A1 Vector Algebra and Calculus

Prof David Murray

david.murray@eng.ox.ac.uk
www.robots.ox.ac.uk/~dwm/Courses/2VA

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Vector Algebra and Calculus

1. Revision of vector algebra, scalar product, vector product
2. Triple products, multiple products, applications to geometry
3. **Differentiation of vector functions, applications to mechanics**
4. Scalar and vector fields. Line, surface and volume integrals, curvilinear co-ordinates
5. Vector operators — grad, div and curl
6. Vector Identities, curvilinear co-ordinate systems
7. Gauss' and Stokes' Theorems and extensions
8. Engineering Applications

Differentiating Vector Functions of one Variable

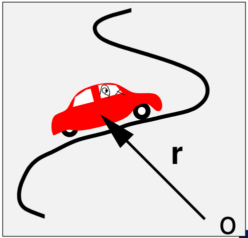
Your experience of differentiation and integration has extended as far as *scalar* functions of single and multiple variables

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) \quad \text{and} \quad \frac{\partial}{\partial x} f(x, y, t)$$

No surprise that we often wish to differentiate *vector functions*.

For example, suppose you were driving along a wiggly road with position $\mathbf{r}(t)$ at time t.

- Differentiating $\mathbf{r}(t)$ gives velocity $\mathbf{v}(t)$.
- Differentiating $\mathbf{v}(t)$ gives acceleration $\mathbf{a}(t)$.
- Differentiating $\mathbf{a}(t)$ gives the jerk $\mathbf{j}(t)$.



Differentiation of a vector

The cypress lined road to your villa in Tuscany ...

Using a fixed — that is $(\hat{\mathbf{\imath}}, \hat{\mathbf{\jmath}}, \hat{\mathbf{k}})$ constant — Cartesian coordinate system the position vector is by definition

$$\mathbf{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}} + z\hat{\mathbf{k}}$$

So as a function of time t

$$\mathbf{r}(t) = x(t)\hat{\mathbf{\imath}} + y(t)\hat{\mathbf{\jmath}} + z(t)\hat{\mathbf{k}}$$

Differentiation of a vector

By analogy with the definition for a scalar function, the derivative of a vector function $\mathbf{a}(p)$ of a single parameter p is

$$\frac{d\mathbf{a}}{dp}(p) = \lim_{\delta p \to 0} \frac{\mathbf{a}(p+\delta p) - \mathbf{a}(p)}{\delta p} \quad .$$

If we write $\mathbf{a}$ in terms of components relative to a FIXED coordinate system ($\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ constant)

$$\mathbf{a}(p) = a_1(p)\hat{\mathbf{i}} + a_2(p)\hat{\mathbf{j}} + a_3(p)\hat{\mathbf{k}}$$

then

$$\frac{d\mathbf{a}}{dp}(p) = \frac{da_1}{dp}\hat{\mathbf{i}} + \frac{da_2}{dp}\hat{\mathbf{j}} + \frac{da_3}{dp}\hat{\mathbf{k}} \quad .$$

> To differentiate a vector function defined wrt a fixed coordinates
> **differentiate each component separately**

Position, velocity and acceleration

- Suppose $\mathbf{r}(t)$ is the position vector of an object moving w.r.t. the orgin.

$$\mathbf{r}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}}$$

- Then the instantaneous velocity is

$$\mathbf{v}(t) = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

- and the acceleration is

$$\mathbf{a}(t) = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} \quad = \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{k}} \ .$$

All the familiar stuff works ...

A moment's thought tells you that ...

- All the familiar rules of differentiation apply
- The rules don't get munged by operations like scalar product and vector products.

For example:

$$\frac{d}{dp}(\mathbf{a} \times \mathbf{b}) = \left(\frac{d\mathbf{a}}{dp} \times \mathbf{b}\right) + \left(\mathbf{a} \times \frac{d\mathbf{b}}{dp}\right)$$

$$\frac{d}{dp}(\mathbf{a} \cdot \mathbf{b}) = \left(\frac{d\mathbf{a}}{dp} \cdot \mathbf{b}\right) + \left(\mathbf{a} \cdot \frac{d\mathbf{b}}{dp}\right) \quad .$$

But note the very obvious facts that:

- $d\mathbf{a}/dp$ has a different direction from $\mathbf{a}$
- $d\mathbf{a}/dp$ has a different magnitude from $\mathbf{a}$.

Chain rule: more good news

Likewise, the chain rule still applies.

If $\mathbf{a} = \mathbf{a}(u)$ and $u = u(p)$, then

$$\frac{d\mathbf{a}}{dp} = \frac{d\mathbf{a}}{du}\,\frac{du}{dp}$$

This follows directly from the fact that the vector derivative is just the vector of derivatives of the components.

♣ Example of chain rule

The position of vehicle is given by $\mathbf{r}(u)$ where u is amount of fuel used by time t, so that $u = u(t)$.

Its velocity must be

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{du}\frac{du}{dt}$$

Its acceleration is (product plust chain rules ...)

$$\begin{aligned}\frac{d^2\mathbf{r}}{dt^2} &= \left(\frac{d^2\mathbf{r}}{du^2}\frac{du}{dt}\right)\frac{du}{dt} + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2} \\ &= \frac{d^2\mathbf{r}}{du^2}\left(\frac{du}{dt}\right)^2 + \frac{d\mathbf{r}}{du}\frac{d^2u}{dt^2}\end{aligned}$$

♣ Example: direction of the derivative

Q:
3D vector **a** has constant magnitude, but is varying over time.
What can you say about the direction of $d\mathbf{a}/dt$?
A:
Using intuition: if only the direction is changing, then the vector must be tracing out points on the surface of a sphere.
So $d\mathbf{a}/dt$ should be orthogonal to **a**.
To prove this write

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}) = \mathbf{a}\cdot\frac{d\mathbf{a}}{dt} + \frac{d\mathbf{a}}{dt}\cdot\mathbf{a} = 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} \quad .$$

But $(\mathbf{a}\cdot\mathbf{a}) = a^2 = \text{const}$.
So

$$\frac{d}{dt}(\mathbf{a}\cdot\mathbf{a}) = 0 \qquad \Rightarrow 2\mathbf{a}\cdot\frac{d\mathbf{a}}{dt} = 0 \qquad \Rightarrow \ldots\ldots\ldots\ldots\ldots\ldots$$

Integration of a vector function

As with scalars, integration of a vector function of a single scalar variable is the reverse of differentiation.

In other words

$$\int_{p_1}^{p_2} \left[\frac{d\mathbf{a}(p)}{dp} \right] dp = \mathbf{a}(p_2) - \mathbf{a}(p_1)$$

Eg, from dynamics-ville

$$\int_{t_1}^{t_2} \mathbf{a} \quad dt = \mathbf{v}(t_2) - \mathbf{v}(t_1)$$

However, other types of integral are possible, especially when the vector is a function of more than one variable.

This requires the introduction of the concepts of *scalar and vector fields*. See lecture 4!

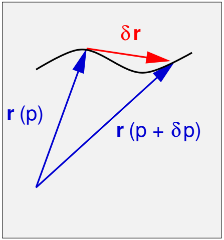
Space curves and Derivatives

Position vector $\mathbf{r}(p)$ traces a space curve.

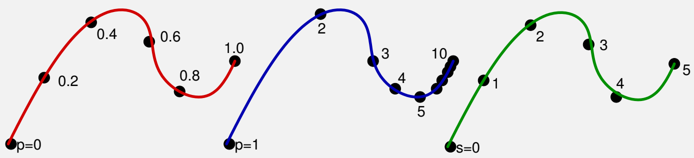
Vector $\delta\mathbf{r}$ is a secant to the curve
$\delta\mathbf{r}/\delta p$ lies in the same direction as $\delta\mathbf{r}(p)$

Take limit as $\delta p \to 0$

$\boxed{\text{$d\mathbf{r}/dp$ **is a tangent to the space curve**}}$



There are countless ways of parametrizing a curve ...



One is special ...

Geometrical interpretation of derivatives /ctd

When the parameter s is **arc-length** or **metric distance**, then we have:

$$|d\mathbf{r}| = ds$$

then

$$\frac{|d\mathbf{r}|}{ds} = \left|\frac{d\mathbf{r}}{ds}\right| = 1$$

and so

$$\boxed{\frac{d\mathbf{r}}{ds} \text{ is the } \textbf{unit tangent} \text{ to } \mathbf{r} \text{ at } s}$$

For s arc-length and p some other parametrization, we have

$$\frac{d\mathbf{r}}{dp} = \frac{d\mathbf{r}}{ds}\frac{ds}{dp} \qquad \Rightarrow \qquad \left|\frac{d\mathbf{r}}{dp}\right| = \left|\frac{d\mathbf{r}}{ds}\right|\frac{ds}{dp} = \frac{ds}{dp}$$

Geometrical interpretation of derivatives /ctd

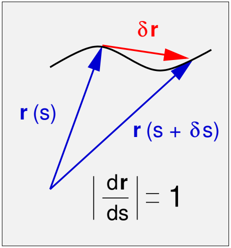
To repeat, the derivative $d\mathbf{r}/dp$ is a vector

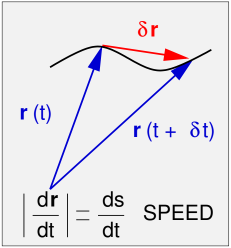
Its direction is **always a tangent to curve** $\mathbf{r}(p)$

Its magnitude is ds/dp, where s is arc length

If the parameter is arc length s, then $d\mathbf{r}/ds$ is the **unit tangential vector**.

If the parameter is time t, then magnitude $|d\mathbf{r}/dt|$ is the speed.





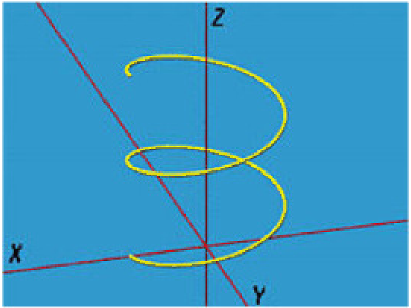
♣ Example

Q: Draw the curve

$$\mathbf{r} = a\cos\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\mathbf{\imath}} + a\sin\left(\frac{s}{\sqrt{a^2+h^2}}\right)\hat{\mathbf{\jmath}} + \frac{hs}{\sqrt{a^2+h^2}}\hat{\mathbf{k}}$$

where s is arc length and a and h are constants.

A:



♣ Example ctd

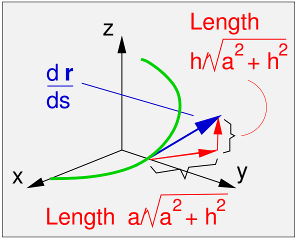
$$\mathbf{r} = a\cos(\frac{s}{\sqrt{a^2+h^2}})\hat{\mathbf{\imath}} + a\sin(\frac{s}{\sqrt{a^2+h^2}})\hat{\mathbf{\jmath}} + \frac{hs}{\sqrt{a^2+h^2}}\hat{\mathbf{k}}$$

Q: Show that the tangent $d\mathbf{r}/ds$ to the curve has a constant elevation angle w.r.t the xy-plane, and determine its magnitude.

A:

$$\frac{d\mathbf{r}}{ds} = -\frac{a}{\sqrt{a^2+h^2}}\sin()\,\hat{\mathbf{\imath}} + \frac{a}{\sqrt{a^2+h^2}}\cos()\,\hat{\mathbf{\jmath}} + \frac{h}{\sqrt{a^2+h^2}}\hat{\mathbf{k}}$$

Projection on the xy plane has magnitude $a/\sqrt{a^2+h^2}$
Projection in the z direction $h/\sqrt{a^2+h^2}$
So the elevation angle is $\tan^{-1}(h/a)$, which is constant.
We are expecting $|d\mathbf{r}/ds| = 1$, and indeed it is!



Why can't we say any old parameter is arc length?

Arc length s parameter is special because $ds = |d\mathbf{r}|$,

Or, in integral form, *whatever the parameter p*,

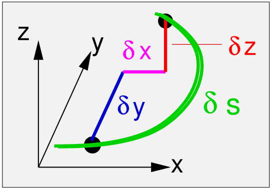
$$\text{Accumulated arc length} = \int_{p_0}^{p_1} \left| \frac{d\mathbf{r}}{dp} \right| dp \ .$$

Using Pythagoras' theorem on a short piece of curve. In the limit as ds tends to zero

$$ds^2 = dx^2 + dy^2 + dz^2 \ .$$

So if a curve is parameterized in terms of p

$$\frac{ds}{dp} = \sqrt{\left[\frac{dx}{dp}\right]^2 + \left[\frac{dy}{dp}\right]^2 + \left[\frac{dz}{dp}\right]^2} \ .$$



Arc length is special /ctd

Suppose we had parameterized our helix as

$$\mathbf{r} = a\cos p\hat{\mathbf{\imath}} \quad + \quad a\sin p\hat{\mathbf{\jmath}} \quad + \quad hp\hat{\mathbf{k}}$$

We could show p is not arc length by testing

$$\left|\frac{d\mathbf{r}}{dp}\right| = \sqrt{\left[\frac{dx}{dp}\right]^2 + \left[\frac{dy}{dp}\right]^2 + \left[\frac{dz}{dp}\right]^2} \quad = \quad \sqrt{a^2\sin^2 p + a^2\cos^2 p + h^2}$$

$$= \quad \sqrt{a^2 + h^2} \qquad \neq 1$$

Here, because this is not a function of p, we know that $s = \alpha p$. But

$$|d\mathbf{r}/dp| = ds/dp = \alpha \qquad \Rightarrow \quad \alpha = \sqrt{a^2 + h^2}$$

If we wanted to re-parameterize in terms of arclength s ...
we would replace p with $s/\sqrt{a^2 + h^2}$.

Curves in 3D

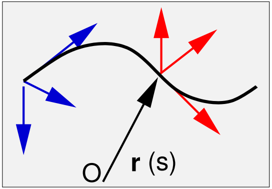
Let's look more closely at parametrizing a 3D space curve in terms of arclength s.

Introduce

- orthogonal coord frames for each value s
- each with its origin at $\mathbf{r}(s)$.

To specify a coordinate frame we need

- three mutually perpendicular directions
- these should be *intrinsic* to the curve
- and NOT fixed in an external reference frame.



Curves in 3D

Rollercoaster will help you see what's going on ...

But *it* has a specially shaped rail or two rails that define the twists and turns.

We are thinking about a 3D curve – just a 3D wire.
Does the curve itself somehow define its own twists and turns?

Yes:
Method due to French mathematicians F-J. Frenet and J. A. Serret

The Frenet-Serret Local Coordinates

Consider $\mathbf{r}(s)$ ...

1. Unit tangent $\hat{\mathbf{t}}$

$$\hat{\mathbf{t}} = d\mathbf{r}/ds$$

2. Principal Normal $\hat{\mathbf{n}}$

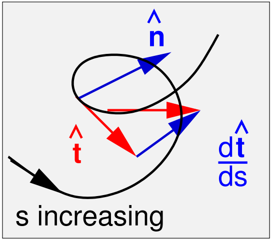
Proved earlier that if $|\mathbf{a}(t)| = \text{const}$ then $\mathbf{a} \cdot d\mathbf{a}/dt = 0$. So

$$\hat{\mathbf{t}} = \hat{\mathbf{t}}(s), \quad |\hat{\mathbf{t}}| = \text{const} \Rightarrow \hat{\mathbf{t}} \cdot d\hat{\mathbf{t}}/ds = 0$$

So the principal normal $\hat{\mathbf{n}}$ is defined from

$$\kappa\hat{\mathbf{n}} = d\hat{\mathbf{t}}/ds$$

where $\kappa \geqslant 0$ is the curve's **curvature**.



3. The Binormal $\hat{\mathbf{b}}$

The third member of a local r-h set is the binormal, $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$.

Deriving the Frenet-Serret relationships

We know: Tangent $\hat{\mathbf{t}} = d\mathbf{r}/ds$	Normal $\hat{\mathbf{n}} = \kappa^{-1} d\hat{\mathbf{t}}/ds$,	Binormal $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$

Since $\hat{\mathbf{b}} \cdot \hat{\mathbf{t}} = 0$, if we differentiate wrt s we still get zero ...

$$\frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \frac{d\hat{\mathbf{t}}}{ds} \quad = \quad \frac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} + \hat{\mathbf{b}} \cdot \kappa\hat{\mathbf{n}} = 0$$

But $\hat{\mathbf{b}} \cdot \hat{\mathbf{n}} = 0$, from which $\dfrac{d\hat{\mathbf{b}}}{ds} \cdot \hat{\mathbf{t}} = 0$.

So $d\hat{\mathbf{b}}/ds$ has no components along both $\hat{\mathbf{b}}$ and $\hat{\mathbf{t}}$.

Hence $d\hat{\mathbf{b}}/ds$ is along the direction of $\hat{\mathbf{n}}$:

$$\frac{d\hat{\mathbf{b}}}{ds} = -\tau(s)\hat{\mathbf{n}}(s)$$

where τ is the space curve's **torsion**. ($-$ sign is convention.)

Deriving the Frenet-Serret relationships

$$\boxed{\begin{array}{c}\text{Tangent } \hat{\mathbf{t}}, \quad \text{Normal } \hat{\mathbf{n}}, \quad \text{Binormal } \hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} \\ d\hat{\mathbf{t}}/ds = \kappa\hat{\mathbf{n}}, \quad d\hat{\mathbf{b}}/ds = -\tau(s)\hat{\mathbf{n}}(s)\end{array}}$$

Differentiating $\hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = 0$:

$$\begin{aligned}(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{t}} + \hat{\mathbf{n}} \cdot (d\hat{\mathbf{t}}/ds) &= 0 \\ (d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{t}} + \hat{\mathbf{n}} \cdot \kappa\hat{\mathbf{n}} &= 0 \\ (d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{t}} &= -\kappa\end{aligned}$$

Now do the same to $\hat{\mathbf{n}} \cdot \hat{\mathbf{b}} = 0$:

$$\begin{aligned}(d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{b}} + \hat{\mathbf{n}} \cdot (d\hat{\mathbf{b}}/ds) &= 0 \\ (d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{b}} + \hat{\mathbf{n}} \cdot (-\tau)\hat{\mathbf{n}} &= 0 \\ (d\hat{\mathbf{n}}/ds) \cdot \hat{\mathbf{b}} &= +\tau\end{aligned}$$

HENCE

$$\frac{d\hat{\mathbf{n}}}{ds} = -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s).$$

Summary of the Frenet-Serret relationships

These three expressions are called

The Frenet-Serret relationships:

$$\begin{aligned} d\hat{\mathbf{t}}/ds &= \kappa\hat{\mathbf{n}} \\ d\hat{\mathbf{n}}/ds &= -\kappa(s)\hat{\mathbf{t}}(s) + \tau(s)\hat{\mathbf{b}}(s) \\ sd\hat{\mathbf{b}}/ds &= -\tau(s)\hat{\mathbf{n}}(s) \end{aligned}$$

They describe by how much the intrinsic coordinate system changes orientation as we move along a space curve.

♣ Example

Q: Derive $\kappa(s)$ and $\tau(s)$ for the helix

$$\mathbf{r}(s) = a\cos\frac{s}{\beta}\hat{\mathbf{i}} + a\sin\frac{s}{\beta}\hat{\mathbf{j}} + \frac{hs}{\beta}\hat{\mathbf{k}} \qquad \text{where} \quad \beta = \sqrt{a^2 + h^2}$$

A: We found the unit tangent earlier as

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \left[-\frac{a}{\beta}\sin\frac{s}{\beta}, \quad \frac{a}{\beta}\cos\frac{s}{\beta}, \quad \frac{h}{\beta}\right] .$$

Hence

$$\kappa\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} = \left[-\frac{a}{\beta^2}\cos\frac{s}{\beta}, \quad -\frac{a}{\beta^2}\sin\frac{s}{\beta}, \quad 0\right]$$

The curvature must be positive, so

$$\kappa = \frac{a}{\beta^2} \qquad \text{and} \qquad \hat{\mathbf{n}} = \left[-\cos\frac{s}{\beta}, \quad -\sin\frac{s}{\beta}, \quad 0\right] .$$

So the curvature is constant, and normal $\hat{\mathbf{n}}$ is parallel to the xy-plane.

♣ Example /continued

Recall

$$\hat{\mathbf{t}} = \left[-\frac{a}{\beta}\sin\frac{s}{\beta}, \frac{a}{\beta}\cos\frac{s}{\beta}, \frac{h}{\beta}\right] \quad \text{and} \quad \hat{\mathbf{n}} = \left[-\cos\frac{s}{\beta}, -\sin\frac{s}{\beta}, 0\right].$$

So the binormal is

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -\frac{a}{\beta}\sin\frac{s}{\beta} & \frac{a}{\beta}\cos\frac{s}{\beta} & \frac{h}{\beta} \\ -\cos\frac{s}{\beta} & -\sin\frac{s}{\beta} & 0 \end{vmatrix} = \left[\frac{h}{\beta}\sin\frac{s}{\beta}, \ -\frac{h}{\beta}\cos\frac{s}{\beta}, \ \frac{a}{\beta}\right]$$

Hence

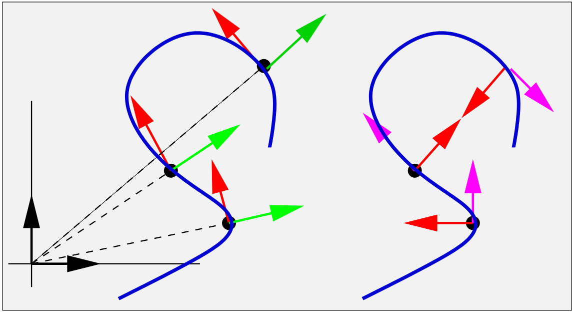
$$\frac{d\hat{\mathbf{b}}}{ds} = \left[\frac{h}{\beta^2}\cos\frac{s}{\beta}, \ \frac{h}{\beta^2}\sin\frac{s}{\beta}, \ 0\right] = -\frac{h}{\beta^2}\hat{\mathbf{n}}$$

So the torsion is also a constant,

$$\tau = \frac{h}{\beta^2}$$

Fixed, varying, and intrinsic coordinates

- Fixed: $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$
- Varying: $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$
- Intrinsic: $\hat{\mathbf{t}}$, $\hat{\mathbf{n}}$, $\hat{\mathbf{b}}$



Derivative of r in plane polars

In plane polar coordinates, the radius vector of any point P is given by

$$\mathbf{r} = r(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) = r\hat{\mathbf{r}}$$

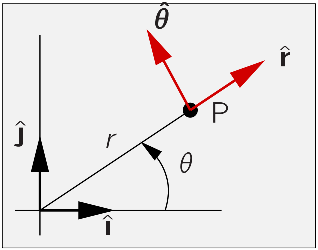
where we have introduced the unit radial vector

$$\hat{\mathbf{r}} = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath} \quad .$$

The other "natural" unit vector in plane polars is orthogonal to $\hat{\mathbf{r}}$ and is

$$\hat{\theta} = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$$

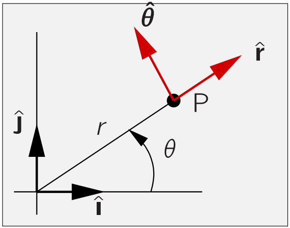
so that $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} = \hat{\theta}\cdot\hat{\theta} = 1$ and $\hat{\mathbf{r}}\cdot\hat{\theta} = 0$.



Derivative of r in plane polars

Now suppose P is moving so that $\mathbf{r}$ is a function of time t.
Its velocity is

$$\begin{aligned}
\dot{\mathbf{r}} &= \frac{d}{dt}(r\hat{\mathbf{r}}) \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d}{dt}\hat{\mathbf{r}} \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d}{dt}(\cos\theta\hat{\mathbf{\imath}} + \sin\theta\hat{\mathbf{\jmath}}) \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}(-\sin\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}}) \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \\
&= \textbf{radial} + \textbf{tangential}
\end{aligned}$$



Note that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d}{dt}(-\sin\theta\hat{\mathbf{\imath}} + \cos\theta\hat{\mathbf{\jmath}}) = -\frac{d\theta}{dt}\hat{\mathbf{r}}$$

Acceleration components in plane polars

Recap ...

$$\dot{\mathbf{r}} = \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \; ; \qquad \frac{d\hat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \; ; \qquad \frac{d\hat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\hat{\mathbf{r}}$$

Differentiating $\dot{\mathbf{r}}$ gives the accel. of P

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d^2r}{dt^2}\hat{\mathbf{r}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + \frac{dr}{dt}\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} - r\frac{d\theta}{dt}\frac{d\theta}{dt}\hat{\mathbf{r}} \\
&= \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} \; + \; \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\boldsymbol{\theta}}
\end{aligned}$$

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Acceleration components in plane polars

We just found that

$$\ddot{\mathbf{r}} = \left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\hat{\mathbf{r}} \;+\; \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\hat{\boldsymbol{\theta}}$$

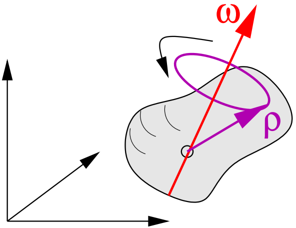
Three obvious cases:

$$\begin{aligned}
\theta \text{ const:} \quad \ddot{\mathbf{r}} &= \frac{d^2 r}{dt^2}\hat{\mathbf{r}} \\
r \text{ const:} \quad \ddot{\mathbf{r}} &= -r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}} + r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} \\
r \text{ and } d\theta/dt \text{ const:} \quad \ddot{\mathbf{r}} &= -r\left(\frac{d\theta}{dt}\right)^2\hat{\mathbf{r}}
\end{aligned}$$

Rotating systems 1

Body rotates with constant $\boldsymbol{\omega}$ about axis passing through the body origin.

Assume the body origin is fixed. We observe from a fixed coord system $Oxyz$.



If $\boldsymbol{\rho}$ is a vector of constant mag and constant direction in the rotating system

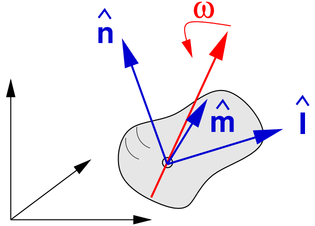
The rate of change of $\boldsymbol{\rho}$ as seen in fixed system is

$$\frac{d\boldsymbol{\rho}}{dt} = \boldsymbol{\omega} \times \boldsymbol{\rho}(t).$$

- $d\boldsymbol{\rho}/dt$ will have fixed magnitude,
- $d\boldsymbol{\rho}/dt$ will always be perpendicular to the axis of rotation, but
- $d\boldsymbol{\rho}/dt$ will vary in direction within those constraints.
- The point $\boldsymbol{\rho}(t)$ will move within a plane in the fixed system.

Rotating systems 2

Instead of ρ, consider a set of mutually orthogonal unit vectors $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$ *attached to the rotating system*.



In the *fixed* frame, each of $\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}}$ has a time dependence

$$\frac{d\hat{\mathbf{l}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{l}}(t) \qquad \frac{d\hat{\mathbf{m}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{m}}(t) \qquad \frac{d\hat{\mathbf{n}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{n}}(t)$$

NB: $\boldsymbol{\omega}$ defines the axis of rotation, and hence is fixed in the rotating frame *and* in the fixed frame.

Rotating systems 2 /ctd

$\Rightarrow$At an instant in the fixed frame, but using $(\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})$ at that instant as the basis

$$\frac{d\hat{\mathbf{l}}}{dt} = \boldsymbol{\omega} \times \hat{\mathbf{l}} = \begin{vmatrix} \hat{\mathbf{l}} & \hat{\mathbf{m}} & \hat{\mathbf{n}} \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix} = \omega_3\hat{\mathbf{m}} - \omega_2\hat{\mathbf{n}}$$

and similarly for $d\hat{\mathbf{m}}/dt$, $d\hat{\mathbf{n}}/dt$.

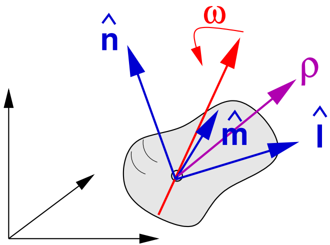
$d\hat{\mathbf{l}}/dt, d\hat{\mathbf{m}}/dt, d\hat{\mathbf{n}}/dt$ are all perpendicular to $\boldsymbol{\omega}$, so they must be...

.... COPLANAR (when treated as free vectors)

Their scalar triple product is zero.

Rotating systems 2 /ctd

Now return to $\boldsymbol{\rho}$ but define it w.r.t. the body's system.



$$\boldsymbol{\rho} = \rho_1 \hat{\mathbf{l}} + \rho_2 \hat{\mathbf{m}} + \rho_3 \hat{\mathbf{n}}$$

$$\frac{d\boldsymbol{\rho}}{dt} = \boldsymbol{\omega} \times \boldsymbol{\rho}(t) = \boldsymbol{\omega} \times \left(\rho_1 \hat{\mathbf{l}}(t) + \rho_2 \hat{\mathbf{m}}(t) + \rho_3 \hat{\mathbf{n}}(t)\right)$$

$$\Rightarrow \frac{d\boldsymbol{\rho}}{dt} = \rho_1 \frac{d\hat{\mathbf{l}}}{dt} + \rho_2 \frac{d\hat{\mathbf{m}}}{dt} + \rho_3 \frac{d\hat{\mathbf{n}}}{dt}$$

Remember that $\boldsymbol{\rho}$ is time-invariant in $(\hat{\mathbf{l}}, \hat{\mathbf{m}}, \hat{\mathbf{n}})$ ...

... so its time dependence in the fixed frame depends on the variation of $\hat{\mathbf{l}}(t)$, $\hat{\mathbf{m}}(t)$, $\hat{\mathbf{n}}(t)$, NOT on the coefficients with respect to this basis set.

Rotating co-ordinate systems 3

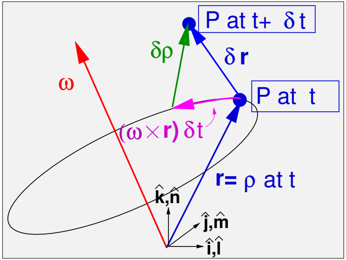
Now $\boldsymbol{\rho}$ is the position vector of a point P in the rotating body, but it is defined in a body-centred coord frame which is, **at the instant**, aligned with our fixed external coord system.
So at t, **and only at time** t, $\mathbf{r} = \boldsymbol{\rho}$

As a result of the rotation, over time δt the point $\mathbf{r} = \boldsymbol{\rho}$ will move by

$$(\boldsymbol{\omega} \times \boldsymbol{\rho})\delta t = (\boldsymbol{\omega} \times \mathbf{r})\delta t$$

Suppose the point is also moving with respect to the rotating system. In time δt it moves $\delta\boldsymbol{\rho}$, so the overall movement in the fixed external frame is

$$\delta\mathbf{r} = \delta\boldsymbol{\rho} \ + \ (\boldsymbol{\omega} \times \mathbf{r})\delta t$$



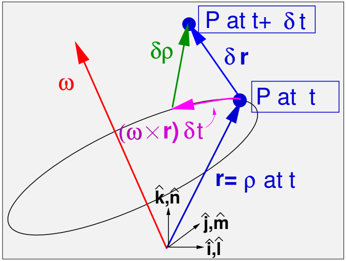
Rotating co-ordinate systems 3

Reminder: The overall movement in the fixed external frame is

$$\delta\mathbf{r} = \delta\boldsymbol{\rho} + (\boldsymbol{\omega} \times \mathbf{r})\delta t$$

So the **instantaneous velocity** wrt the fixed frame is

$$\boxed{\frac{d\mathbf{r}}{dt} = \frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega} \times \mathbf{r}}$$



The capital D is used merely to indicate differentiation in the rotating frame, not the fixed frame.

This result is general one.

It relates the instantaneous time derivatives with respect to the rotating and non-rotating frames for any vector.

Rotating co-ordinate systems 4

Reminder ... It relates the instantaneous time derivatives with respect to the rotating and non-rotating frames for *any* vector.

Justification?

Because the frames are aligned at t, any vector in the fixed frame has the same value in the rotating frame, just as $\mathbf{r} = \boldsymbol{\rho}$.

So we can devise an operator [ ] ...

$$\frac{d\mathbf{r}}{dt} = \frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega} \times \mathbf{r} = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\mathbf{r}$$

Note when we say any vector is the same, DON'T think that the "labels" are the same. As we have already seen, velocity in the fixed frame is not velocity in the rotating frame.

Rotating co-ordinate systems 4

We can therefore find the instantaneous acceleration as

$$\ddot{\mathbf{r}} = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\dot{\mathbf{r}} \quad = \left[\frac{D}{Dt} + \boldsymbol{\omega}\times\right]\left(\frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega}\times\mathbf{r}\right)$$

The **instantaneous acceleration in the fixed frame** is

$$\ddot{\mathbf{r}} = \frac{D^2\boldsymbol{\rho}}{Dt^2} + 2\boldsymbol{\omega}\times\frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$$

Term 1 is P's acceleration in the rotating frame.

Term 3 is the centripetal accel: magnitude $\omega^2(\textit{radius})$ and towards centre of rotation.

Term 2 is a SURPRISE: a coupling of rotation and velocity of P in the rotating frame. It is the **CORIOLIS acceleration**.

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♣ Example

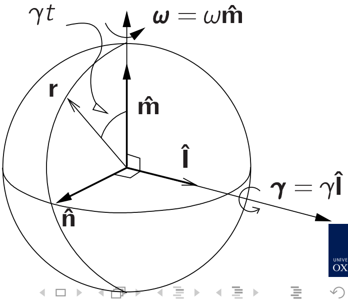
Q: Find the instantaneous acceleration as observed in a fixed frame of a projectile fired along a line of longitude (with angular velocity of $\boldsymbol{\gamma}$ constant relative to the sphere) if the sphere is rotating about its poles with angular velocity $\boldsymbol{\omega}$.

A: We will need to use

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{D^2\boldsymbol{\rho}}{Dt^2} + 2\boldsymbol{\omega}\times\frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\\ &= \text{Local frame} + \text{Coriolis} + \text{Centripetal}\end{aligned}$$

First, find velocity and acceleration in the local rotating frame.
For this specific example ...

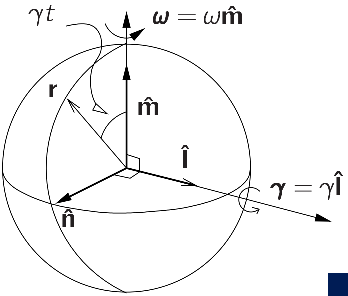
$$\begin{aligned}\frac{D\boldsymbol{\rho}}{Dt} &= \boldsymbol{\gamma}\times\boldsymbol{\rho} = \boldsymbol{\gamma}\times\mathbf{r}\\ \frac{D^2\boldsymbol{\rho}}{Dt^2} &= \boldsymbol{\gamma}\times\frac{D\boldsymbol{\rho}}{Dt} = \boldsymbol{\gamma}\times(\boldsymbol{\gamma}\times\mathbf{r})\ .\end{aligned}$$



♣ Example /ctd

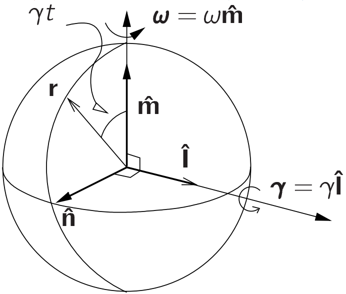
Hence, in the fixed frame, the instantaneous acceleration is

$$
\begin{aligned}
\ddot{\mathbf{r}} &= \text{Local frame} + \text{Coriolis} + \text{Centripetal} \\
&= \frac{D^2\boldsymbol{\rho}}{Dt^2} + 2\boldsymbol{\omega} \times \frac{D\boldsymbol{\rho}}{Dt} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \\
&= \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \mathbf{r}) + 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \mathbf{r}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})
\end{aligned}
$$



♣ Example /ctd

Reminder: $\ddot{\mathbf{r}} = \boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \mathbf{r}) \quad + \quad 2\boldsymbol{\omega} \times (\boldsymbol{\gamma} \times \mathbf{r}) \quad + \quad \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$



1) As $\boldsymbol{\gamma} = \gamma\hat{\mathbf{l}}$, and
$\mathbf{r} = \boldsymbol{\rho} = R\cos(\gamma t)\hat{\mathbf{m}} + R\sin(\gamma t)\hat{\mathbf{n}}$
$\Rightarrow$acceleration in rotating frame is

$$\boldsymbol{\gamma} \times (\boldsymbol{\gamma} \times \boldsymbol{\rho}) = -\gamma^2\mathbf{r}$$

2) Centripetal accel due to rotation of sphere is

$$\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = -\omega^2 R\sin(\gamma t)\hat{\mathbf{n}}$$

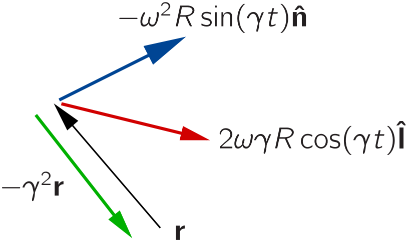
3) The Coriolis acceleration is

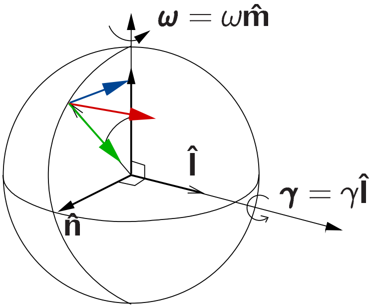
$$2\boldsymbol{\omega} \times \frac{D\boldsymbol{\rho}}{dt} = 2\begin{bmatrix}0\\ \omega\\ 0\end{bmatrix} \times \left(\begin{bmatrix}\gamma\\ 0\\ 0\end{bmatrix} \times \begin{bmatrix}0\\ R\cos(\gamma t)\\ R\sin(\gamma t)\end{bmatrix}\right) = 2\omega\gamma R\cos(\gamma t)\hat{\mathbf{l}}$$

♣ Example /ctd

Recap:

- Accel in rotating frame $-\gamma^2\mathbf{r}$
- Centripetal due to sphere rotating $-\omega^2 R\sin(\gamma t)\hat{\mathbf{n}}$
- Coriolis acceleration: $2\omega\gamma R\cos(\gamma t)\hat{\mathbf{l}}$



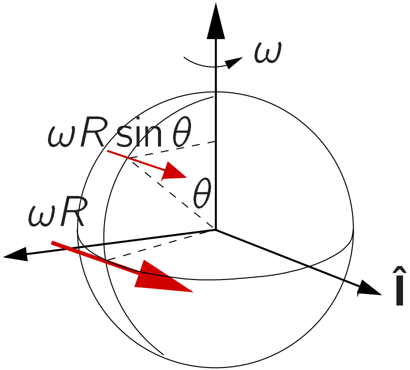


♣ Example /ctd

Consider a rocket on rails which stretch north from the equator. As rocket travels **north** it experiences the Coriolis force exerted by the rails:

2	γ	ω	$R\cos(\gamma t)$	$\hat{\mathbf{I}}$
+ve	-ve	+ve	+ve	

Force is opposed to $\hat{\mathbf{I}}$ (i.e. opposing earth's rotation).



Summary

We started by differentiating vectors wrt to a fixed coordinates.

Then looked at the properties of the derivative of a position vector **r** with respect to a general parameter p, then arc-length s, and time t

Derived Frenet-Serret relationships — a method of describing a 3D space curve by describing the change in a intrinsic coordinate system as it moves along the curve.

Discussed rotating coordinate systems, deriving an operator to evaluate the instantaneous rate of change of a vector as view in a fixed coordinate system.

We saw that there is coupled term in the acceleration, called the Coriolis acceleration.