Lecture 7

Gauss' and Stokes' Theorems

This section finally begins to deliver on why we introduced div grad and curl. Two theorems, both of them over two hundred years old, are explained:

Gauss' Theorem enables an integral taken over a volume to be replaced by one taken over the surface bounding that volume, and vice versa. Why would we want to do that? Computational efficiency and/or numerical accuracy!

Stokes' Law enables an integral taken around a closed curve to be replaced by one taken over *any* surface bounded by that curve.

7.1 Gauss' Theorem

Suppose that $\mathbf{a}(\mathbf{r})$ is a vector field and we want to compute the total flux of the field across the surface S that bounds a volume V. That is, we are interested in calculating:

$$\int_S \mathbf{a} \cdot d\mathbf{S} \tag{7.1}$$

where recall that $d\mathbf{S}$ is normal to the locally planar surface element and **must everywhere point out of the volume** as shown in Figure 7.1.

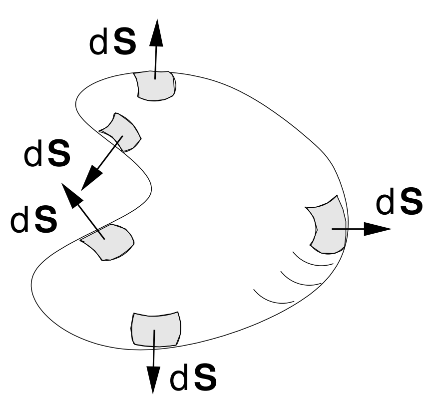


Figure 7.1: The surface element $d\mathbf{S}$ must stick out of the surface.

Gauss' Theorem tells us that we can do this by considering the total flux generated inside the volume V:

Gauss' Theorem

$$\int_S \mathbf{a} \cdot d\mathbf{S} = \int_V \text{div } \mathbf{a} \; dV \tag{7.2}$$

obtained by integrating the divergence over the entire volume.

7.1.1 Informal proof

An non-rigorous proof can be realized by recalling that we defined div by considering the efflux dE from the surfaces of an infinitesimal volume element

$$dE = \mathbf{a} \cdot d\mathbf{S} \tag{7.3}$$

and defining it as

$$\text{div } \mathbf{a} \; dV \quad = \quad dE \quad = \quad \mathbf{a} \cdot d\mathbf{S} \; . \tag{7.4}$$

If we sum over the volume elements, this results in a sum over the surface elements. But if two elemental surface touch, their $d\mathbf{S}$ vectors are in opposing direction and cancel as shown in Figure 7.2. Thus the sum over surface elements gives the overall bounding surface.

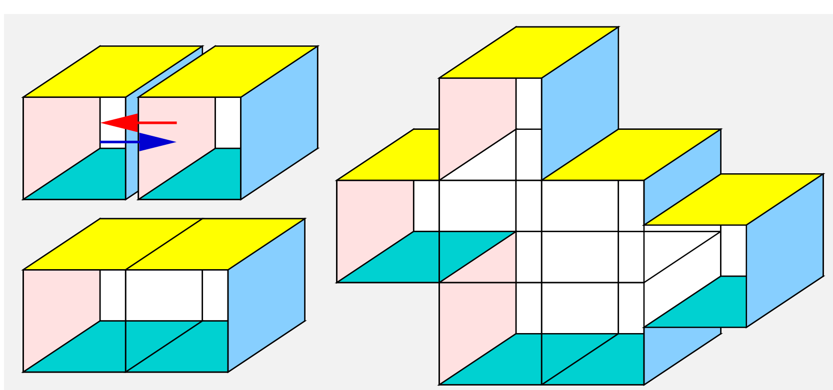


Figure 7.2: When two elements touch, the $d\mathbf{S}$ vectors at the common surface cancel out. One can imagine building the entire volume up from the infinitesimal units.

♣ Example of Gauss' Theorem

This is a typical example, in which the surface integral is rather tedious, whereas the volume integral is straightforward.

Q: Derive $\int_S \mathbf{a}\cdot d\mathbf{S}$ where $\mathbf{a} = z^3\hat{\mathbf{k}}$ and S is the surface of a sphere of radius R centred on the origin:

1. directly;
2. by applying Gauss' Theorem

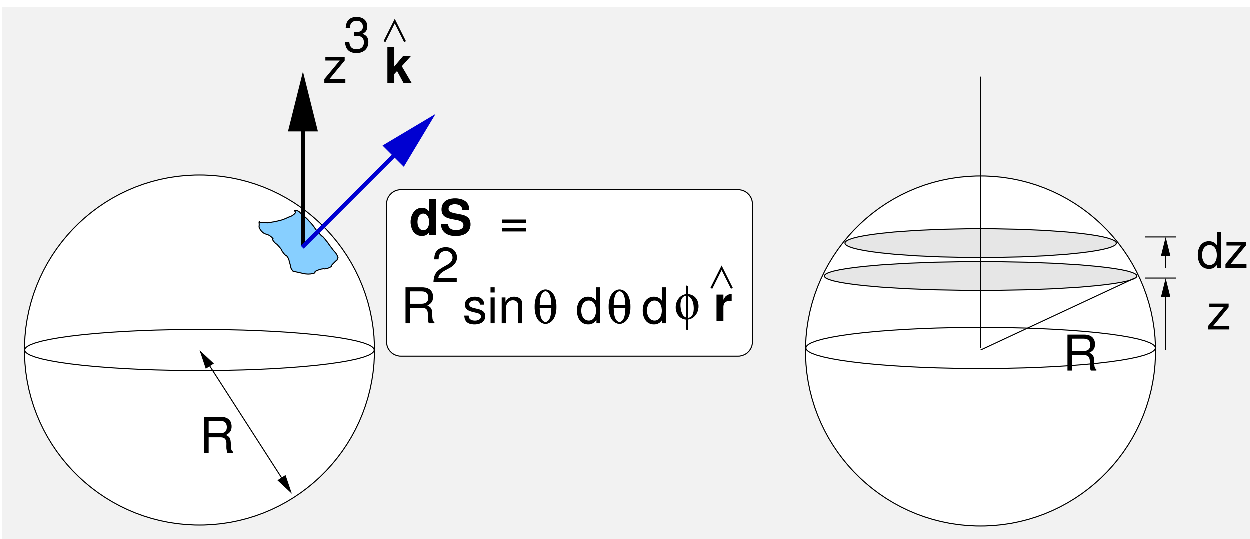


Figure 7.3:

A:

(1) On the surface of the sphere, $\mathbf{a} = R^3\cos^3\theta\hat{\mathbf{k}}$ and $d\mathbf{S} = R^2\sin\theta d\theta d\phi\hat{\mathbf{r}}$. Everywhere $\hat{\mathbf{r}}\cdot\hat{\mathbf{k}} = \cos\theta$.

$$\begin{aligned}
\Rightarrow \int_S \mathbf{a}\cdot d\mathbf{S} &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} R^3\cos^3\theta\ .\ R^2\sin\theta d\theta d\phi\ \hat{\mathbf{r}}\cdot\hat{\mathbf{k}} \qquad (7.5)\\
&= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} R^3\cos^3\theta\ .\ R^2\sin\theta d\theta d\phi\ .\ \cos\theta\\
&= 2\pi R^5\int_0^{\pi}\cos^4\theta\sin\theta d\theta\\
&= \frac{2\pi R^5}{5}\left[-\cos^5\theta\right]_0^{\pi} = \frac{4\pi R^5}{5}
\end{aligned}$$

(2) To apply Gauss' Theorem, we need to figure out div $\mathbf{a}$ and decide how to compute

the volume integral. The first is easy:

$$\mathrm{div}\mathbf{a} = 3z^2 \tag{7.6}$$

For the second, because div**a** involves just z, we can divide the sphere into discs of constant z and thickness dz, as shown in Fig. 7.3. Then

$$dV = \pi(R^2 - z^2)dz \tag{7.7}$$

and

$$\begin{aligned} \int_V \mathrm{div}\ \mathbf{a}dV &= 3\pi \int_{-R}^{R} z^2(R^2 - z^2)dz \\ &= 3\pi \left[\frac{R^2z^3}{3} - \frac{z^5}{5}\right]_{-R}^{R} \quad = \quad \frac{4\pi R^5}{5} \end{aligned} \tag{7.8}$$

7.2 Surface versus volume integrals

At first sight, it might seem that with a computer performing surface integrals might be better than a volume integral, perhaps because there are, somehow, "fewer elements". However, this is not the case. Imagine doing a surface integral over a wrinkly surface, say that of the moon. All the elements involved in the integration are "difficult" and must be modelled correctly. With a volume integral, most of the elements are not at the surface, and so the bulk of the integral is done without accurate modelling. The computation easier, faster, and better conditioned numerically.

7.3 Extension to Gauss' Theorem

Suppose the vector field $\mathbf{a}(\mathbf{r})$ is of the form $\mathbf{a} = U(\mathbf{r})\mathbf{c}$, where $U(\mathbf{r})$ as scalar field and $\mathbf{c}$ is a constant vector. Then, as we showed in the previous lecture,

$$\begin{aligned} \mathrm{div}\ \mathbf{a} &= \mathrm{grad}U \cdot \mathbf{c} + U\mathrm{div}\ \mathbf{c} \\ &= \mathrm{grad}U \cdot \mathbf{c} \end{aligned} \tag{7.9}$$

since div$\mathbf{c} = 0$ because $\mathbf{c}$ is constant.

Gauss' Theorem becomes

$$\int_S U\mathbf{c} \cdot d\mathbf{S} = \int_V \mathrm{grad}\ U \cdot \mathbf{c}dV \tag{7.10}$$

or, alternatively, taking the constant $\mathbf{c}$ out of the integrals

$$\mathbf{c} \cdot \left(\int_S Ud\mathbf{S}\right) = \mathbf{c} \cdot \left(\int_V \mathrm{grad}\ UdV\right) \tag{7.11}$$

This is still a scalar equation but we now note that the vector $\mathbf{c}$ is arbitrary so that the result must be true for any vector $\mathbf{c}$. This can be true only if the vector equation

$$\int_S U d\mathbf{S} = \int_V \text{grad } U dV \tag{7.12}$$

is satisfied.

If you think this is fishy, just write $\mathbf{c} = \hat{\mathbf{\imath}}$, then $\mathbf{c} = \hat{\mathbf{\jmath}}$, and $\mathbf{c} = \hat{\mathbf{k}}$ in turn, and you must obtain the three components of $\int_S U d\mathbf{S}$ in turn.

Further "extensions" can be obtained of course. For example one might be able to write the vector field of interest as

$$\mathbf{a}(\mathbf{r}) = \mathbf{b}(\mathbf{r}) \times \mathbf{c} \tag{7.13}$$

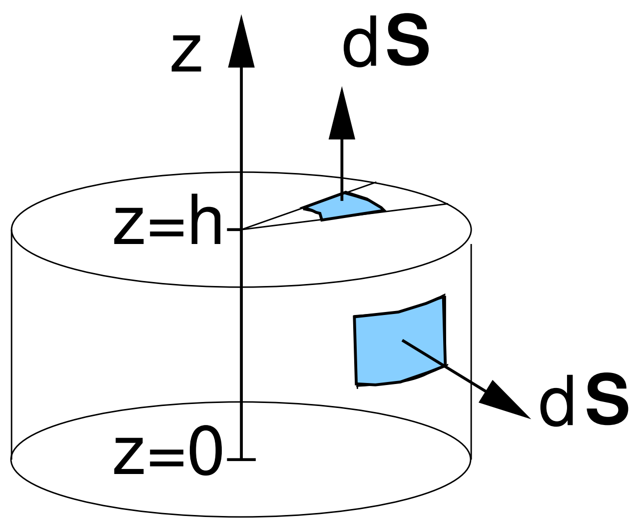
where $\mathbf{c}$ is a constant vector.

♣ Example of extension to Gauss' Theorem

Q $U = x^2 + y^2 + z^2$ is a scalar field, and volume V is the cylinder $x^2 + y^2 \leq a^2$, $0 \leq z \leq h$. Compute the surface integral

$$\int_S U d\mathbf{S} \tag{7.14}$$

over the surface of the cylinder.



A It is immediately clear from symmetry that there is no contribution from the curved surface of the cylinder since for every vector surface element there exists an equal and opposite element with the same value of U. We therefore need consider only the top and bottom faces.

Top face:

$$U = x^2 + y^2 + z^2 = r^2 + h^2 \text{ and } d\mathbf{S} = r dr d\phi \hat{\mathbf{k}} \tag{7.15}$$

so

$$\int U d\mathbf{S} = \int_{r=0}^{a} (h^2 + r^2) 2\pi r dr \int_{\phi=0}^{2\pi} d\phi \hat{\mathbf{k}} = \hat{\mathbf{k}}\pi \left[h^2 r^2 + \frac{1}{2} r^4 \right]_0^a = \pi [h^2 a^2 + \frac{1}{2} a^4] \hat{\mathbf{k}} \tag{7.16}$$

Bottom face:

$$U = r^2 \text{ and } d\mathbf{S} = -r\,dr\,d\phi\hat{\mathbf{k}} \tag{7.17}$$

The contribution from this face is thus $-\frac{\pi a^4}{2}\hat{\mathbf{k}}$, and the total integral is $\pi h^2 a^2 \hat{\mathbf{k}}$.

On the other hand, using Gauss' Theorem we have to compute

$$\int_V \text{grad } U dV \tag{7.18}$$

In this case, grad $U = 2\mathbf{r}$,

$$2\int_V (x\hat{\mathbf{i}} + y\hat{\mathbf{k}} + z\hat{\mathbf{k}})r\ dr\ dz\ d\phi \tag{7.19}$$

The integrations over x and y are zero by symmetry, so that the only remaining part is

$$2\int_{z=0}^{h} z dz \int_{r=0}^{a} r\ dr \int_{\phi=0}^{2\pi} d\phi\hat{\mathbf{k}} = \pi a^2 h^2 \hat{\mathbf{k}} \tag{7.20}$$

7.4 Stokes' Theorem

Stokes' Theorem relates a line integral around a closed path to a surface integral over what is called a *capping surface* of the path.

Stokes' Theorem states:

$$\oint_C \mathbf{a}\cdot d\mathbf{l} = \int_S \text{curl } \ \mathbf{a}\cdot d\mathbf{S} \tag{7.21}$$

where S is *any* surface capping the curve C.

Why have we used $d\mathbf{l}$ rather than $d\mathbf{r}$, where $\mathbf{r}$ is the position vector?

There is no good reason for this, as $d\mathbf{l} = d\mathbf{r}$. It just seems to be common usage in line integrals!

7.5 Informal proof

You will recall that in Lecture 5 that we defined curl as the circulation per unit area, and showed that

$$\sum_{\text{around elemental loop}} \mathbf{a} \cdot d\mathbf{l} = dC = (\nabla \times \mathbf{a}) \cdot d\mathbf{S} \quad . \tag{7.22}$$

Now if we add these little loops together, the internal line sections cancel out because the $d\mathbf{l}$'s are in opposite direction but the field $\mathbf{a}$ is not. This gives the larger surface and the larger bounding contour as shown in Fig. 7.4.

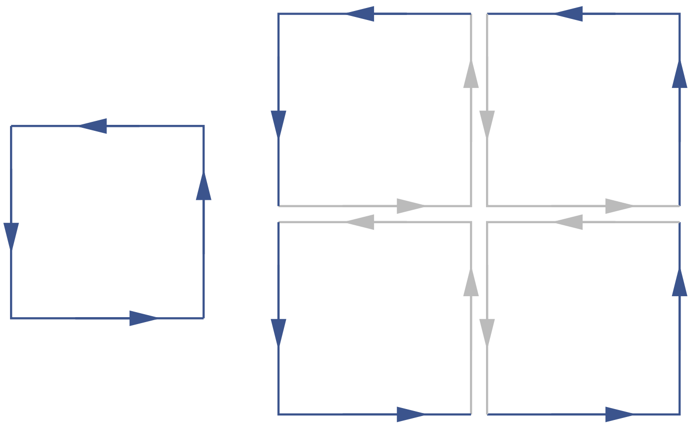


Figure 7.4: An example of an elementary loop, and how they combine together.

For a given contour, the capping surface can be ANY surface bound by the contour. The only requirement is that the surface element vectors point in the "general direction" of a right-handed screw with respect to the sense of the contour integral. See Fig. 7.5.

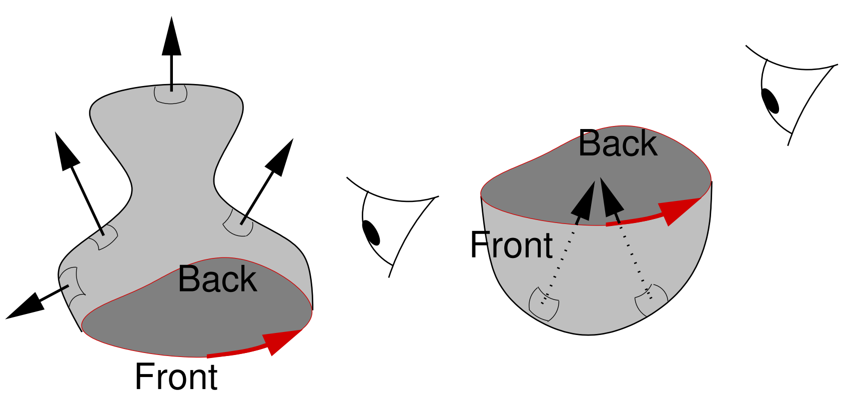


Figure 7.5: For a given contour, the bounding surface can be any shape. $d\mathbf{S}$'s must have a positive component in the sense of a r-h screw wrt the contour sense.

♣ Example of Stokes' Theorem

In practice, (and especially in exam questions!) the bounding contour is often planar, and the capping surface flat or hemispherical or cylindrical.

Q Vector field $\mathbf{a} = x^3\hat{\mathbf{j}} - y^3\hat{\mathbf{i}}$ and C is the circle of radius R centred on the origin. Derive

$$\oint_C \mathbf{a} \cdot d\mathbf{l} \tag{7.23}$$

directly and (ii) using Stokes' theorem where the surface is the planar surface bounded by the contour.

A(i) Directly. On the circle of radius R

$$\mathbf{a} = R^3(-\sin^3\theta\hat{\mathbf{i}} + \cos^3\theta\hat{\mathbf{j}}) \tag{7.24}$$

and

$$d\mathbf{l} = Rd\theta(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) \tag{7.25}$$

so that:

$$\oint_C \mathbf{a} \cdot d\mathbf{l} = \int_0^{2\pi} R^4(\sin^4\theta + \cos^4\theta)d\theta = \frac{3\pi}{2}R^4, \tag{7.26}$$

since

$$\int_0^{2\pi} \sin^4\theta d\theta = \int_0^{2\pi} \cos^4\theta d\theta = \frac{3\pi}{4} \tag{7.27}$$

A(ii) Using Stokes' theorem ...

$$\text{curl }\ \mathbf{a} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = 3(x^2+y^2)\hat{\mathbf{k}} = 3r^2\hat{\mathbf{k}} \tag{7.28}$$

We choose area elements to be circular strips of radius r thickness dr. Then

$$d\mathbf{S} = 2\pi r dr\hat{\mathbf{k}} \qquad \text{and} \qquad \int_S \text{curl } \mathbf{a} \cdot d\mathbf{S} = 6\pi \int_0^R r^3 dr = \frac{3\pi}{2}R^4 \tag{7.29}$$

7.6 An Extension to Stokes' Theorem

Just as we considered one extension to Gauss' theorem (not *really* an extension, more of a re-expression), so we will try something similar with Stoke's Theorem.

Again let $\mathbf{a}(\mathbf{r}) = U(\mathbf{r})\mathbf{c}$, where $\mathbf{c}$ is a constant vector. Then

$$\text{curl } \mathbf{a} = U\text{curl } \mathbf{c} + \text{grad } U \times \mathbf{c}) \tag{7.30}$$

Again, curl $\mathbf{c}$ is zero. Stokes' Theorem becomes in this case:

$$\oint_C U(\mathbf{c}\cdot d\mathbf{l}) = \int_S (\text{grad } U \times \mathbf{c}\cdot d\mathbf{S} = \int_S \mathbf{c}\cdot(d\mathbf{S}\times \text{grad } U) \tag{7.31}$$

or, rearranging the triple scalar products and taking the constant $\mathbf{c}$ out of the integrals gives

$$\mathbf{c}\cdot\oint_C U d\mathbf{l} = -\mathbf{c}\cdot\int_S \mathrm{grad} U \times d\mathbf{S} \quad . \tag{7.32}$$

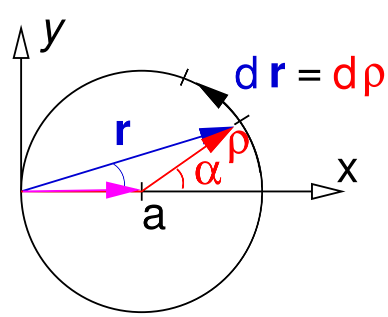
But $\mathbf{c}$ is arbitrary and so

$$\oint_C U d\mathbf{l} = -\int_S \text{grad } U \times d\mathbf{S} \tag{7.33}$$

7.7 ♣ Example of extension to Stokes' Theorem

Q Derive $\oint_C U d\mathbf{r}$ (i) directly and (ii) using Stokes', where $U = x^2+y^2+z^2$ and the line integral is taken around C the circle $(x-a)^2+y^2=a^2$ and $z=0$.

Note that, for no special reason, we have used $d\mathbf{r}$ here not $d\mathbf{l}$.



A(i) First some preamble.

If the circle were centred at the origin, we would write $d\mathbf{r} = ad\theta\hat{\boldsymbol{\theta}} = ad\theta(-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}})$. For such a circle the magnitude $r = |\mathbf{r}| = a$, a constant and so $dr = 0$.

However, in this example $d\mathbf{r}$ is not always in the direction of $\hat{\boldsymbol{\theta}}$, and $dr \neq 0$. Could you write down $d\mathbf{r}$? If not, revise Lecture 3, where we saw that in plane polars $x = r\cos\theta$, $y = r\sin\theta$ and the general expression is

$$d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} = (\cos\theta dr - r\sin\theta d\theta)\hat{\mathbf{i}} + (\sin\theta dr + r\cos\theta d\theta)\hat{\mathbf{j}} \tag{7.34}$$

To avoid having to find an expression for r in terms of θ, we will perform a coordinate transformation by writing $\mathbf{r} = [a, 0]^\top + \boldsymbol{\rho}$. So, $x = (a + \rho\cos\alpha)$ and $y = \rho\sin\alpha$, and on the circle itself where $\rho = a$

$$\mathbf{r} = a(1 + \cos\alpha)\hat{\mathbf{i}} + a\sin\alpha\hat{\mathbf{j}} \quad , \tag{7.35}$$

$$d\mathbf{r} = ad\alpha(-\sin\alpha\hat{\mathbf{i}} \ + \ \cos\alpha\hat{\mathbf{j}}) \quad , \tag{7.36}$$

and, as $z = 0$ on the circle,

$$U = a^2(1 + \cos\alpha)^2 + a^2\sin^2\alpha = 2a^2(1 + \cos\alpha) \ . \tag{7.37}$$

The line integral becomes

$$\oint U d\mathbf{r} = 2a^3 \int_{\alpha=0}^{2\pi} (1 + \cos\alpha)(-\sin\alpha\hat{\mathbf{i}} + \cos\alpha\hat{\mathbf{j}})d\alpha = 2\pi a^3\hat{\mathbf{j}} \tag{7.38}$$

A(ii) Now using Stokes' ...

For a planar surface covering the disc, the surface element can be written using the new parametrization as

$$d\mathbf{S} = \rho \ d\rho \ d\alpha\hat{\mathbf{k}} \tag{7.39}$$

Remember that $U = x^2 + y^2 + z^2 = r^2$, and as $z = 0$ in the plane

$$\text{grad } U = 2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = 2(a + \rho\cos\alpha)\hat{\mathbf{i}} + 2\rho\sin\alpha\hat{\mathbf{j}} \ . \tag{7.40}$$

Be careful to note that x, y are specified for any point on the disc, not on its circular boundary!

So

$$\begin{aligned} d\mathbf{S} \times \text{grad} U &= 2\rho \ d\rho \ d\alpha \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & 1 \\ (a + \rho\cos\alpha) & \rho\sin\alpha & 0 \end{vmatrix} \\ &= 2\rho[-\rho\sin\alpha\hat{\mathbf{i}} + (a + \rho\cos\alpha)\hat{\mathbf{j}}] \ d\rho \ d\alpha \end{aligned} \tag{7.41}$$

Both $\int_0^{2\pi} \sin\alpha d\alpha = 0$ and $\int_0^{2\pi} \cos\alpha d\alpha = 0$, so we are left with

$$\int_S d\mathbf{S} \times \text{grad} U = \int_{\rho=0}^{a} \int_{\alpha=0}^{2\pi} 2\rho a\hat{\mathbf{j}} \ d\rho \ d\alpha = 2\pi a^3\hat{\mathbf{j}} \tag{7.42}$$

Hurrah!

Lecture 8

Engineering applications

In Lecture 6 we saw one classic example of the application of vector calculus to Maxwell's equation.

In this lecture we explore a few more examples from fluid mechanics and heat transfer. As with Maxwell's eqations, the examples show how vector calculus provides a powerful way of representing underlying physics.

The power come from the fact that div, grad and curl have a significance or meaning which is more immediate than a collection of partial derivatives. Vector calculus will, with practice, become a convenient shorthand for you.

- Electricity – Ampère's Law
- Fluid Mechanics - The Continuity Equation
- Thermo: The Heat Conduction Equation
- Mechanics/Electrostatics - Conservative fields
- The Inverse Square Law of force
- Gravitational field due to distributed mass
- Gravitational field inside body
- Pressure forces in non-uniform flows

8.1 Electricity – Ampère's Law

If the frequency is low, the displacement current in Maxwell's equation curl$\mathbf{H} = \mathbf{J} + \partial\mathbf{D}/\partial t$ is negligible, and we find

$$\mathrm{curl}\mathbf{H} = \mathbf{J}$$

Hence

$$\int_S \mathrm{curl}\mathbf{H} \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

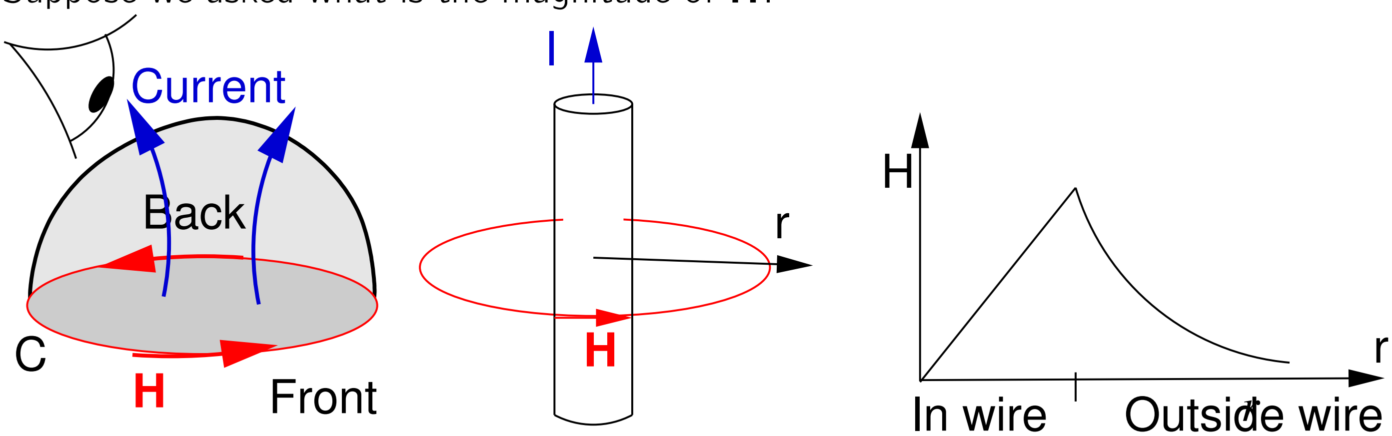
or

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$$

where $\int_S \mathbf{J} \cdot d\mathbf{S}$ is total current through the surface.

Now consider the $\mathbf{H}$ around a straight wire carrying current I. Symmetry tells us the $\mathbf{H}$ is in the $\hat{\boldsymbol{\theta}}$ direction, in a rhs screw sense with respect to the current. (You might check this against Biot-Savart's law.)

Suppose we asked what is the magnitude of $\mathbf{H}$?



Inside the wire, the bounding contour only encloses a fraction $(\pi r^2)/(\pi a^2)$ of the current, and so

$$\begin{aligned} H2\pi r &= \int \mathbf{J} \cdot d\mathbf{S} = I(r^2/A^2) \\ \Rightarrow H &= Ir/2\pi A^2 \end{aligned}$$

whereas outside we enclose all the current, and so

$$\begin{aligned} H2\pi r &= \int \mathbf{J} \cdot d\mathbf{S} = I \\ \Rightarrow H &= I/2\pi r \end{aligned}$$

A plot is shown in the Figure.

8.2 Fluid Mechanics - The Continuity Equation

The **Continuity Equation** expresses the condition of conservation of mass in a fluid flow. The continuity principle applied to *any* volume (called a *control volume*) may be expressed in words as follows:

> "The net rate of mass flow of fluid out of the control volume must equal the rate of decrease of the mass of fluid within the control volume"

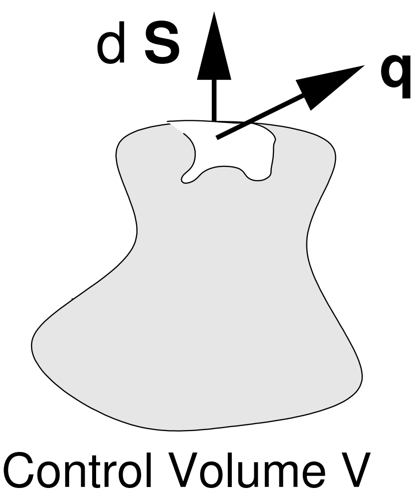


Figure 8.1:

To express the above as a mathematical equation, we denote the velocity of the fluid at each point of the flow by $\mathbf{q}(\mathbf{r})$ (a vector field) and the density by $\rho(\mathbf{r})$ (a scalar field). The element of rate-of-volume-loss through surface $d\mathbf{S}$ is $d\dot{V} = \mathbf{q} \cdot d\mathbf{S}$, so the rate of mass loss is

$$d\dot{M} = \rho\mathbf{q} \cdot d\mathbf{S},$$

so that the total rate of mass loss from the volume is

$$-\frac{\partial}{\partial t}\int_V \rho(\mathbf{r})dV = \int_S \rho\mathbf{q} \cdot d\mathbf{S}.$$

Assuming that the volume of interest is fixed, this is the same as

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_S \rho\mathbf{q} \cdot d\mathbf{S} \ .$$

Now we use Gauss' Theorem to transform the RHS into a volume integral

$$-\int_V \frac{\partial\rho}{\partial t}dV = \int_V \text{div }(\rho\mathbf{q})dV \ .$$

The two volume integrals can be equal for any control volume V only if the two integrands are equal at each point of the flow. This leads to the mathematical formulation of

The Continuity Equation:

$$\text{div } (\rho \mathbf{q}) = -\frac{\partial \rho}{\partial t}$$

Notice that if the density doesn't vary with time, div $(\rho \mathbf{q}) = 0$, and if the density doesn't vary with position then

The Continuity Equation for uniform, time-invariant density:

$$\text{div } (\mathbf{q}) = 0 \quad .$$

In this last case, we can say that the flow **q** is solenoidal.

8.3 Thermodynamics - The Heat Conduction Equation

Flow of heat is very similar to flow of fluid, and heat flow satisfies a similar continuity equation. The flow is characterized by the heat current density $\mathbf{q}(\mathbf{r})$ (heat flow per unit area and time), sometimes misleadingly called heat flux.

Assuming that there is no mass flow across the boundary of the control volume and no source of heat inside it, the rate of flow of heat out of the control volume by conduction must equal the rate of decrease of internal energy (constant volume) or enthalpy (constant pressure) within it. This leads to the equation

$$\text{div } \ \mathbf{q} = -\rho c \frac{\partial T}{\partial t},$$

where ρ is the density of the conducting medium, c its specific heat (both are assumed constant) and T is the temperature.

In order to solve for the temperature field another equation is required, linking q to the temperature gradient. This is

$$\mathbf{q} = -\kappa \text{grad } \ T,$$

where κ is the thermal conductivity of the medium. Combining the two equations gives the *heat conduction equation*:

$$-\text{div } \mathbf{q} = \kappa \text{div } \ \text{grad } \ T = \kappa \nabla^2 T = \rho c \frac{\partial T}{\partial t}$$

where it has been assumed that κ is a constant. In steady flow the temperature field satisfies Laplace's Equation $\nabla^2 T = 0$.

8.4 Mechanics - Conservative fields of force

A conservative field of force is one for which the work done

$$\int_A^B \mathbf{F} \cdot d\mathbf{r},$$

moving from A to B is indep. of path taken. As we saw in Lecture 4, conservative fields must satisfy the condition

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0,$$

Stokes' tells us that this is

$$\int_S \text{curl } \mathbf{F} \cdot d\mathbf{S} = 0,$$

where S is *any* surface bounded by C.

But if true for *any* C containing A and B, it must be that

$$\text{curl } \mathbf{F} = 0$$

Conservative fields are irrotational
All radial fields are irrotational

One way (actually the only way) of satisfying this condition is for

$$\mathbf{F} = \nabla\ U$$

The scalar field $U(\mathbf{r})$ is the Potential Function

8.5 The Inverse Square Law of force

Radial forces are found in electrostatics and gravitation — so they are certainly irrotational and conservative.

But in nature these radial forces are also inverse square laws. One reason why this may be so is that it turns out to be the only central force field which is **solenoidal**, i.e. has zero divergence.

If $\mathbf{F} = f(r)\mathbf{r}$,

$$\text{div } \mathbf{F} = 3f(r) + rf'(r).$$

For div $\mathbf{F} = 0$ we conclude

$$r\frac{df}{dr} + 3f = 0$$

or

$$\frac{df}{f} + 3\frac{dr}{r} = 0.$$

Integrating with respect to r gives $fr^3 = const = A$, so that

$$\mathbf{F} = \frac{A\mathbf{r}}{r^3}, \quad |\mathbf{F}| = \frac{A}{r^2}.$$

The condition of zero divergence of the inverse square force field applies everywhere except at $\mathbf{r} = 0$, where the divergence is infinite.

To show this, calculate the outward normal flux out of a sphere of radius R centered on the origin when $\mathbf{F} = F\hat{\mathbf{r}}$. This is

$$\int_S \mathbf{F} \cdot d\mathbf{S} = F \int_{\text{Sphere}} \hat{\mathbf{r}} \cdot d\mathbf{S} = F \int_{\text{Sphere}} d = F4\pi R^2 = 4\pi A = \text{Constant}.$$

Gauss tells us that this flux must be equal to

$$\int_V \text{div } \mathbf{F} dV = \int_0^R \text{div } \mathbf{F} 4\pi r^2 dr$$

where we have done the volume integral as a summation over thin shells of surface area $4\pi r^2$ and thickness dr.

But for all finite r, div$\mathbf{F} = 0$, so div$\mathbf{F}$ must be infinite at the origin.

The flux integral is thus

- zero — for any volume which does not contain the origin
- $4\pi A$ for any volume which does contain it.

8.6 Gravitational field due to distributed mass: Poisson's Equation

If one tried the same approach as §8.4 for the gravitational field, $A = Gm$, where m is the mass at the origin and G the universal gravitational constant, one would run into the problem that there is no such thing as point mass.

We can make progress though by considering distributed mass.

The mass contained in each small volume element dV is ρdV and this will make a contribution $-4\pi\rho G dV$ to the flux integral from the control volume. Mass outside the control volume makes no contribution, so that we obtain the equation

$$\int_S \mathbf{F} \cdot d\mathbf{S} = -4\pi G \int_V \rho dV.$$

Transforming the left hand integral by Gauss' Theorem gives

$$\int_V \text{div } \mathbf{F} dV = -4\pi G \int_V \rho dV$$

which, since it is true for any V, implies that

$$-\text{div } \mathbf{F} = 4\pi\rho G.$$

Since the gravitational field is also conservative (i.e. irrotational) it must have an associated potential function U, so that $\mathbf{F} = \text{grad } U$. It follows that the gravitational potential U satisfies

Poisson's Equation

$$\nabla^2 U = 4\pi\rho G \ .$$

Using the integral form of Poisson's equation, it is possible to calculate the gravitational field inside a spherical body whose density is a function of radius only. We have

$$4\pi R^2 F = 4\pi G \int_0^R 4\pi r^2 \rho dr,$$

where $F = |\mathbf{F}|$, or

$$|F| = \frac{G}{R^2} \int_0^R 4\pi r^2 \rho dr = \frac{MG}{R^2},$$

where M is the total mass inside radius R. For the case of uniform density, this is equal to $M = \frac{4}{3}\pi\rho R^3$ and $|F| = \frac{4}{3}\pi\rho G R$.

8.7 Pressure forces in non-uniform flows

When a body is immersed in a flow it experiences a net pressure force

$$\mathbf{F}_p = -\int_S p d\mathbf{S},$$

where S is the surface of the body. If the pressure p is non-uniform, this integral is not zero. The integral can be transformed using Gauss' Theorem to give the alternative expression

$$\mathbf{F}_p = -\int_V \text{grad } p \; dV,$$

where V is the volume of the body. In the simple hydrostatic case $p + \rho g z = constant$, so that

$$\text{grad } p = -\rho g k$$

and the net pressure force is simply

$$\mathbf{F}_p = g\hat{\mathbf{k}} \int_V \rho dV$$

which, in agreement with Archimedes' principle, is equal to the weight of fluid displaced.