

Chapter 3 Vectors

3.1 Vector Analysis	1
3.1.1 Introduction to Vectors	1
3.1.2 Properties of Vectors	1
3.2 Cartesian Coordinate System	5
3.2.1 Cartesian Coordinates	6
3.3 Application of Vectors	8
Example 3.1 Vector Addition.....	12
Example 3.2 Sinking Sailboat	13
Example 3.3 Vector Description of a Point on a Line.....	14
Example 3.4 Rotated Coordinate Systems.....	15
Example 3.5 Vector Description in Rotated Coordinate Systems	15
Example 3.6 Vector Addition.....	17

Chapter 3 Vectors

Philosophy is written in this grand book, the universe which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a single word of it; without these, one wanders about in a dark labyrinth.¹

Galileo Galilei

3.1 Vector Analysis

3.1.1 Introduction to Vectors

Certain physical quantities such as mass or the absolute temperature at some point only have magnitude. Numbers alone can represent these quantities, with the appropriate units, and they are called scalars. There are, however, other physical quantities that have both magnitude and direction: the magnitude can stretch or shrink, and the direction can reverse. These quantities can be added in such a way that takes into account both direction and magnitude. Force is an example of a quantity that acts in a certain direction with some magnitude that we measure in newtons. When two forces act on an object, the sum of the forces depends on both the direction and magnitude of the two forces. Position, displacement, velocity, acceleration, force, momentum and torque are all physical quantities that can be represented mathematically by vectors. We begin by defining precisely what we mean by a vector.

3.1.2 Properties of Vectors

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol \vec{A} . The magnitude of \vec{A} is $|\vec{A}| \equiv A$. We can represent vectors as geometric objects using arrows. The length of the arrow corresponds to the magnitude of the vector. The arrow points in the direction of the vector (Figure 3.1).

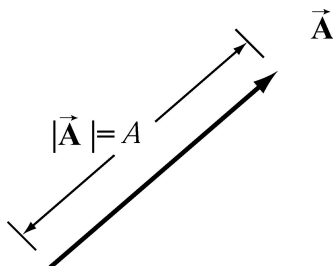


Figure 3.1 Vectors as arrows.

¹ Galileo Galilei, *The Assayer*, tr. Stillman Drake (1957), *Discoveries and Opinions of Galileo* pp. 237-8.

There are two defining operations for vectors:

(1) Vector Addition:

Vectors can be added. Let \vec{A} and \vec{B} be two vectors. We define a new vector, $\vec{C} = \vec{A} + \vec{B}$, the “vector addition” of \vec{A} and \vec{B} , by a geometric construction. Draw the arrow that represents \vec{A} . Place the tail of the arrow that represents \vec{B} at the tip of the arrow for \vec{A} as shown in Figure 3.2a. The arrow that starts at the tail of \vec{A} and goes to the tip of \vec{B} is defined to be the “vector addition” $\vec{C} = \vec{A} + \vec{B}$. There is an equivalent construction for the law of vector addition. The vectors \vec{A} and \vec{B} can be drawn with their tails at the same point. The two vectors form the sides of a parallelogram. The diagonal of the parallelogram corresponds to the vector $\vec{C} = \vec{A} + \vec{B}$, as shown in Figure 3.2b.



Figure 3.2 (a) Geometric sum of vectors. **Figure 3.2 (b)** Geometric sum of vectors.

Vector addition satisfies the following four properties:

(i) Commutativity:

The order of adding vectors does not matter;

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} . \tag{3.1.1}$$

Our geometric definition for vector addition satisfies the commutative property (i) since in the parallelogram representation for the addition of vectors, it doesn't matter which side you start with, as seen in Figure 3.3.

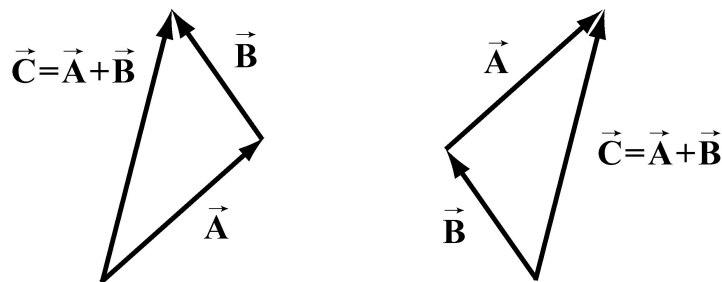


Figure 3.3 Commutative property of vector addition.

(ii) Associativity:

When adding three vectors, it doesn't matter which two you start with

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}). \quad (3.1.2)$$

In Figure 3.4, we add $(\vec{A} + \vec{B}) + \vec{C}$, and $\vec{A} + (\vec{B} + \vec{C})$ to arrive at the same vector sum in either case.

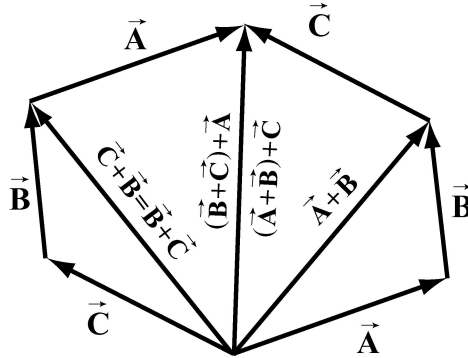


Figure 3.4 Associative law.

(iii) Identity Element for Vector Addition:

There is a unique vector, $\vec{0}$, that acts as an identity element for vector addition. For all vectors \vec{A} ,

$$\vec{A} + \vec{0} = \vec{0} + \vec{A} = \vec{A}. \quad (3.1.3)$$

(iv) Inverse Element for Vector Addition:

For every vector \vec{A} , there is a unique inverse vector

$$(-1)\vec{A} \equiv -\vec{A}, \quad (3.1.4)$$

such that

$$\vec{A} + (-\vec{A}) = \vec{0}.$$

The vector $-\vec{A}$ has the same magnitude as \vec{A} , $|\vec{A}| = |-\vec{A}| = A$, but they point in opposite directions (Figure 3.5).

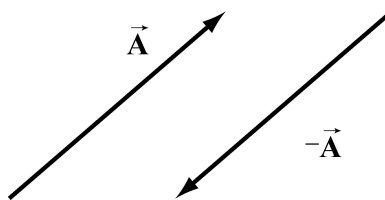


Figure 3.5 Additive inverse

(2) Scalar Multiplication of Vectors:

Vectors can be multiplied by real numbers. Let \vec{A} be a vector. Let c be a real positive number. Then the multiplication of \vec{A} by c is a new vector, which we denote by the symbol $c\vec{A}$. The magnitude of $c\vec{A}$ is c times the magnitude of \vec{A} (Figure 3.6a),

$$|c\vec{A}| = c|\vec{A}|. \quad (3.1.5)$$

Let $c > 0$, then the direction of $c\vec{A}$ is the same as the direction of \vec{A} . However, the direction of $-c\vec{A}$ is opposite of \vec{A} (Figure 3.6b).

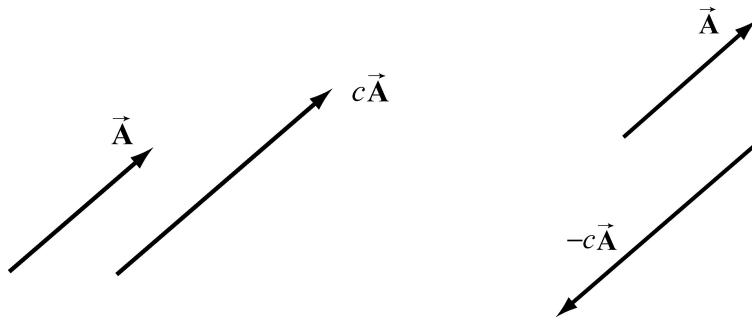


Figure 3.6 Multiplication of vector \vec{A} by (a) $c > 0$, and (b) $-c < 0$.

Scalar multiplication of vectors satisfies the following properties:

(i) Associative Law for Scalar Multiplication:

The order of multiplying numbers is doesn't matter. Let b and c be real numbers. Then

$$b(c\vec{A}) = (bc)\vec{A} = (cb\vec{A}) = c(b\vec{A}). \quad (3.1.6)$$

(ii) Distributive Law for Vector Addition:

Vector addition satisfies a distributive law for multiplication by a number. Let c be a real number. Then

$$c(\vec{A} + \vec{B}) = c\vec{A} + c\vec{B}. \quad (3.1.7)$$

Figure 3.7 illustrates this property.

(iii) Distributive Law for Scalar Addition:

The multiplication operation also satisfies a distributive law for the addition of numbers. Let b and c be real numbers. Then

$$(b+c)\vec{A} = b\vec{A} + c\vec{A} \quad (3.1.8)$$

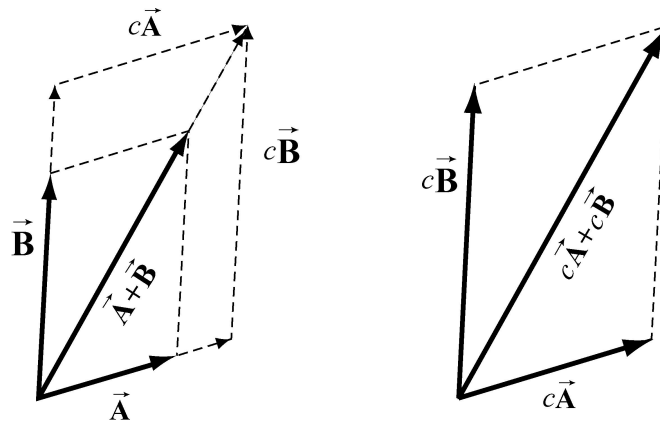


Figure 3.7 Distributive Law for vector addition.

Our geometric definition of vector addition satisfies this condition as seen in Figure 3.8.

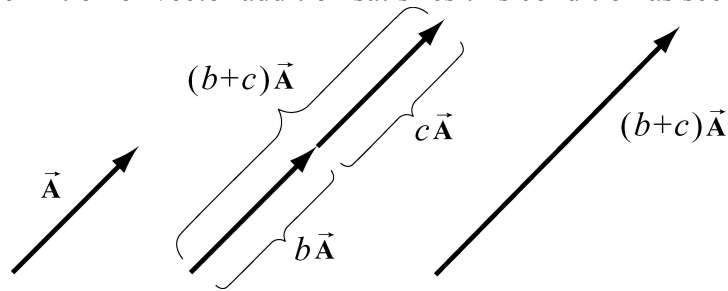


Figure 3.8 Distributive law for scalar multiplication.

(iv) Identity Element for Scalar Multiplication:

The number 1 acts as an identity element for multiplication,

$$1 \vec{A} = \vec{A}. \quad (3.1.9)$$

3.2 Cartesian Coordinate System

Physics involve the study of phenomena that we observe in the world. In order to connect the phenomena to mathematics we begin by introducing the concept of a coordinate system. A coordinate system consists of four basic elements:

- (1) Choice of origin
- (2) Choice of axes
- (3) Choice of positive direction for each axis
- (4) Choice of unit vectors for each axis

There are three commonly used coordinate systems: Cartesian, cylindrical and spherical. What makes these systems extremely useful is the associated set of infinitesimal line, area, and volume elements that are key to making many integration calculations in classical mechanics, such as finding the center of mass and moment of inertia.

3.2.1 Cartesian Coordinates

Cartesian coordinates consist of a set of mutually perpendicular axes, which intersect at a common point, the origin O . We live in a three-dimensional spatial world; for that reason, the most common system we will use has three axes, for which we choose the directions of the axes and position of the origin are.

(1) Choice of Origin

Choose an origin O . If you are given an object, then your choice of origin may coincide with a special point in the body. For example, you may choose the mid-point of a straight piece of wire.

(2) Choice of Axis

Now we shall choose a set of axes. The simplest set of axes is known as the Cartesian axes, x -axis, y -axis, and the z -axis. Once again, we adapt our choices to the physical object. For example, we select the x -axis so that the wire lies on the x -axis, as shown in Figure 3.9

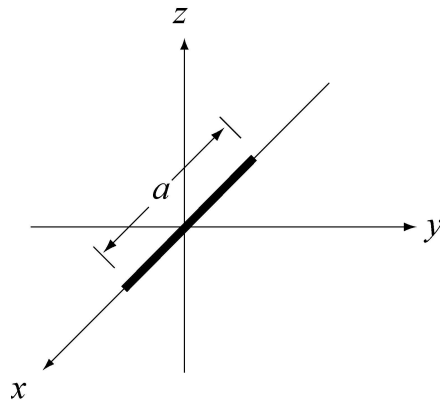


Figure 3.9 A segment of wire of length a lying along the x -axis of a Cartesian coordinate system.

Then each point P in our space S can be assigned a triplet of values (x_p, y_p, z_p) , the Cartesian coordinates of the point P . The ranges of these values are: $-\infty < x_p < +\infty$, $-\infty < y_p < +\infty$, $-\infty < z_p < +\infty$.

The collection of points that have the same the coordinate y_p is called a level surface. For example, the set of points with the same value of $y = y_p$ is

$$S_{y_p} = \{(x, y, z) \in S \text{ such that } y = y_p\}. \quad (3.2.1)$$

This set S_{y_p} is a plane, the xz -plane (Figure 3.10), called a **level set** for constant y_p . Thus, the y -coordinate of any point actually describes a plane of points perpendicular to the y -axis.

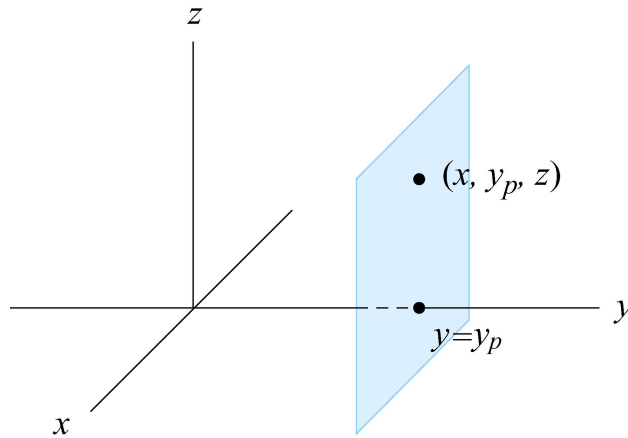


Figure 3.10 Level surface set for constant value y_p .

(3) Choice of Positive Direction

Our third choice is an assignment of positive direction for each coordinate axis. We shall denote this choice by the symbol + along the positive axis. Conventionally, Cartesian coordinates are drawn with the yz -plane corresponding to the plane of the paper. The horizontal direction from left to right is taken as the positive y -axis, and the vertical direction from bottom to top is taken as the positive z -axis. In physics problems we are free to choose our axes and positive directions any way that we decide best fits a given problem. Problems that are very difficult using the conventional choices may turn out to be much easier to solve by making a thoughtful choice of axes. The endpoints of the wire now have coordinates $(a/2, 0, 0)$ and $(-a/2, 0, 0)$.

(4) Choice of Unit Vectors

We now associate to each point P in space, a set of three unit directions vectors $(\hat{\mathbf{i}}_p, \hat{\mathbf{j}}_p, \hat{\mathbf{k}}_p)$. A unit vector has magnitude one: $|\hat{\mathbf{i}}_p| = 1$, $|\hat{\mathbf{j}}_p| = 1$, and $|\hat{\mathbf{k}}_p| = 1$. We assign the direction of $\hat{\mathbf{i}}_p$ to point in the direction of the increasing x -coordinate at the point P . We define the directions for $\hat{\mathbf{j}}_p$ and $\hat{\mathbf{k}}_p$ in the direction of the increasing y -coordinate and z -coordinate respectively, (Figure 3.11). If we choose a different point S , the units

vectors $(\hat{\mathbf{i}}_S, \hat{\mathbf{j}}_S, \hat{\mathbf{k}}_S)$ at S , are equal to the unit vectors $(\hat{\mathbf{i}}_P, \hat{\mathbf{j}}_P, \hat{\mathbf{k}}_P)$ at P . This fact only holds true for a Cartesian coordinate system and does not hold for cylindrical coordinates, as we shall soon see. We therefore can drop the reference to the point and use $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to represent the unit vectors in a Cartesian coordinate system.

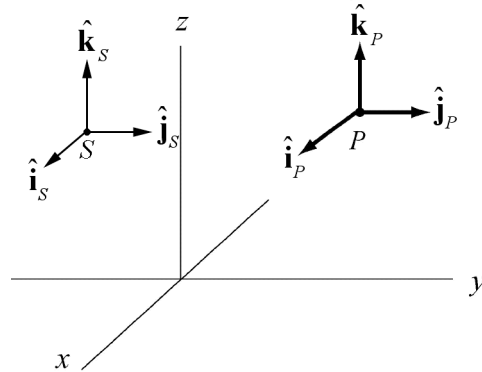


Figure 3.11 Choice of unit vectors at points P and S .

3.3 Application of Vectors

When we apply vectors to physical quantities it's nice to keep in the back of our minds all these formal properties. However from the physicist's point of view, we are interested in representing physical quantities such as displacement, velocity, acceleration, force, impulse, momentum, torque, and angular momentum as vectors. We can't add force to velocity or subtract momentum from torque. We must always understand the physical context for the vector quantity. Thus, instead of approaching vectors as formal mathematical objects we shall instead consider the following essential properties that enable us to represent physical quantities as vectors.

- (1) Vectors can exist at any point P in space.
- (2) Vectors have direction and magnitude.
- (3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.
- (4) Vector Decomposition: Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis. In Figure 3.12 we choose Cartesian coordinates for the x - y plane (we ignore the z -direction for simplicity but we can extend our results when we need to). A vector $\vec{\mathbf{A}}$ at P can be decomposed into the vector sum,

$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y, \quad (3.3.1)$$

where $\vec{\mathbf{A}}_x$ is the x -component vector pointing in the positive or negative x -direction, and $\vec{\mathbf{A}}_y$ is the y -component vector pointing in the positive or negative y -direction (Figure 3.12).

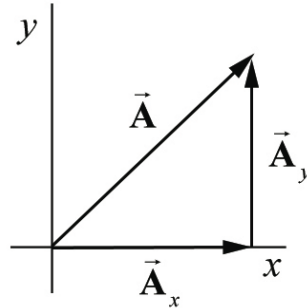


Figure 3.12 Vector decomposition.

(5) Vector Components: Once we have defined unit vectors ($\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$), we then define the x -component and y -component of a vector. Recall our vector decomposition, $\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y$. We define the x -component vector, $\vec{\mathbf{A}}_x$, as

$$\vec{\mathbf{A}}_x = A_x \hat{\mathbf{i}}. \quad (3.3.2)$$

In this expression the term A_x , (without the arrow above) is called the x -component of the vector $\vec{\mathbf{A}}$. The x -component A_x can be positive, zero, or negative. It is not the magnitude of $\vec{\mathbf{A}}_x$ which is given by $(A_x^2)^{1/2}$. Note the difference between the x -component, A_x , and the x -component vector, $\vec{\mathbf{A}}_x$.

In a similar fashion we define the y -component, A_y , and the z -component, A_z , of the vector $\vec{\mathbf{A}}$

$$\vec{\mathbf{A}}_y = A_y \hat{\mathbf{j}}, \quad \vec{\mathbf{A}}_z = A_z \hat{\mathbf{k}}. \quad (3.3.3)$$

A vector $\vec{\mathbf{A}}$ can be represented by its three components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$. We can also write the vector as

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}. \quad (3.3.4)$$

In Figure 3.13, we show the vector components $\vec{\mathbf{A}} = (A_x, A_y, A_z)$.

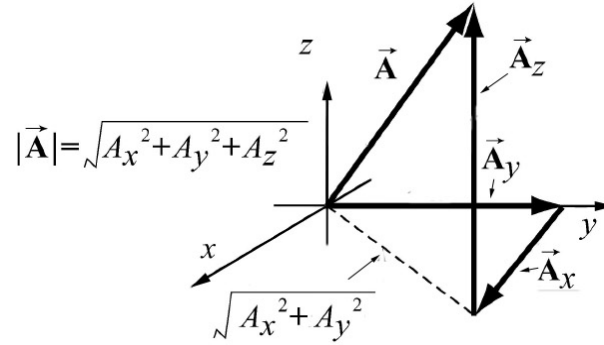


Figure 3.13 Component vectors in Cartesian coordinates.

(7) Magnitude: Using the Pythagorean theorem, the magnitude of $\vec{\mathbf{A}}$ is,

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}. \quad (3.3.5)$$

(8) Direction: Let's consider a vector $\vec{\mathbf{A}} = (A_x, A_y, 0)$. Since the z -component is zero, the vector $\vec{\mathbf{A}}$ lies in the x - y plane. Let θ denote the angle that the vector $\vec{\mathbf{A}}$ makes in the counterclockwise direction with the positive x -axis (Figure 3.14). Then the x -component and y -component are

$$A_x = A \cos(\theta), \quad A_y = A \sin(\theta). \quad (3.3.6)$$

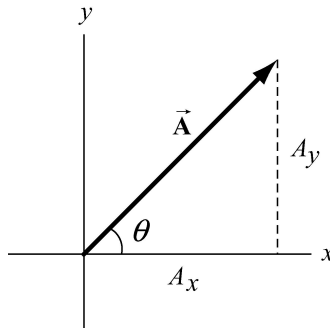


Figure 3.14 Components of a vector in the xy -plane.

We now write a vector in the xy -plane as

$$\vec{\mathbf{A}} = A \cos(\theta) \hat{\mathbf{i}} + A \sin(\theta) \hat{\mathbf{j}} \quad (3.3.7)$$

Once the components of a vector are known, the tangent of the angle θ can be determined by

$$\frac{A_y}{A_x} = \frac{A \sin(\theta)}{A \cos(\theta)} = \tan(\theta), \quad (3.3.8)$$

and hence the angle θ is given by

$$\theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \quad (3.3.9)$$

Clearly, the direction of the vector depends on the sign of A_x and A_y . For example, if both $A_x > 0$ and $A_y > 0$, then $0 < \theta < \pi/2$, and the vector lies in the first quadrant. If, however, $A_x > 0$ and $A_y < 0$, then $-\pi/2 < \theta < 0$, and the vector lies in the fourth quadrant.

(9) Unit vector in the direction of $\vec{\mathbf{A}}$: Let $\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}$. Let $\hat{\mathbf{A}}$ denote a unit vector in the direction of $\vec{\mathbf{A}}$. Then

$$\hat{\mathbf{A}} = \frac{\vec{\mathbf{A}}}{|\vec{\mathbf{A}}|} = \frac{A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}}{(A_x^2 + A_y^2 + A_z^2)^{1/2}}. \quad (3.3.10)$$

(10) Vector Addition: Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors in the x - y plane. Let θ_A and θ_B denote the angles that the vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ make (in the counterclockwise direction) with the positive x -axis. Then

$$\vec{\mathbf{A}} = A \cos(\theta_A) \hat{\mathbf{i}} + A \sin(\theta_A) \hat{\mathbf{j}}, \quad (3.3.11)$$

$$\vec{\mathbf{B}} = B \cos(\theta_B) \hat{\mathbf{i}} + B \sin(\theta_B) \hat{\mathbf{j}} \quad (3.3.12)$$

In Figure 3.15, the vector addition $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ is shown. Let θ_C denote the angle that the vector $\vec{\mathbf{C}}$ makes with the positive x -axis.

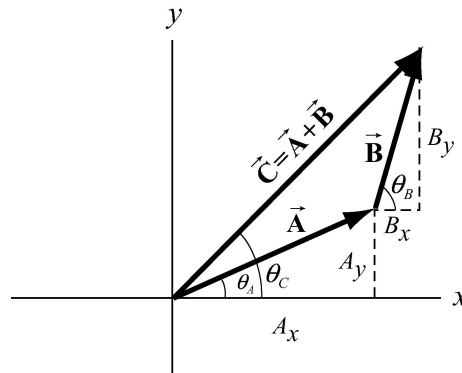


Figure 3.15 Vector addition using components.

Then the components of \vec{C} are

$$C_x = A_x + B_x, \quad C_y = A_y + B_y. \quad (3.3.13)$$

In terms of magnitudes and angles, we have

$$\begin{aligned} C_x &= C \cos(\theta_C) = A \cos(\theta_A) + B \cos(\theta_B) \\ C_y &= C \sin(\theta_C) = A \sin(\theta_A) + B \sin(\theta_B). \end{aligned} \quad (3.3.14)$$

We can write the vector \vec{C} as

$$\vec{C} = (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} = C \cos(\theta_C) \hat{i} + C \sin(\theta_C) \hat{j}, \quad (3.3.15)$$

Example 3.1 Vector Addition

Given two vectors, $\vec{A} = 2\hat{i} - 3\hat{j} + 7\hat{k}$ and $\vec{B} = 5\hat{i} + \hat{j} + 2\hat{k}$, find: (a) $|\vec{A}|$; (b) $|\vec{B}|$; (c) $\vec{A} + \vec{B}$; (d) $\vec{A} - \vec{B}$; (e) a unit vector \hat{A} pointing in the direction of \vec{A} ; (f) a unit vector \hat{B} pointing in the direction of \vec{B} .

Solution:

$$(a) |\vec{A}| = (2^2 + (-3)^2 + 7^2)^{1/2} = \sqrt{62} = 7.87. \quad (b) |\vec{B}| = (5^2 + 1^2 + 2^2)^{1/2} = \sqrt{30} = 5.48.$$

$$\begin{aligned} \vec{A} + \vec{B} &= (A_x + B_x) \hat{i} + (A_y + B_y) \hat{j} + (A_z + B_z) \hat{k} \\ (c) \quad &= (2+5) \hat{i} + (-3+1) \hat{j} + (7+2) \hat{k} \\ &= 7 \hat{i} - 2 \hat{j} + 9 \hat{k}. \end{aligned}$$

$$\begin{aligned} \vec{A} - \vec{B} &= (A_x - B_x) \hat{i} + (A_y - B_y) \hat{j} + (A_z - B_z) \hat{k} \\ (d) \quad &= (2-5) \hat{i} + (-3-1) \hat{j} + (7-2) \hat{k} \\ &= -3 \hat{i} - 4 \hat{j} + 5 \hat{k}. \end{aligned}$$

(e) A unit vector \hat{A} in the direction of \vec{A} can be found by dividing the vector \vec{A} by the magnitude of \vec{A} . Therefore

$$\hat{A} = \vec{A} / |\vec{A}| = (2 \hat{i} - 3 \hat{j} + 7 \hat{k}) / \sqrt{62}.$$

(f) In a similar fashion, $\hat{B} = \vec{B} / |\vec{B}| = (5 \hat{i} + \hat{j} + 2 \hat{k}) / \sqrt{30}$.

Example 3.2 Sinking Sailboat

A Coast Guard ship is located 35 km away from a checkpoint in a direction 42° north of west. A distressed sailboat located in still water 20 km from the same checkpoint in a direction 36° south of east is about to sink. Draw a diagram indicating the position of both ships. In what direction and how far must the Coast Guard ship travel to reach the sailboat?

Solution: The diagram of the set-up is Figure 3.16.

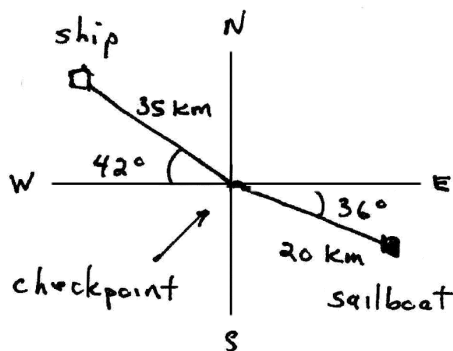


Figure 3.16 Example 3.2

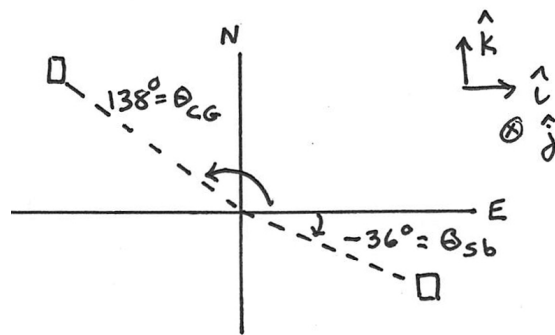


Figure 3.17 Coordinate system for sailboat and ship

Choose the checkpoint as the origin, with North as the positive $\hat{\mathbf{k}}$ -direction and East as the positive $\hat{\mathbf{i}}$ -direction (see Figure 3.17). The Coast Guard ship is then at an angle $\theta_{CG} = 180^\circ - 42^\circ = 138^\circ$ from the checkpoint, and the sailboat is at an angle $\theta_{sb} = -36^\circ$ from the checkpoint. The position of the Coast Guard ship is then

$$\begin{aligned}\vec{\mathbf{r}}_{CG} &= r_{CG} (\cos\theta_{CG} \hat{\mathbf{i}} + \sin\theta_{CG} \hat{\mathbf{k}}) \\ &= -26.0 \text{ km } \hat{\mathbf{i}} + 23.4 \text{ km } \hat{\mathbf{k}},\end{aligned}$$

and the position of the sailboat is

$$\begin{aligned}\vec{\mathbf{r}}_{sb} &= r_{sb} (\cos\theta_{sb} \hat{\mathbf{i}} + \sin\theta_{sb} \hat{\mathbf{k}}) \\ &= 16.2 \text{ km } \hat{\mathbf{i}} - 11.8 \text{ km } \hat{\mathbf{k}}.\end{aligned}$$

Note that an extra significant figure has been kept for the intermediate calculations. The position vector from the Coast Guard ship to the sailboat is (Figure 3.18)

$$\begin{aligned}\vec{r}_{sb} - \vec{r}_{CG} &= (16.2\text{ km}\hat{i} - 11.8\text{ km}\hat{k}) - (-26.0\text{ km}\hat{i} + 23.4\text{ km}\hat{k}) \\ &= 42.2\text{ km}\hat{i} - 35.2\text{ km}\hat{k}.\end{aligned}$$

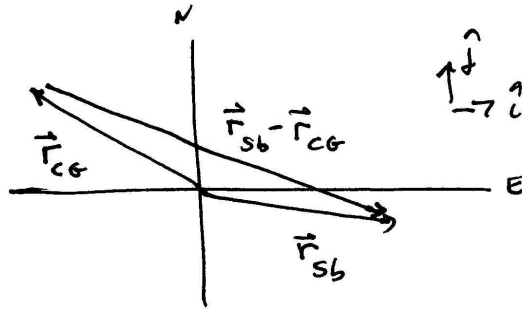


Figure 3.18 Relative position vector from ship to sailboat

The rescue ship's heading would be the inverse tangent of the ratio of the North and East components of the relative position,

$$\theta_{\text{rescue}} = \tan^{-1}(-35.2/42.2) = -39.8^\circ,$$

roughly 40° South of East.

Example 3.3 Vector Description of a Point on a Line

Consider two points located at \vec{r}_1 and \vec{r}_2 , separated by distance $r_{12} = |\vec{r}_1 - \vec{r}_2|$. Find a vector \vec{A} from the origin to the point on the line between \vec{r}_1 and \vec{r}_2 at a distance xr_{12} from the point at \vec{r}_1 , where x is some number.

Solution: Consider the unit vector pointing from \vec{r}_2 to \vec{r}_1 given by $\hat{r}_{12} = (\vec{r}_1 - \vec{r}_2) / |\vec{r}_1 - \vec{r}_2| = (\vec{r}_1 - \vec{r}_2) / r_{12}$. The vector $\vec{\alpha}$ in Figure 3.19 connects \vec{A} to the point at \vec{r}_1 , therefore we can write $\vec{\alpha} = xr_{12}\hat{r}_{12} = xr_{12}(\vec{r}_1 - \vec{r}_2) / r_{12} = x(\vec{r}_1 - \vec{r}_2)$. The vector $\vec{r}_1 = \vec{A} + \vec{\alpha}$. Therefore $\vec{A} = \vec{r}_1 - \vec{\alpha} = \vec{r}_1 - x(\vec{r}_1 - \vec{r}_2) = \vec{r}_1(1-x) + x\vec{r}_2$.

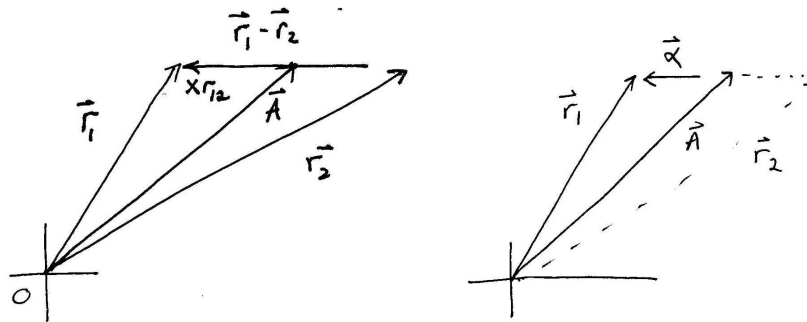


Figure 3.19 Vector geometry for Example 3.3

Example 3.4 Rotated Coordinate Systems

Consider two Cartesian coordinate systems S and S' with the same origin (Figure 3.20). Show that if the axes x', y' axes are rotated by an angle θ relative to the axes x, y , then the corresponding unit vectors are related according to $\hat{\mathbf{i}}' = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta$, and $\hat{\mathbf{j}}' = \hat{\mathbf{j}}\cos\theta - \hat{\mathbf{i}}\sin\theta$.

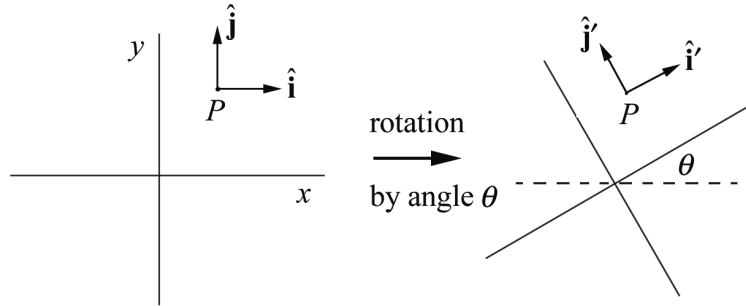


Figure 3.20 Example 3.4

Solution: This is simple vector decomposition. The components of $\hat{\mathbf{i}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by $i'_x = |\hat{\mathbf{i}}'| \cos\theta = \cos\theta$ and $i'_y = |\hat{\mathbf{i}}'| \sin\theta = \sin\theta$. Therefore

$$\hat{\mathbf{i}}' = i'_x \hat{\mathbf{i}} + i'_y \hat{\mathbf{j}} = \cos\theta \hat{\mathbf{i}} + \sin\theta \hat{\mathbf{j}}. \quad (3.3.16)$$

A similar argument holds for the components of $\hat{\mathbf{j}}'$. The components of $\hat{\mathbf{j}}'$ in the $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ direction are given by $j'_x = -|\hat{\mathbf{j}}'| \sin\theta = -\sin\theta$ and $j'_y = |\hat{\mathbf{j}}'| \cos\theta = \cos\theta$. Therefore

$$\hat{\mathbf{j}}' = j'_x \hat{\mathbf{i}} + j'_y \hat{\mathbf{j}} = -\sin\theta \hat{\mathbf{i}} + \cos\theta \hat{\mathbf{j}}. \quad (3.3.17)$$

Example 3.5 Vector Description in Rotated Coordinate Systems

With respect to a given Cartesian coordinate system S , a vector has components $A_x = 5$, $A_y = -3$, $A_z = 0$. (a) What are the components $A_{x'}$ and $A_{y'}$ of this vector in a second coordinate system S' whose x' and y' axes make angles of 60° with the x and y axes respectively of the first coordinate system S ? (b) Calculate the magnitude of the vector from its A_x and A_y components and from its $A_{x'}$ and $A_{y'}$ components. Does the result agree with what you expect?

Solution: We begin with a sketch of the rotated coordinate systems and the vector \vec{A} (Figure 3.21).

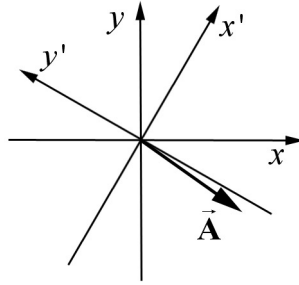


Figure 3.21 Example 3.5

We then take the vector decomposition of \vec{A} with respect to the xy -coordinate system,

$$\vec{A} = A_x \hat{i} + A_y \hat{j}. \quad (3.3.18)$$

Now we can use our results from Example 3.5 to solve for the unit vectors \hat{i} and \hat{j} in terms of \hat{i}' and \hat{j}' . Multiply Eq. (3.3.16) for \hat{i}' by $\sin\theta$ and Eq.(3.3.17) for \hat{j}' by $\cos\theta$ yielding

$$\begin{aligned} \sin\theta \hat{i}' &= \sin\theta \cos\theta \hat{i} + \sin^2\theta \hat{j} \\ \cos\theta \hat{j}' &= -\sin\theta \cos\theta \hat{i} + \cos^2\theta \hat{j}. \end{aligned}$$

Now add these equations, using the identity $\sin^2\theta + \cos^2\theta = 1$ yielding

$$\hat{j} = \sin\theta \hat{i}' + \cos\theta \hat{j}'. \quad (3.3.19)$$

Similarly multiply the equation for \hat{i}' by $\cos\theta$ and the equation for \hat{j}' by $-\sin\theta$ yielding

$$\begin{aligned} \cos\theta \hat{i}' &= \cos^2\theta \hat{i} + \cos\theta \sin\theta \hat{j} \\ -\sin\theta \hat{j}' &= \sin^2\theta \hat{i} - \sin\theta \cos\theta \hat{j}. \end{aligned}$$

Now add these equations, using the identity $\sin^2\theta + \cos^2\theta = 1$

$$\hat{i} = \cos\theta \hat{i}' - \sin\theta \hat{j}'. \quad (3.3.20)$$

Now we can rewrite the vector \vec{A} as

$$\begin{aligned}
\vec{\mathbf{A}} &= A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} = A_x (\cos \theta \hat{\mathbf{i}}' - \sin \theta \hat{\mathbf{j}}') + A_y (\sin \theta \hat{\mathbf{i}}' + \cos \theta \hat{\mathbf{j}}') \\
&= (A_x \cos \theta + A_y \sin \theta) \hat{\mathbf{i}}' + (-A_x \sin \theta + A_y \cos \theta) \hat{\mathbf{j}}' \\
&= A_{x'} \hat{\mathbf{i}}' + A_{y'} \hat{\mathbf{j}}',
\end{aligned}$$

where

$$A_{x'} = A_x \cos \theta + A_y \sin \theta \quad (3.3.21)$$

$$A_{y'} = -A_x \sin \theta + A_y \cos \theta. \quad (3.3.22)$$

We can use the given information that $A_x = 5$, $A_y = -3$, and $\theta = 60^\circ$ to solve for the components of $\vec{\mathbf{A}}$ in the prime coordinate system

$$\begin{aligned}
A_{x'} &= A_x \cos \theta + A_y \sin \theta = \frac{1}{2}(5 - 3\sqrt{3}), \\
A_{y'} &= -A_x \sin \theta + A_y \cos \theta = \frac{1}{2}(-5\sqrt{3} - 3).
\end{aligned}$$

c) The magnitude can be calculated in either coordinate system

$$\begin{aligned}
|\vec{\mathbf{A}}| &= \sqrt{(A_x)^2 + (A_y)^2} = \sqrt{(5)^2 + (-3)^2} = \sqrt{34} \\
|\vec{\mathbf{A}}| &= \sqrt{(A_{x'})^2 + (A_{y'})^2} = \sqrt{\left(\frac{1}{2}(5 - 3\sqrt{3})\right)^2 + \left(\frac{1}{2}(-5\sqrt{3} - 3)\right)^2} = \sqrt{34}.
\end{aligned}$$

This result agrees with what I expect because the length of vector is independent of the choice of coordinate system.

Example 3.6 Vector Addition

Two force vectors $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$, such that $|\vec{\mathbf{B}}| = 2|\vec{\mathbf{A}}|$ have a resultant $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$ of magnitude 26.5 N, which makes an angle of 41° with respect to the smaller vector $\vec{\mathbf{A}}$. Find the magnitude of each vector and the angle between them.

Solution: We begin by making a sketch of the three vectors, choosing $\vec{\mathbf{A}}$ to point in the positive x -direction (Figure 3.22).

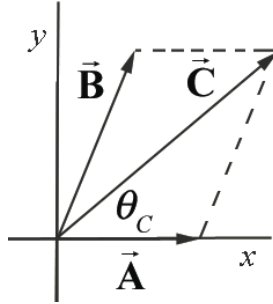


Figure 3.22 Choice of coordinates system for Example 3.6

Let's denote the magnitude of \vec{C} by $C \equiv |\vec{C}| = \sqrt{(C_x)^2 + (C_y)^2} = 26.5 \text{ N}$. We first note that the components of $\vec{C} = \vec{A} + \vec{B}$ are given by

$$C_x = A_x + B_x = C \cos \theta_C = (26.5 \text{ N}) \cos(41^\circ) = 20 \text{ N} \quad (3.3.23)$$

$$C_y = B_y = C \sin \theta_C = (26.5 \text{ N}) \sin(41^\circ) = 17.39 \text{ N}. \quad (3.3.24)$$

Let's denote the magnitude of \vec{B} by $B \equiv |\vec{B}|$. From the condition that $|\vec{B}| = 2|\vec{A}|$, we know that

$$(B_x)^2 + (B_y)^2 = 4(A_x)^2. \quad (3.3.25)$$

Using Eqs. (3.3.23) and (3.3.24), Eq. (3.3.25) becomes

$$\begin{aligned} (C_x - A_x)^2 + (C_y)^2 &= 4(A_x)^2 \\ (C_x)^2 - 2C_x A_x + (A_x)^2 + (C_y)^2 &= 4(A_x)^2. \end{aligned}$$

This is a quadratic equation

$$0 = 3(A_x)^2 + 2C_x A_x - C^2$$

with solution

$$\begin{aligned} A_x &= \frac{-2C_x \pm \sqrt{(2C_x)^2 + (4)(3)(C^2)}}{6} = \frac{-2(20 \text{ N}) \pm \sqrt{(40 \text{ N})^2 + (4)(3)(26.5 \text{ N})^2}}{6} \\ &= 10.0 \text{ N}, \end{aligned}$$

where we choose the positive square root because we originally chose $A_x > 0$. Then

$$\begin{aligned} B_x = C_x - A_x &= 20.0 \text{ N} - 10.0 \text{ N} = 10.0 \text{ N} \\ B_y &= 17.39 \text{ N}. \end{aligned}$$

Note that the magnitude of $|\vec{\mathbf{B}}| = \sqrt{(B_x)^2 + (B_y)^2} = 20.0 \text{ N}$ is equal to two times the magnitude of $|\vec{\mathbf{A}}| = 10.0 \text{ N}$. The angle between $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ is given by

$$\theta = \sin^{-1}(B_y / |\vec{\mathbf{B}}|) = \sin^{-1}(17.39 \text{ N} / 20.0 \text{ N}) = 60^\circ.$$