta K_1 + \Delta K_2 = W_{\text{c}} = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\,\vec{\mathbf{r}}_1 + \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\,\vec{\mathbf{r}}_2 . \tag{14.3.2}$$

(In Equation (14.3.2), the labels "A" and "B" refer to initial and final states, not paths.)

From Newton's Third Law, Equation (14.3.1), the sum in Equation (14.3.2) becomes

$$\Delta K_{\text{sys}} = W_c = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_1 - \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_2 = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot (d\,\vec{\mathbf{r}}_1 - d\,\vec{\mathbf{r}}_2) = \int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\,\vec{\mathbf{r}}_{2,1} \tag{14.3.3}$$

where $d\,\vec{\mathbf{r}}_{1,2} = d\,\vec{\mathbf{r}}_1 - d\,\vec{\mathbf{r}}_2$ is the relative displacement of the two objects. Note that since $\vec{\mathbf{F}}_{2,1} = -\vec{\mathbf{F}}_{1,2}$ and $d\vec{\mathbf{r}}_{2,1} = -d\,\vec{\mathbf{r}}_{1,2}$, $\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\,\vec{\mathbf{r}}_{2,1} = \int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\,\vec{\mathbf{r}}_{1,2}$.

Consider a system consisting of two objects interacting through a conservative force. Let $\vec{\mathbf{F}}_{2,1}$ *denote the force on object 1 due to the interaction with object 2 and let* $d\vec{\mathbf{r}}_{2,1} = d\vec{\mathbf{r}}_1 - d\vec{\mathbf{r}}_2$ *be the relative displacement of the two objects. The* ***change in internal potential energy of the system*** *is defined to be the negative of the work done by the conservative force when the objects undergo a relative displacement from the initial state* A *to the final state* B *along any displacement that changes the initial state* A *to the final state* B*,*

$$\Delta U_{\text{sys}} = -W_{\text{c}} = -\int_A^B \vec{\mathbf{F}}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\int_A^B \vec{\mathbf{F}}_{1,2} \cdot d\vec{\mathbf{r}}_{1,2} . \tag{14.3.4}$$

Our definition of potential energy only holds for conservative forces, because the work done by a conservative force does not depend on the path but only on the initial and final positions. Because the work done by the conservative force is equal to the change in kinetic energy, we have that

$$\Delta U_{\text{sys}} = -\Delta K_{\text{sys}}, \text{ (closed system with no non-conservative forces)} . \tag{14.3.5}$$

Recall that the work done by a conservative force in going around a closed path is zero (Equation (14.2.16)); therefore the change in kinetic energy when a system returns to its initial state is zero. This means that the kinetic energy is completely recoverable.

In the *Appendix 13A: Work Done on a System of Two Particles,* we showed that the work done by an internal force in changing a system of two particles of masses m_1 and m_2 respectively from an initial state A to a final state B is equal to

$$W = \frac{1}{2}\mu(v_B^2 - v_A^2) = \Delta K_{\text{sys}} , \tag{14.3.6}$$

where v_B^2 is the square of the relative velocity in state B, v_A^2 is the square of the relative velocity in state A, and $\mu = m_1 m_2 / (m_1 + m_2)$ is a quantity known as the *reduced mass* of the system.

14.3.1 Change in Potential Energy for Several Conservative Forces

When there are several internal conservative forces acting on the system we define a separate change in potential energy for the work done by each conservative force,

$$\Delta U_{\text{sys},\,i} = -W_{c,i} = -\int_A^B \vec{\mathbf{F}}_{\text{c},\,i} \cdot d\vec{\mathbf{r}}_i . \tag{14.3.7}$$

where $\vec{\mathbf{F}}_{c,i}$ is a conservative internal force and $d\vec{\mathbf{r}}_i$ a change in the relative positions of the objects on which $\vec{\mathbf{F}}_{c,i}$ when the system is changed from state A to state B. The work done is the sum of the work done by the individual conservative forces,

$$W_{c} = W_{c,1} + W_{c,2} + \cdots. \qquad (14.3.8)$$

Hence, the sum of the changes in potential energies for the system is the sum

$$\Delta U_{sys} = \Delta U_{sys,1} + \Delta U_{sys,2} + \cdots. \qquad (14.3.9)$$

Therefore the change in potential energy of the system is equal to the negative of the work done

$$\Delta U_{sys} = -W_{c} = -\sum_{i} \int_{A}^{B} \vec{\mathbf{F}}_{c,i} \cdot d\vec{\mathbf{r}}_i\,. \qquad (14.3.10)$$

If the system is closed (external forces do no work), and there are no non-conservative internal forces then Eq. (14.3.5) holds.

14.4 Change in Potential Energy and Zero Point for Potential Energy

We already calculated the work done by different conservative forces: constant gravity near the surface of the earth, the spring force, and the universal gravitation force. We chose the system in each case so that the conservative force was an external force. In each case, there was no change of potential energy and the work done was equal to the change of kinetic energy,

$$W_{ext} = \Delta K_{sys}\,. \qquad (14.4.1)$$

We now treat each of these conservative forces as internal forces and calculate the change in potential energy of the system according to our definition

$$\Delta U_{sys} = -W_{c} = -\int_{A}^{B} \vec{\mathbf{F}}_{c} \cdot d\vec{\mathbf{r}}\,. \qquad (14.4.2)$$

We shall also choose a ***zero reference potential*** for the potential energy of the system, so that we can consider all changes in potential energy relative to this reference potential.

14.4.1 Change in Gravitational Potential Energy Near Surface of the Earth

Let's consider the example of an object falling near the surface of the earth. Choose our system to consist of the earth and the object. The gravitational force is now an internal conservative force acting inside the system. The distance separating the object and the center of mass of the earth, and the velocities of the earth and the object specifies the initial and final states.

Let's choose a coordinate system with the origin on the surface of the earth and the $+y$-direction pointing away from the center of the earth. Because the displacement of the earth is negligible, we need only consider the displacement of the object in order to calculate the change in potential energy of the system.

Suppose the object starts at an initial height y_i above the surface of the earth and ends at final height y_f. The gravitational force on the object is given by $\vec{\mathbf{F}}^g = -mg\,\hat{\mathbf{j}}$, the displacement is given by $d\vec{\mathbf{r}} = dy\,\hat{\mathbf{j}}$, and the scalar product is given by $\vec{\mathbf{F}}^g \cdot d\vec{\mathbf{r}} = -mg\,\hat{\mathbf{j}} \cdot dy\hat{\mathbf{j}} = -mgdy$. The work done by the gravitational force on the object is then

$$W^g = \int_{y_i)}^{y_f} \vec{\mathbf{F}}^g \cdot d\vec{\mathbf{r}} = \int_{y_i)}^{y_f} -mg\,dy = -mg(y_f - y_i) \ . \tag{14.4.3}$$

The change in potential energy is then given by

$$\Delta U^g = -W^g = mg\,\Delta y = mg\,y_f - mg\,y_i \,. \tag{14.4.4}$$

We introduce a potential energy function U so that

$$\Delta U^g \equiv U_f^g - U_i^g \,. \tag{14.4.5}$$

Only differences in the function U^g have a physical meaning. We can choose a zero reference point for the potential energy anywhere we like. We have some flexibility to adapt our choice of zero for the potential energy to best fit a particular problem. Because the change in potential energy only depended on the displacement, Δy. In the above expression for the change of potential energy (Eq. (14.4.4)), let $y_f = y$ be an arbitrary point and $y_i = 0$ denote the surface of the earth. Choose the zero reference potential for the potential energy to be at the surface of the earth corresponding to our origin $y = 0$, with $U^g(0) = 0$. Then

$$\Delta U^g = U^g(y) - U^g(0) = U^g(y) \,. \tag{14.4.6}$$

Substitute $y_i = 0$, $y_f = y$ and Eq. (14.4.6) into Eq. (14.4.4) yielding a potential energy as a function of the height y above the surface of the earth,

$$U^g(y) = mgy, \text{ with } U^g(y = 0) = 0 \,. \tag{14.4.7}$$

14.4.2 Hooke's Law Spring-Object System

Consider a spring-object system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to an object of mass m (Figure 14.7). The spring force is an internal conservative force. The wall exerts an external force on the spring-object system but since the point of contact of the wall with the spring undergoes no displacement, this external force does no work.

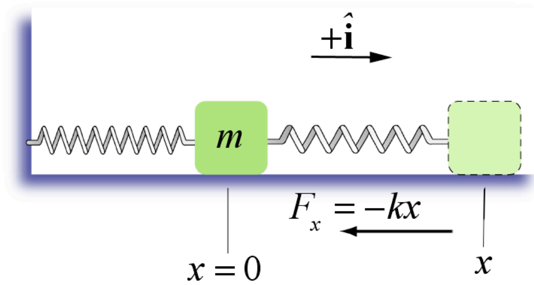

Figure 14.7 A spring-object system.

Choose the origin at the position of the center of the object when the spring is relaxed (the equilibrium position). Let x be the displacement of the object from the origin. We choose the $+\hat{\mathbf{i}}$ unit vector to point in the direction the object moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on a mass is then given by $\vec{\mathbf{F}}^s = F_x^s\,\hat{\mathbf{i}} = -kx\,\hat{\mathbf{i}}$. The displacement is $d\vec{\mathbf{r}} = dx\,\hat{\mathbf{i}}$. The scalar product is $\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -kx\,\hat{\mathbf{i}} \cdot dx\,\hat{\mathbf{i}} = -kx\,dx$. The work done by the spring force on the mass is

$$W^s = \int_{x=x_i}^{x=x_f} \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = -\frac{1}{2} \int_{x=x_i}^{x=x_f} -\frac{1}{2}(-kx)\,dx = -\frac{1}{2}k({x_f}^2 - {x_i}^2)\,. \qquad (14.4.8)$$

We then define the change in potential energy in the spring-object system in moving the object from an initial position x_i from equilibrium to a final position x_f from equilibrium by

$$\Delta U^s \equiv U^s(x_f) - U^s(x_i) = -W^s = \frac{1}{2}k(x_f^2 - x_i^2)\,. \qquad (14.4.9)$$

Therefore an arbitrary stretch or compression of a spring-object system from equilibrium $x_i = 0$ to a final position $x_f = x$ changes the potential energy by

$$\Delta U^s = U^s(x_f) - U^s(0) = \frac{1}{2}k\,x^2\,. \qquad (14.4.10)$$

For the spring-object system, there is an obvious choice of position where the potential energy is zero, the equilibrium position of the spring- object,

$$U^{s}(0) \equiv 0 . \tag{14.4.11}$$

Then with this choice of zero reference potential, the potential energy as a function of the displacement x from the equilibrium position is given by

$$U^{s}(x) = \frac{1}{2} k x^{2}, \text{ with } U^{s}(0) \equiv 0 . \tag{14.4.12}$$

14.4.3 Inverse Square Gravitation Force

Consider a system consisting of two objects of masses m_1 and m_2 that are separated by a center-to-center distance $r_{2,1}$. A coordinate system is shown in the Figure 14.8. The internal gravitational force on object 1 due to the interaction between the two objects is given by

$$\vec{\mathbf{F}}^{G}_{2,1} = -\frac{G m_1 m_2}{r_{2,1}^{2}} \hat{\mathbf{r}}_{2,1} . \tag{14.4.13}$$

The displacement vector is given by $d\vec{\mathbf{r}}_{2,1} = dr_{2,1}\, \hat{\mathbf{r}}_{2,1}$. So the scalar product is

$$\vec{\mathbf{F}}^{G}_{2,1} \cdot d\vec{\mathbf{r}}_{2,1} = -\frac{G m_1 m_2}{r_{2,1}^{2}} \hat{\mathbf{r}}_{2,1} \cdot dr_{2,1}\, \hat{\mathbf{r}}_{2,1} = -\frac{G m_1 m_2}{r_{2,1}^{2}} dr_{2,1} \ . \tag{14.4.14}$$

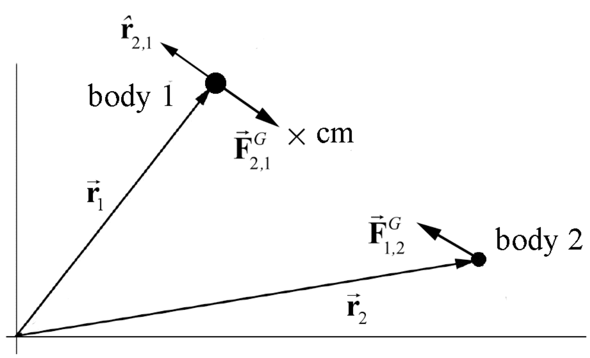

Figure 14.8 Gravitational interaction

Using our definition of potential energy (Eq. (14.3.4)), we have that the change in the gravitational potential energy of the system in moving the two objects from an initial position in which the center of mass of the two objects are a distance r_i apart to a final position in which the center of mass of the two objects are a distance r_f apart is given by

$$\Delta U^G = -\int_A^B \vec{\mathbf{F}}_{2,1}^G \cdot d\vec{\mathbf{r}}_{2,1} = -\int_{r_i}^{f} -\frac{G\, m_1\, m_2}{r_{2,1}^2} dr_{2,1} = -\left.\frac{G\, m_1\, m_2}{r_{2,1}}\right|_{r_i}^{r_f} = -\frac{G\, m_1\, m_2}{r_f} + \frac{G\, m_1\, m_2}{r_i}. \quad (14.4.15)$$

We now choose our reference point for the zero of the potential energy to be at infinity, $r_i = \infty$, with the choice that $U^G(\infty) \equiv 0$. By making this choice, the term $1/r$ in the expression for the change in potential energy vanishes when $r_i = \infty$. The gravitational potential energy as a function of the relative distance r between the two objects is given by

$$U^G(r) = -\frac{G\, m_1\, m_2}{r}, \text{ with } U^G(\infty) \equiv 0. \quad (14.4.16)$$

14.5 Mechanical Energy and Conservation of Mechanical Energy

The total change in the ***mechanical energy*** *of the system is defined to be the sum of the changes of the kinetic and the potential energies,*

$$\Delta E_m = \Delta K_{\text{sys}} + \Delta U_{\text{sys}}. \quad (14.4.17)$$

For a closed system with only conservative internal forces, the total change in the mechanical energy is zero,

$$\Delta E_m = \Delta K_{\text{sys}} + \Delta U_{\text{sys}} = 0. \quad (14.4.18)$$

Equation (14.4.18) is the symbolic statement of what is called ***conservation of mechanical energy*****.** Recall that the work done by a conservative force in going around a closed path is zero (Equation (14.2.16)), therefore both the changes in kinetic energy and potential energy are zero when a closed system with only conservative internal forces returns to its initial state. Throughout the process, the kinetic energy may change into internal potential energy but if the system returns to its initial state, the kinetic energy is completely recoverable. We shall refer to a closed system in which processes take place in which only conservative forces act as ***completely reversible processes***.

14.5.1 Change in Gravitational potential Energy Near Surface of the Earth

Let's consider the example of an object of mass m_o falling near the surface of the earth (mass m_e). Choose our system to consist of the earth and the object. The gravitational force is now an internal conservative force acting inside the system. The initial and final states are specified by the distance separating the object and the center of mass of the earth, and the velocities of the earth and the object. The change in kinetic energy between the initial and final states for the system is

$$\Delta K_{\text{sys}} = \Delta K_e + \Delta K_o, \quad (14.4.19)$$

$$\Delta K_{\text{sys}} = \left(\frac{1}{2}m_{\text{e}}(v_{e,f})^2 - \frac{1}{2}m_{\text{e}}(v_{\text{e,i}})^2\right) + \left(\frac{1}{2}m_o(v_{o,f})^2 - \frac{1}{2}m_o(v_{o,i})^2\right). \quad (14.4.20)$$

The change of kinetic energy of the earth due to the gravitational interaction between the earth and the object is negligible. The change in kinetic energy of the system is approximately equal to the change in kinetic energy of the object,

$$\Delta K_{\text{sys}} \cong \Delta K_o = \frac{1}{2}m_o(v_{o,f})^2 - \frac{1}{2}m_o(v_{o,i})^2. \quad (14.4.21)$$

We now define the mechanical energy function for the system

$$E_m = K + U^g = \frac{1}{2}m_o(v_b)^2 + m_o g y, \text{ with } U^g(0) = 0, \quad (14.4.22)$$

where K is the kinetic energy and U^g is the potential energy. The change in mechanical energy is then

$$\Delta E_m \equiv E_{m,f} - E_{m,i} = (K_f + U_f^g) - (K_i + U_i^g). \quad (14.4.23)$$

When the work done by the external forces is zero and there are no internal non-conservative forces, the total mechanical energy of the system is constant,

$$E_{m,f} = E_{m,i}, \quad (14.4.24)$$

or equivalently

$$(K_f + U_f) = (K_i + U_i). \quad (14.4.25)$$

14.6 Spring Force Energy Diagram

The spring force on an object is a restoring force $\vec{\mathbf{F}}^s = F_x^s\,\hat{\mathbf{i}} = -k\,x\,\hat{\mathbf{i}}$ where we choose a coordinate system with the equilibrium position at $x_i = 0$ and x is the amount the spring has been stretched $(x > 0)$ or compressed $(x < 0)$ from its equilibrium position. We calculate the potential energy difference Eq. (14.4.9) and found that

$$U^s(x) - U^s(x_i) = -\int_{x_i}^{x} F_x^s\, dx = \frac{1}{2}k(x^2 - x_i^2). \quad (14.5.1)$$

The first fundamental theorem of calculus states that

$$U(x) - U(x_i) = \int_{x'=x_i}^{x'=x} \frac{dU}{dx'}\, dx'. \quad (14.5.2)$$

Comparing Equation (14.5.1) with Equation (14.5.2) shows that the force is the negative derivative (with respect to position) of the potential energy,

$$F_x^s = -\frac{dU^s(x)}{dx}. \tag{14.5.3}$$

Choose the zero reference point for the potential energy to be at the equilibrium position, $U^s(0) \equiv 0$. Then the potential energy function becomes

$$U^s(x) = \frac{1}{2}k\,x^2. \tag{14.5.4}$$

From this, we obtain the spring force law as

$$F_x^s = -\frac{dU^s(x)}{dx} = -\frac{d}{dx}\left(\frac{1}{2}k\,x^2\right) = -k\,x. \tag{14.5.5}$$

In Figure 14.9 we plot the potential energy function $U^s(x)$ for the spring force as function of x with $U^s(0) \equiv 0$ (the units are arbitrary).

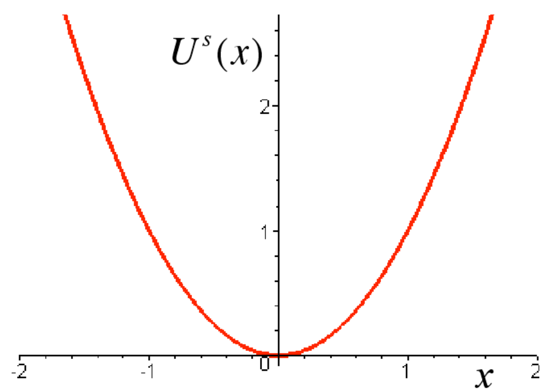

Figure 14.9 Graph of potential energy function as function of x for the spring.

The minimum of the potential energy function occurs at the point where the first derivative vanishes

$$\frac{dU^s(x)}{dx} = 0. \tag{14.5.6}$$

From Equation (14.5.4), the minimum occurs at $x = 0$,

$$0 = \frac{dU^s(x)}{dx} = k\,x. \tag{14.5.7}$$

Because the force is the negative derivative of the potential energy, and this derivative vanishes at the minimum, we have that the spring force is zero at the minimum $x = 0$ agreeing with our force law, $F_x^s\big|_{x=0} = -k\,x\big|_{x=0} = 0$.

The potential energy function has positive curvature in the neighborhood of a minimum equilibrium point. If the object is extended a small distance $x > 0$ away from equilibrium, the slope of the potential energy function is positive, $dU(x)/dx > 0$, hence the component of the force is negative because $F_x = -dU(x)/dx < 0$. Thus the object experiences a restoring force towards the minimum point of the potential. If the object is compresses with $x < 0$ then $dU(x)/dx < 0$, hence the component of the force is positive, $F_x = -dU(x)/dx > 0$, and the object again experiences a restoring force back towards the minimum of the potential energy as in Figure 14.10.

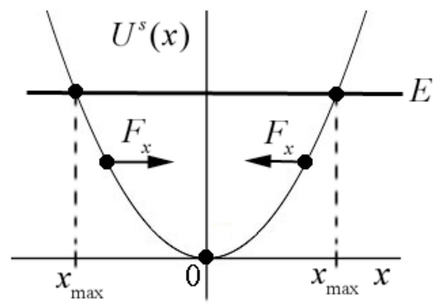

Figure 14.10 Stability diagram for the spring force.

The mechanical energy at any time is the sum of the kinetic energy $K(x)$ and the potential energy $U^s(x)$

$$E_m = K(x) + U^s(x). \tag{14.5.8}$$

Suppose our spring-object system has no loss of mechanical energy due to dissipative forces such as friction or air resistance. Both the kinetic energy and the potential energy are functions of the position of the object with respect to equilibrium. The energy is a constant of the motion and with our choice of $U^s(0) \equiv 0$, the energy can be either a positive value or zero. When the energy is zero, the object is at rest at the equilibrium position.

In Figure 14.10, we draw a straight horizontal line corresponding to a non-zero positive value for the energy E_m on the graph of potential energy as a function of x. The energy intersects the potential energy function at two points $\{-x_{\max}, x_{\max}\}$ with $x_{\max} > 0$. These points correspond to the maximum compression and maximum extension of the spring, which are called the ***turning points***. The kinetic energy is the difference between the energy and the potential energy,

$$K(x) = E_m - U^s(x). \tag{14.5.9}$$

At the turning points, where $E_m = U^s(x)$, the kinetic energy is zero. Regions where the kinetic energy is negative, $x < -x_{\max}$ or $x > x_{\max}$ are called the ***classically forbidden regions***, which the object can never reach if subject to the laws of classical mechanics. In quantum mechanics, with similar energy diagrams for quantum systems, there is a very small probability that the quantum object can be found in a classically forbidden region.

Example 14.1 Energy Diagram

The potential energy function for a particle of mass m, moving in the x-direction is given by

$$U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right), \tag{14.5.10}$$

where U_1 and x_1 are positive constants and $U(0) = 0$. (a) Sketch $U(x)/U_1$ as a function of x/x_1. (b) Find the points where the force on the particle is zero. Classify them as stable or unstable. Calculate the value of $U(x)/U_1$ at these equilibrium points. (c) For energies E that lies in $0 < E < (4/27)U_1$ find an equation whose solution yields the turning points along the x-axis about which the particle will undergo periodic motion. (d) Suppose $E = (4/27)U_1$ and that the particle starts at $x = 0$ with speed v_0. Find v_0.

Solution: a) Figure 14.11 shows a graph of $U(x)$ vs. x, with the choice of values $x_1 = 1.5\text{ m}$, $U_1 = 27/4\text{ J}$, and $E = 0.2\text{ J}$.

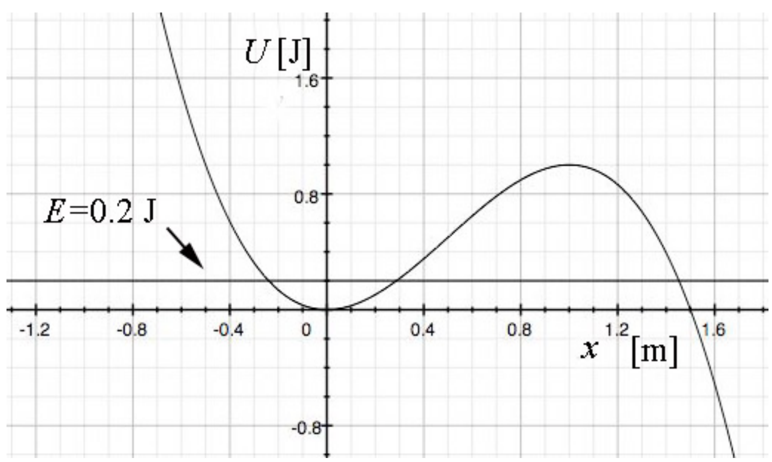

Figure 14.11 Energy diagram for Example 14.1

b) The force on the particle is zero at the minimum of the potential which occurs at

$$F_x(x) = -\frac{dU}{dx}(x) = U_1\left(\left(\frac{3}{x_1^{\,3}}\right)x^2 - \left(\frac{2}{x_1^{\,2}}\right)x\right) = 0 \qquad (14.5.11)$$

which becomes

$$x^2 = (2x_1 / 3)x\,. \qquad (14.5.12)$$

We can solve Eq. (14.5.12) for the extrema. This has two solutions

$$x = (2x_1 / 3) \quad \text{and} \quad x = 0\,. \qquad (14.5.13)$$

The second derivative is given by

$$\frac{d^2U}{dx^2}(x) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)x - \left(\frac{2}{x_1^{\,2}}\right)\right). \qquad (14.5.14)$$

Evaluating the second derivative at $x = (2x_1 / 3)$ yields a negative quantity

$$\frac{d^2U}{dx^2}(x = (2x_1 / 3)) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)\frac{2x_1}{3} - \left(\frac{2}{x_1^{\,2}}\right)\right) = -\frac{2U_1}{x_1^{\,2}} < 0\,, \qquad (14.5.15)$$

indicating the solution $x = (2x_1 / 3)$ represents a local maximum and hence is an unstable point. At $x = (2x_1 / 3)$, the potential energy is given by the value $U((2x_1 / 3)) = (4 / 27)U_1$. Evaluating the second derivative at $x = 0$ yields a positive quantity

$$\frac{d^2U}{dx^2}(x = 0) = -U_1\left(\left(\frac{6}{x_1^{\,3}}\right)0 - \left(\frac{2}{x_1^{\,2}}\right)\right) = \frac{2U_1}{x_1^{\,2}} > 0\,, \qquad (14.5.16)$$

indicating the solution $x = 0$ represents a local minimum and is a stable point. At the local minimum $x = 0$, the potential energy $U(0) = 0$.

c) Consider a fixed value of the energy of the particle within the range

$$U(0) = 0 < E < U(2x_1 / 3) = \frac{4U_1}{27}\,. \qquad (14.5.17)$$

If the particle at any time is found in the region $x_a < x < x_b < 2x_1 / 3$, where x_a and x_b are the turning points and are solutions to the equation

$$E = U(x) = -U_1\left(\left(\frac{x}{x_1}\right)^3 - \left(\frac{x}{x_1}\right)^2\right). \qquad (14.5.18)$$

then the particle will undergo periodic motion between the values $x_a < x < x_b$. Within this region $x_a < x < x_b$, the kinetic energy is always positive because $K(x) = E - U(x)$. There is another solution x_c to Eq. (14.5.18) somewhere in the region $x_c > 2x_1/3$. If the particle at any time is in the region $x > x_c$ then it at any later time it is restricted to the region $x_c < x < +\infty$.

For $E > U(2x_1/3) = (4/27)U_1$, Eq. (14.5.18) has only one solution x_d. For all values of $x > x_d$, the kinetic energy is positive, which means that the particle can "escape" to infinity but can never enter the region $x < x_d$.

For $E < U(0) = 0$, the kinetic energy is negative for the range $-\infty < x < x_e$ where x_e satisfies Eq. (14.5.18) and therefore this region of space is forbidden.

(d) If the particle has speed v_0 at $x = 0$ where the potential energy is zero, $U(0) = 0$, the energy of the particle is constant and equal to kinetic energy

$$E = K(0) = \frac{1}{2}mv_0^2. \qquad (14.5.19)$$

Therefore

$$(4/27)U_1 = \frac{1}{2}mv_0^2, \qquad (14.5.20)$$

which we can solve for the speed

$$v_0 = \sqrt{8U_1/27m}. \qquad (14.5.21)$$

14.7 Change of Mechanical Energy for Closed System with Internal Non-conservative Forces

Consider a closed system (energy of the system is constant) that undergoes a transformation from an initial state to a final state by a prescribed set of changes.

> *Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, the force is called a* ***non-conservative force****.*

Suppose the internal forces are both conservative and non-conservative. The work W done by the forces is a sum of the conservative work W_c, which is path-independent, and the non-conservative work W_{nc}, which is path-dependent,

$$W = W_c + W_{nc}\,. \tag{14.6.1}$$

The work done by the conservative forces is equal to the negative of the change in the potential energy

$$\Delta U = -W_c\,. \tag{14.6.2}$$

Substituting Equation (14.6.2) into Equation (14.6.1) yields

$$W = -\Delta U + W_{nc}\,. \tag{14.6.3}$$

The work done is equal to the change in the kinetic energy,

$$W = \Delta K\,. \tag{14.6.4}$$

Substituting Equation (14.6.4) into Equation (14.6.3) yields

$$\Delta K = -\Delta U + W_{nc}\,. \tag{14.6.5}$$

which we can rearrange as

$$W_{nc} = \Delta K + \Delta U\,. \tag{14.6.6}$$

We can now substitute Equation (14.6.4) into our expression for the change in the mechanical energy, Equation (14.4.17), with the result

$$W_{nc} = \Delta E_m\,. \tag{14.6.7}$$

The mechanical energy is no longer constant. The total change in energy of the system is zero,

$$\Delta E_{\text{system}} = \Delta E_m - W_{nc} = 0\,. \tag{14.6.8}$$

Energy is conserved but some mechanical energy has been transferred into non-recoverable energy W_{nc}. We shall refer to processes in which there is non-zero non-recoverable energy as ***irreversible processes***.

14.7.1 Change of Mechanical Energy for a Non-closed System

When the system is no longer closed but in contact with its surroundings, the change in energy of the system is equal to the negative of the change in energy of the surroundings (Eq. (14.1.1)),

$$\Delta E_{\text{system}} = -\Delta E_{\text{surroundings}} \tag{14.6.9}$$

If the system is not isolated, the change in energy of the system can be the result of external work done by the surroundings on the system (which can be positive or negative)

$$W_{\text{ext}} = \int_A^B \vec{\mathbf{F}}_{\text{ext}} \cdot d\vec{\mathbf{r}}\ . \tag{14.6.10}$$

This work will result in the system undergoing ***coherent motion***. Note that $W_{\text{ext}} > 0$ if work is done on the system ($\Delta E_{\text{surroundings}} < 0$) and $W_{\text{ext}} < 0$ if the system does work on the surroundings ($\Delta E_{\text{surroundings}} > 0$). If the system is in thermal contact with the surroundings, then energy can flow into or out of the system. This energy flow due to thermal contact is often denoted by Q with the convention that $Q > 0$ if the energy flows into the system ($\Delta E_{\text{surroundings}} < 0$) and $Q < 0$ if the energy flows out of the system ($\Delta E_{\text{surroundings}} > 0$). Then Eq. (14.6.9) can be rewritten as

$$W^{\text{ext}} + Q = \Delta E_{\text{sys}} \tag{14.6.11}$$

Equation (14.6.11) is also called ***the first law of thermodynamics*****.**

This will result in either an increase or decrease in random thermal motion of the molecules inside the system, There may also be other forms of energy that enter the system, for example *radiative energy*.

Several questions naturally arise from this set of definitions and physical concepts. Is it possible to identify all the conservative forces and calculate the associated changes in potential energies? How do we account for non-conservative forces such as friction that act at the boundary of the system?

14.8 Dissipative Forces: Friction

Suppose we consider an object moving on a rough surface. As the object slides it slows down and stops. While the sliding occurs both the object and the surface increase in temperature. The increase in temperature is due to the molecules inside the materials increasing their kinetic energy. This random kinetic energy is called *thermal energy*. Kinetic energy associated with the coherent motion of the molecules of the object has been dissipated into kinetic energy associated with random motion of the molecules composing the object and surface.

If we define the system to be just the object, then the friction force acts as an external force on the system and results in the dissipation of energy into both the block and the surface. Without knowing further properties of the material we cannot determine the exact changes in the energy of the system.

Friction introduces a problem in that the point of contact is not well defined because the surface of contact is constantly deforming as the object moves along the surface. If we considered the object and the surface as the system, then the friction force is an internal force, and the decrease in the kinetic energy of the moving object ends up as an increase in the internal random kinetic energy of the constituent parts of the system. When there is dissipation at the boundary of the system, we need an additional model (thermal equation of state) for how the dissipated energy distributes itself among the constituent parts of the system.

14.8.1 Source Energy

Consider a person walking. The frictional force between the person and the ground does no work because the point of contact between the person's foot and the ground undergoes no displacement as the person applies a force against the ground, (there may be some slippage but that would be opposite the direction of motion of the person). However the kinetic energy of the object increases. Have we disproved the work-energy theorem? The answer is no! The chemical energy stored in the body tissue is converted to kinetic energy and thermal energy. Because the person-air-ground can be treated as a closed system, we have that

$$0 = \Delta E_{\text{sys}} = \Delta E_{\text{chemical}} + \Delta E_{\text{thermal}} + \Delta E_{\text{mechanical}}, \quad (\text{closed system})\,. \qquad (14.7.1)$$

If we assume that there is no change in the potential energy of the system, then $\Delta E_{\text{mechanical}} = \Delta K$. Therefore some of the internal chemical energy has been transformed into thermal energy and the rest has changed into the kinetic energy of the system,

$$-\Delta E_{\text{chemical}} = \Delta E_{\text{thermal}} + \Delta K\,. \qquad (14.7.2)$$

14.9 Worked Examples

Example 14.2 Escape Velocity of Toro

The asteroid Toro, discovered in 1964, has a radius of about $R = 5.0\,\text{km}$ and a mass of about $m_t = 2.0\times 10^{15}\,\text{kg}$. Let's assume that Toro is a perfectly uniform sphere. What is the escape velocity for an object of mass m on the surface of Toro? Could a person reach this speed (on earth) by running?

Solution: The only potential energy in this problem is the gravitational potential energy. We choose the zero point for the potential energy to be when the object and Toro are an infinite distance apart, $U^G(\infty) \equiv 0$. With this choice, the potential energy when the object and Toro are a finite distance r apart is given by

$$U^G(r) = -\frac{Gm_t\,m}{r} \qquad (14.8.1)$$

with $U^G(\infty) \equiv 0$. The expression ***escape velocity*** refers to the minimum speed necessary for an object to escape the gravitational interaction of the asteroid and move off to an infinite distance away. If the object has a speed less than the escape velocity, it will be unable to escape the gravitational force and must return to Toro. If the object has a speed greater than the escape velocity, it will have a non-zero kinetic energy at infinity. The condition for the escape velocity is that the object will have exactly zero kinetic energy at infinity.

We choose our initial state, at time t_i, when the object is at the surface of the asteroid with speed equal to the escape velocity. We choose our final state, at time t_f, to occur when the separation distance between the asteroid and the object is infinite.

The initial kinetic energy is $K_i = (1/2)m v_{\text{esc}}{}^2$. The initial potential energy is $U_i = -Gm_t m / R$, and so the initial mechanical energy is

$$E_i = K_i + U_i = \frac{1}{2} m v_{\text{esc}}^2 - \frac{Gm_t m}{R}. \tag{14.8.2}$$

The final kinetic energy is $K_f = 0$, because this is the condition that defines the escape velocity. The final potential energy is zero, $U_f = 0$ because we chose the zero point for potential energy at infinity. The final mechanical energy is then

$$E_f = K_f + U_f = 0. \tag{14.8.3}$$

There is no non-conservative work, so the change in mechanical energy is zero

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i. \tag{14.8.4}$$

Therefore

$$0 = -\left(\frac{1}{2} m v_{\text{esc}}^2 - \frac{Gm_t m}{R}\right). \tag{14.8.5}$$

This can be solved for the escape velocity,

$$\begin{aligned} v_{\text{esc}} &= \sqrt{\frac{2Gm_t}{R}} \\ &= \sqrt{\frac{2(6.67 \times 10^{-11} \text{N} \cdot \text{m}^2 \cdot \text{kg}^{-2})(2.0 \times 10^{15} \text{kg})}{(5.0 \times 10^3 \text{m})}} = 7.3 \text{ m} \cdot \text{s}^{-1}. \end{aligned} \tag{14.8.6}$$

Considering that Olympic sprinters typically reach velocities of $12\ \text{m}\cdot\text{s}^{-1}$, this is an easy speed to attain by running on earth. It may be harder on Toro to generate the acceleration necessary to reach this speed by pushing off the ground, since any slight upward force will raise the runner's center of mass and it will take substantially more time than on earth to come back down for another push off the ground.

Example 14.3 Spring-Block-Loop-the-Loop

A small block of mass m is pushed against a spring with spring constant k and held in place with a catch. The spring is compressed an unknown distance x (Figure 14.12). When the catch is removed, the block leaves the spring and slides along a frictionless circular loop of radius r. When the block reaches the top of the loop, the force of the loop on the block (the normal force) is equal to twice the gravitational force on the mass. (a) Using conservation of energy, find the kinetic energy of the block at the top of the loop. (b) Using Newton's Second Law, derive the equation of motion for the block when it is at the top of the loop. Specifically, find the speed v_{top} in terms of the gravitation constant g and the loop radius r. (c) What distance was the spring compressed?

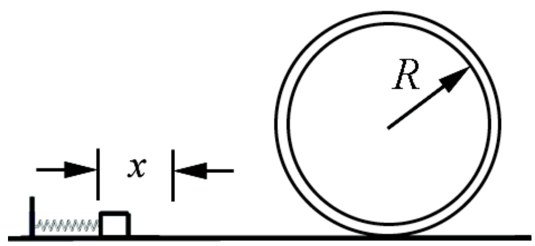

Figure 14.12 Initial state for spring-block-loop-the-loop system

Solution: a) Choose for the initial state the instant before the catch is released. The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = (1/2)k\,x^2$. The initial mechanical energy is then

$$E_i = K_i + U_i = \frac{1}{2}k\,x^2\,. \qquad (14.8.7)$$

Choose for the final state the instant the block is at the top of the loop. The final kinetic energy is $K_f = (1/2)m\,v_{\text{top}}^2$; the block is in motion with speed v_{top}. The final potential energy is non-zero, $U_f = (mg)(2R)$. The final mechanical energy is then

$$E_f = K_f + U_f = 2mgR + \frac{1}{2}m\,v_{\text{top}}^2\,. \qquad (14.8.8)$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i\,. \tag{14.8.9}$$

Mechanical energy is conserved, $E_f = E_i$, therefore

$$2mgR + \frac{1}{2}m v_{\text{top}}^2 = \frac{1}{2}k x^2\,. \tag{14.8.10}$$

From Equation (14.8.10), the kinetic energy at the top of the loop is

$$\frac{1}{2}m v_{\text{top}}^2 = \frac{1}{2}k x^2 - 2mgR\,. \tag{14.8.11}$$

b) At the top of the loop, the forces on the block are the gravitational force of magnitude mg and the normal force of magnitude N, both directed down. Newton's Second Law in the radial direction, which is the downward direction, is

$$-mg - N = -\frac{m v_{\text{top}}^2}{R}\,. \tag{14.8.12}$$

In this problem, we are given that when the block reaches the top of the loop, the force of the loop on the block (the normal force, *downward* in this case) is equal to twice the weight of the block, $N = 2mg$**.** The Second Law, Eq. (14.8.12), then becomes

$$3mg = \frac{m v_{\text{top}}^2}{R}\,. \tag{14.8.13}$$

We can rewrite Equation (14.8.13) in terms of the kinetic energy as

$$\frac{3}{2}mg\,R = \frac{1}{2}m v_{\text{top}}^2\,. \tag{14.8.14}$$

The speed at the top is therefore

$$v_{\text{top}} = \sqrt{3mg\,R}\,. \tag{14.8.15}$$

c) Combing Equations (14.8.11) and (14.8.14) yields

$$\frac{7}{2}mg\,R = \frac{1}{2}k x^2\,. \tag{14.8.16}$$

Thus the initial displacement of the spring from equilibrium is

$$x=\sqrt{\frac{7mgR}{k}}\,. \tag{14.8.17}$$

Example 14.4 Mass-Spring on a Rough Surface

A block of mass m slides along a horizontal table with speed v_0. At $x=0$ it hits a spring with spring constant k and begins to experience a friction force. The coefficient of friction is variable and is given by $\mu=bx$, where b is a positive constant. Find the loss in mechanical energy when the block first momentarily comes to rest.

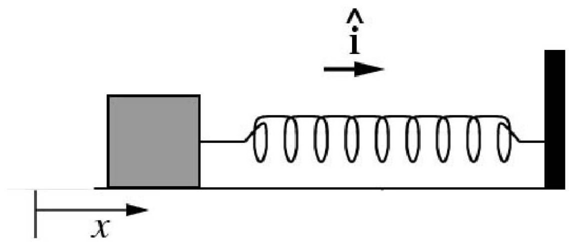

Figure 14.13 Spring-block system

Solution: From the model given for the frictional force, we could find the non-conservative work done, which is the same as the loss of mechanical energy, if we knew the position x_f where the block first comes to rest. The most direct (and easiest) way to find x_f is to use the work-energy theorem. The initial mechanical energy is $E_i=mv_i^2/2$ and the final mechanical energy is $E_f=k\,x_f^2/2$ (note that there is no potential energy term in E_i and no kinetic energy term in E_f). The difference between these two mechanical energies is the non-conservative work done by the frictional force,

$$\begin{aligned} W_{\text{nc}} &= \int_{x=0}^{x=x_f} F_{\text{nc}}\,dx = \int_{x=0}^{x=x_f} -F_{\text{friction}}\,dx = \int_{x=0}^{x=x_f} -\mu\,N\,dx \\ &= -\int_0^{x_f} b\,x\,mg\,dx = -\frac{1}{2}bmg\,x_f^2. \end{aligned} \tag{14.8.18}$$

We then have that

$$\begin{aligned} W_{\text{nc}} &= \Delta E_m \\ W_{\text{nc}} &= E_f - E_i \\ -\frac{1}{2}bmg\,x_f^2 &= \frac{1}{2}k\,x_f^2 - \frac{1}{2}mv_i^2. \end{aligned} \tag{14.8.19}$$

Solving the last of these equations for x_f^2 yields

$$x_f^2 = \frac{m v_0^2}{k + bmg}. \tag{14.8.20}$$

Substitute Eq. (14.8.20) into Eq. (14.8.18) gives the result that

$$W_{\mathrm{nc}} = -\frac{bmg}{2}\frac{mv_0^2}{k+bmg} = -\frac{mv_0^2}{2}\left(1+\frac{k}{bmg}\right)^{-1}. \tag{14.8.21}$$

It is worth checking that the above result is dimensionally correct. From the model, the parameter b must have dimensions of inverse length (the coefficient of friction μ must be dimensionless), and so the product bmg has dimensions of force per length, as does the spring constant k; the result is dimensionally consistent.

Example 14.5 Cart-Spring on an Inclined Plane

An object of mass m slides down a plane that is inclined at an angle θ from the horizontal (Figure 14.14). The object starts out at rest. The center of mass of the cart is a distance d from an unstretched spring that lies at the bottom of the plane. Assume the spring is massless, and has a spring constant k. Assume the inclined plane to be frictionless. (a) How far will the spring compress when the mass first comes to rest? (b) Now assume that the inclined plane has a coefficient of kinetic friction μ_{k}. How far will the spring compress when the mass first comes to rest? The friction is primarily between the wheels and the bearings, not between the cart and the plane, but the friction force may be modeled by a coefficient of friction μ_{k}. (c) In case (b), how much energy has been lost to friction?

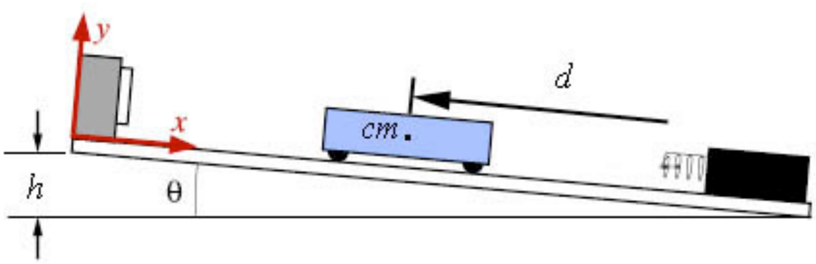

Figure 14.14 Cart on inclined plane

Solution: Let x denote the displacement of the spring from the equilibrium position. Choose the zero point for the gravitational potential energy $U^g(0) = 0$ not at the very bottom of the inclined plane, but at the location of the end of the unstretched spring. Choose the zero point for the spring potential energy where the spring is at its equilibrium position, $U^s(0) = 0$.

a) Choose for the initial state the instant the object is released (Figure 14.15). The initial kinetic energy is $K_i = 0$. The initial potential energy is non-zero, $U_i = mg\,d\sin\theta$. The initial mechanical energy is then

$$E_i = K_i + U_i = mg\,d\sin\theta \qquad (14.8.22)$$

Choose for the final state the instant when the object first comes to rest and the spring is compressed a distance x at the bottom of the inclined plane (Figure 14.16). The final kinetic energy is $K_f = 0$ since the mass is not in motion. The final potential energy is non-zero, $U_f = k\,x^2/2 - x\,mg\sin\theta$. Notice that the gravitational potential energy is negative because the object has dropped below the height of the zero point of gravitational potential energy.

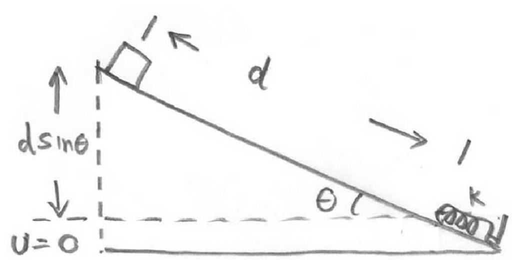

Figure 14.15 Initial state

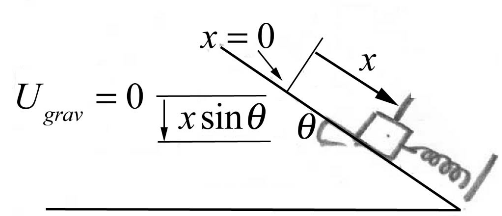

Figure 14.16 Final state

The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}k\,x^2 - x\,mg\sin\theta\,. \qquad (14.8.23)$$

Because we are assuming the track is frictionless and neglecting air resistance, there is no non- conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i\,. \qquad (14.8.24)$$

Therefore

$$d\,mg\sin\theta = \frac{1}{2}k\,x^2 - x\,mg\sin\theta\,. \qquad (14.8.25)$$

This is a quadratic equation in x,

$$x^2 - \frac{2mg\sin\theta}{k}x - \frac{2d\,mg\sin\theta}{k} = 0\,. \qquad (14.8.26)$$

In the quadratic formula, we want the positive choice of square root for the solution to ensure a positive displacement of the spring from equilibrium,

$$x = \frac{mg\sin\theta}{k} + \left(\frac{m^2 g^2 \sin^2\theta}{k^2} + \frac{2d\,mg\sin\theta}{k}\right)^{1/2}$$
$$= \frac{mg}{k}(\sin\theta + \sqrt{1 + 2(k\,d / mg)\sin\theta}). \tag{14.8.27}$$

(What would the solution with the negative root represent?)

b) The effect of kinetic friction is that there is now a non-zero non-conservative work done on the object, which has moved a distance, $d + x$, given by

$$W_{\text{nc}} = -f_{\text{k}}(d + x) = -\mu_{\text{k}} N(d + x) = -\mu_{\text{k}} mg\cos\theta(d + x). \tag{14.8.28}$$

Note the normal force is found by using Newton's Second Law in the perpendicular direction to the inclined plane,

$$N - mg\cos\theta = 0. \tag{14.8.29}$$

The change in mechanical energy is therefore

$$W_{\text{nc}} = \Delta E_m = E_f - E_i, \tag{14.8.30}$$

which becomes

$$-\mu_{\text{k}} mg\cos\theta(d + x) = \left(\frac{1}{2}k\,x^2 - x\,mg\sin\theta\right) - d\,mg\sin\theta. \tag{14.8.31}$$

Equation (14.8.31) simplifies to

$$0 = \left(\frac{1}{2}k\,x^2 - x\,mg(\sin\theta - \mu_{\text{k}}\cos\theta)\right) - d\,mg(\sin\theta - \mu_{\text{k}}\cos\theta). \tag{14.8.32}$$

This is the same as Equation (14.8.25) above, but with $\sin\theta \to \sin\theta - \mu_k\cos\theta$. The maximum displacement of the spring is when there is friction is then

$$x = \frac{mg}{k}((\sin\theta - \mu_{\text{k}}\cos\theta) + \sqrt{1 + 2(k\,d / mg)(\sin\theta - \mu_{\text{k}}\cos\theta)}). \tag{14.8.33}$$

.

c) The energy lost to friction is given by $W_{\text{nc}} = -\mu_{\text{k}} mg\cos\theta(d + x)$, where x is given in part b).

Example 14.6 Object Sliding on a Sphere

A small point like object of mass m rests on top of a sphere of radius R. The object is released from the top of the sphere with a negligible speed and it slowly starts to slide (Figure 14.17). Let g denote the gravitation constant. (a) Determine the angle θ_1 with respect to the vertical at which the object will lose contact with the surface of the sphere. (b) What is the speed v_1 of the object at the instant it loses contact with the surface of the sphere.

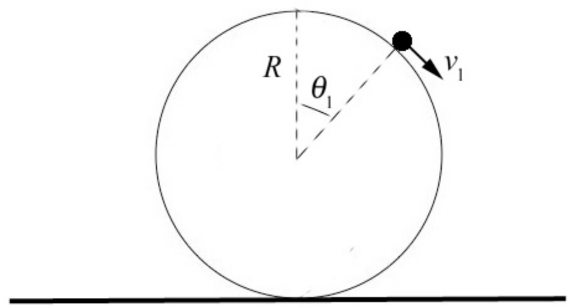

Figure 14.17 Object sliding on surface of sphere

Solution: We begin by identifying the forces acting on the object. There are two forces acting on the object, the gravitation and radial normal force that the sphere exerts on the particle that we denote by N. We draw a free-body force diagram for the object while it is sliding on the sphere. We choose polar coordinates as shown in Figure 14.18.

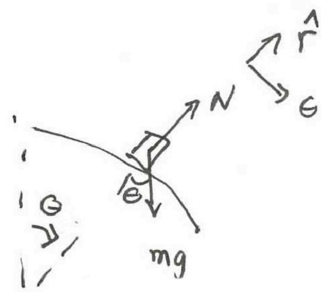

Figure 14.18 Free-body force diagram on object

The key constraint is that when the particle just leaves the surface the normal force is zero,

$$N(\theta_1) = 0, \tag{14.8.34}$$

where θ_1 denotes the angle with respect to the vertical at which the object will just lose contact with the surface of the sphere. Because the normal force is perpendicular to the displacement of the object, it does no work on the object and hence conservation of energy does not take into account the constraint on the motion imposed by the normal force. In order to analyze the effect of the normal force we must use the radial component of Newton's Second Law,

$$N - mg\cos\theta = -m\frac{v^2}{R}. \tag{14.8.35}$$

Then when the object just loses contact with the surface, Eqs. (14.8.34) and (14.8.35) require that

$$mg\cos\theta_1 = m\frac{v_1^{\,2}}{R}. \qquad (14.8.36)$$

where v_1 denotes the speed of the object at the instant it loses contact with the surface of the sphere. Note that the constrain condition Eq. (14.8.36) can be rewritten as

$$mgR\cos\theta_1 = mv_1^{\,2}. \qquad (14.8.37)$$

We can now apply conservation of energy. Choose the zero reference point $U = 0$ for potential energy to be the midpoint of the sphere.

Identify the initial state as the instant the object is released (Figure 14.19). We can neglect the very small initial kinetic energy needed to move the object away from the top of the sphere and so $K_i = 0$. The initial potential energy is non-zero, $U_i = mgR$. The initial mechanical energy is then

$$E_i = K_i + U_i = mgR. \qquad (14.8.38)$$

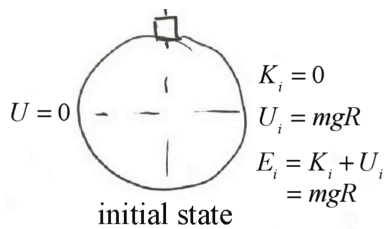

Figure 14.19 Initial state

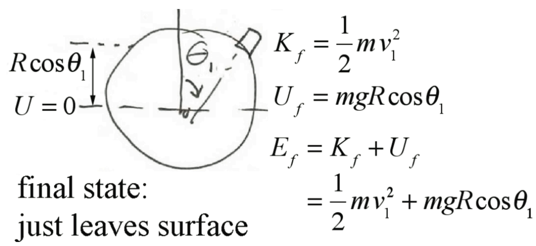

Figure 14.20 Final state

Choose for the final state the instant the object leaves the sphere (Figure 14.20). The final kinetic energy is $K_f = mv_1^2/2$; the object is in motion with speed v_1. The final potential energy is non-zero, $U_f = mgR\cos\theta_1$. The final mechanical energy is then

$$E_f = K_f + U_f = \frac{1}{2}mv_1^2 + mgR\cos\theta_1. \qquad (14.8.39)$$

Because we are assuming the contact surface is frictionless and neglecting air resistance, there is no non-conservative work. The change in mechanical energy is therefore zero,

$$0 = W_{\text{nc}} = \Delta E_m = E_f - E_i. \qquad (14.8.40)$$

Therefore

$$\frac{1}{2}mv_1^2 + mgR\cos\theta_1 = mgR. \qquad (14.8.41)$$

We now solve the constraint condition Eq. (14.8.37) into Eq. (14.8.41) yielding

$$\frac{1}{2} mgR\cos\theta_1 + mgR\cos\theta_1 = mgR\,. \tag{14.8.42}$$

We can now solve for the angle at which the object just leaves the surface

$$\theta_1 = \cos^{-1}(2/3)\,. \tag{14.8.43}$$

We now substitute this result into Eq. (14.8.37) and solve for the speed

$$v_1 = \sqrt{2gR/3}\,. \tag{14.8.44}$$