Classical Mechanics

Rigid Body Motion

- ☞ Inertia Tensor
- ☞ Rotational Kinetic Energy
- ☞ Principal Axes of Rotation
- ☞ Steiner's Theorem
- ☞ Euler's Equations for a Rigid Body
- ☞ Eulerian Angles

Review of Fundamental Equations

- **Rigid body ☞ collection of a large number of small mass elements which all maintain a fixed spatial relationship with respect to one another**

If there are N elements and the ith element:

- has mass m_i
- instantaneous position vector $\vec{r}_i$

$$\Downarrow$$

the equation of motion of the ith element is written

$$m_i \frac{d^2 \vec{r}_i}{dt^2} = \sum_{j=1,N}^{j \neq i} \vec{f}_{ij} + \vec{F}_i$$

- $\vec{f}_{ij}$ ☞ internal force exerted on the ith element by the jth element
- $\vec{F}_i$ the external force acting on the ith element

- **Internal forces $\vec{f}_{ij}$ ☞ stresses which develop within the body to ensure its various elements maintain a constant spatial relationship with respect to one another**
- **Of course ☞ $\vec{f}_{ij} = -\vec{f}_{ji}$ by Newton's third law**
- **The external forces represent forces which originate outside the body**

Review of Fundamental Equations (cont'd)

Recall that summing over all mass elements

$$\Downarrow$$

$$M\,\frac{d^2\vec{r}_{cm}}{dt^2} = \vec{F}$$

- $M = \sum_{i=1,N} m_i$ ☞ the total mass
- $\vec{r}_{\rm cm}$ ☞ position vector of the center of mass
- $\vec{F} = \sum_{i=1,N} \vec{F}_i$ ☞ total external force

Recall that the center of mass of a rigid body moves under the action of the external forces as a point particle whose mass is identical with that of the body

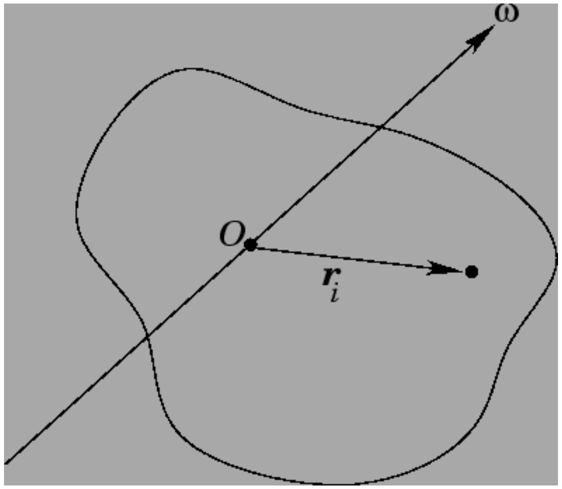
Likewise

$$\frac{d\vec{L}}{dt} = \vec{\tau}$$

- $\vec{L} = \sum_{i=1,N} m_i\,\vec{r}_i \times d\vec{r}_i/dt$ ☞ total angular momentum of the body
- $\vec{\tau} = \sum_{i=1,N} \vec{r}_i \times \vec{F}_i$ the total external torque

Moment of Inertia Tensor

Consider a rigid body rotating with fixed angular velocity ω
about an axis which passes through the origin



Let $\vec{r}_i$ be the position vector of the ith mass element whose mass is m_i

This position vector to precesses about the axis of rotation with angular velocity $\vec{\omega}$

$$\frac{d\vec{r}_i}{dt} = \vec{\omega} \times \vec{r}_i$$

This equation specifies the velocity $\vec{v}_i = d\vec{r}_i/dt$ of each mass element as the body rotates with fixed angular velocity $\vec{\omega}$ about an axis passing through the origin

The total angular momentum of the body (about the origin) is written

$$\vec{L} = \sum_{i=1,N} m_i \, \vec{r}_i \times \frac{d\vec{r}_i}{dt} = \sum_{i=1,N} m_i \, \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \sum_{i=1,N} m_i \, \left[r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega}) \, \vec{r}_i \right]$$

Moment of Inertia Tensor (cont'd)

The ith component reads

$$
\begin{aligned}
L_i &= \sum_\alpha m_\alpha \left(\omega_i \sum_k x_{\alpha,k}^2 - x_{\alpha,i} \sum_j x_{\alpha,j} \omega_j \right) \\
&= \sum_\alpha m_\alpha \sum_j \left(\omega_j \delta_{ij} \sum_k x_{\alpha,k}^2 - \omega_j x_{\alpha,i} x_{\alpha,j} \right) \\
&= \sum_j \omega_j \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j} \right)
\end{aligned}
$$

$$\Downarrow$$

$$L_i = \sum_j I_{ij}\, \omega_j$$

Moment of Inertia Tensor (cont'd)

The previous equation can be written in a matrix form

$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$I_{xx} = \sum_{i=1,N} (y_i^2 + z_i^2)\, m_i = \int (y^2 + z^2)\, dm$$

$$I_{yy} = \sum_{i=1,N} (x_i^2 + z_i^2)\, m_i = \int (x^2 + z^2)\, dm$$

$$I_{zz} = \sum_{i=1,N} (x_i^2 + y_i^2)\, m_i = \int (x^2 + y^2)\, dm$$

$$I_{xy} = I_{yx} = -\sum_{i=1,N} x_i\, y_i\, m_i = -\int x\, y\, dm$$

$$I_{yz} = I_{zy} = -\sum_{i=1,N} y_i\, z_i\, m_i = -\int y\, z\, dm$$

$$I_{xz} = I_{zx} = -\sum_{i=1,N} x_i\, z_i\, m_i = -\int x\, z\, dm$$

Moment of Inertia Tensor (cont'd)

- I_{xx} ☞ moment of inertia about the x-axis
- I_{yy} ☞ moment of inertia about the y-axis
- I_{zz} ☞ moment of inertia about the z-axis
- I_{ij} ☞ the ij product of inertia $i \neq j$

The matrix of the I_{ij} values is known as the moment of inertia tensor

☛ a sum over separate mass elements

Each component of the moment of inertia tensor can be written as

☛ an integral over infinitesimal mass elements

In the integrals

- $dm = \rho \, dV$
- ρ is the mass density
- dV a volume element

The total angular momentum of the body can be written more succinctly as

$$\vec{L} = \mathbf{I}\,\vec{\omega} \qquad (*)$$

$\vec{L}$ and $\vec{\omega}$ are both column vectors and $\mathbf{I}$ is the matrix of the I_{ij} values

Note that $\mathbf{I}$ is a real symmetric matrix ☞ $I^*_{ij} = I_{ij}$ and $I_{ji} = I_{ij}$

Although the above results were obtained assuming a fixed angular velocity they remain valid at each instant in time even if the angular velocity varies

Rotational Kinetic Energy

The instantaneous rotational kinetic energy of a rotating rigid body is

$$T = \frac{1}{2} \sum_{i=1,N} m_i \left(\frac{d\vec{r}_i}{dt} \right)^2$$

$$\Downarrow$$

$$T = \frac{1}{2} \sum_{i=1,N} m_i \left(\vec{\omega} \times \vec{r}_i\right) \cdot \left(\vec{\omega} \times \vec{r}_i\right) = \frac{1}{2} \vec{\omega} \cdot \sum_{i=1,N} m_i \vec{r}_i \times \left(\vec{\omega} \times \vec{r}_i\right)$$

$$\Downarrow$$

$$T = \frac{1}{2}\, \vec{\omega} \cdot \vec{L}$$

or equivalently

$$T = \frac{1}{2}\, \vec{\omega}^T\ \mathbf{I}\ \vec{\omega}$$

$\vec{\omega}^T$ ☞ row vector of the Cartesian components (ω_x, ω_y, ω_z) which is the transpose (denoted ^{T}) of the column vector $\vec{\omega}$

when written in component form

$$T = \frac{1}{2} \left(I_{xx}\, \omega_x^2 + I_{yy}\, \omega_y^2 + I_{zz} \omega_z^2 + 2 I_{xy}\, \omega_x\, \omega_y + 2 I_{yz}\, \omega_y\, \omega_z + 2\, I_{xz}\, \omega_x\, \omega_z \right)$$

Matrix Theory

It is time to review a little matrix theory

If $\mathbf{A}$ is a real symmetric matrix of dimension n

$$\Downarrow$$

$$\mathbf{A}^* = \mathbf{A} \qquad \text{and} \qquad \mathbf{A}^T = \mathbf{A}$$

- * denotes a complex conjugate
- ^{T} denotes a transpose

Consider the matrix equation

$$\mathbf{A}\,\mathbf{x} = \lambda\,\mathbf{x}$$

- vector $\mathbf{x}$ ☞ eigenvector of $\mathbf{A}$
- the associated number λ ☞ eigenvalue of $\mathbf{A}$

Properties of the eigenvectors and eigenvalues of a real symmetric matrix

the matrix equation can be rearranged to give

$$(\mathbf{A} - \lambda\,\mathbf{1})\,\mathbf{x} = \mathbf{0}$$

where $\mathbf{1}$ is the unit matrix

Matrix Theory (cont'd)

Matrix Equation

$\Downarrow$

set of n homogeneous simultaneous algebraic equations for the n components of $\mathbf{x}$

such a set of equations only has a non-trivial solution
when the determinant of the associated matrix is set to zero

$\Downarrow$

a necessary condition for the set of equations to have a non-trivial solution is

$$|\mathbf{A} - \lambda\,\mathbf{1}| = 0$$

The above formula is essentially an nth-order polynomial equation for λ

We know that such an equation has n (possibly complex) roots

$\Downarrow$

CONCLUSION

there are n eigenvalues and n associated eigenvectors of the n-D matrix $\mathbf{A}$

Matrix Theory (cont'd)

THEOREM

The n eigenvalues and eigenvectors of the real symmetric matrix $\mathbf{A}$ are all real

$$\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i \qquad (\dagger)$$

taking the transpose and complex conjugate

$$\mathbf{x}_i^{*\,T}\,\mathbf{A} = \lambda_i^*\,\mathbf{x}_i^{*\,T} \qquad (\ddagger)$$

where $\mathbf{x}_i$ and λ_i are the ith eigenvector and eigenvalue of $\mathbf{A}$ respectively

Left multiplying Eq. $(\dagger)$ by $\mathbf{x}_i^{*\,T}$ we obtain

$$\mathbf{x}_i^{*\,T}\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i^{*\,T}\mathbf{x}_i$$

Right multiplying $(\ddagger)$ by $\mathbf{x}_i$ we get

$$\mathbf{x}_i^{*\,T}\,\mathbf{A}\,\mathbf{x}_i = \lambda_i^*\,\mathbf{x}_i^{*\,T}\mathbf{x}_i$$

The difference of the previous two equations yields

$$(\lambda_i - \lambda_i^*)\,\mathbf{x}_i^{*\,T}\mathbf{x}_i = 0$$

$\lambda_i = \lambda_i^* \Leftrightarrow \mathbf{x}_i^{*\,T}\mathbf{x}_i$ (which is $\mathbf{x}_i^* \cdot \mathbf{x}_i$ in vector notation) is positive definite $\downarrow$

λ_i is real ☞ it follows that $\mathbf{x}_i$ is real

Matrix Theory (cont'd)

THEOREM

2 eigenvectors corresponding to 2 different eigenvalues are mutually orthogonal

Let $\mathbf{A}\,\mathbf{x}_i = \lambda_i\,\mathbf{x}_i$ and $\mathbf{A}\,\mathbf{x}_j = \lambda_j\,\mathbf{x}_j$ ☞ $\lambda_i \neq \lambda_j$

Taking the transpose of the first equation and right multiplying by $\mathbf{x}_j$ and left multiplying the second equation by $\mathbf{x}_i^T$ we obtain

$$\mathbf{x}_i^T\,\mathbf{A}\,\mathbf{x}_j = \lambda_i\,\mathbf{x}_i^T\mathbf{x}_j \quad \text{and} \quad \mathbf{x}_i^T\,\mathbf{A}\,\mathbf{x}_j = \lambda_j\,\mathbf{x}_i^T\mathbf{x}_j$$

Taking the difference of these equations we get

$$(\lambda_i - \lambda_j)\,\mathbf{x}_i^T\mathbf{x}_j = 0$$

Since by hypothesis $\lambda_i \neq \lambda_j$ ☞ it follows that $\mathbf{x}_i^T\mathbf{x}_j = 0$

In vector notation ☞ $\mathbf{x}_i \cdot \mathbf{x}_j = 0$

$$\Downarrow$$

the eigenvectors x_i and x_j are mutually orthogonal

Matrix Theory (cont'd)

If $\lambda_i = \lambda_j = \lambda$

$\Downarrow$

we cannot conclude that $\mathbf{x}_i^T \mathbf{x}_j = 0$ by the above argument

NEVERTHELESS

it is easily seen that any linear combination of $\mathbf{x}_i$ and $\mathbf{x}_j$
is an eigenvector of $\mathbf{A}$ with eigenvalue λ

$\Downarrow$

it is possible to define two new eigenvectors of $\mathbf{A}$ which are mutually orthogonal
(with the eigenvalue λ)

For example

$$\mathbf{x}_i' = \mathbf{x}_i \qquad \text{and} \qquad \mathbf{x}_j' = \mathbf{x}_j - \left(\frac{\mathbf{x}_i^T \mathbf{x}_j}{\mathbf{x}_i^T \mathbf{x}_i} \right) \mathbf{x}_i$$

This can be generalized to any number of eigenvalues which take the same value

CONCLUSION

A real symmetric n-dimensional matrix possesses
n real eigenvalues with n associated real eigenvectors
which are or can be chosen to be mutually orthogonal

Principal Axes of Rotation

The moment of inertia tensor $\mathbf{I}$
takes the form of a real symmetric three-dimensional matrix

$\Downarrow$

from the matrix theory which we have just reviewed
the moment of inertia tensor possesses
3 mutually orthogonal eigenvectors which are associated with 3 real eigenvalues

Let the ith eigenvector be denoted $\hat{\boldsymbol{\omega}}_i$ and the ith eigenvalue λ_i
(which can be normalized to be a unit vector)

$\Downarrow$

$$\mathbf{I}\,\hat{\boldsymbol{\omega}}_i = \lambda_i\,\hat{\boldsymbol{\omega}}_i \qquad i = 1, 2, 3 \qquad (\star)$$

The directions of the 3 mutually orthogonal unit vectors $\hat{\boldsymbol{\omega}}_i$
define the 3 so-called principal axes of rotation of the rigid body

These axes are special because when the body rotates about one of them
(i.e., when ω is parallel to one of them)
the angular momentum vector $\mathbf{L}$ becomes parallel to the angular velocity vector ω

This can be seen from a comparison of Eq. $(*)$ and Eq. $(\star)$

Principal Axes of Rotation (cont'd)

Next ☞ reorient the Cartesian coordinate axes
so the they coincide with the mutually orthogonal principal axes of rotation

In this new reference frame the eigenvectors of $\mathbf{I}$ are the unit vectors

$$\mathbf{e}_x \qquad \mathbf{e}_y \qquad \mathbf{e}_z$$

and the eigenvalues are the moments of inertia about these axes

$$I_{xx} \qquad I_{yy} \qquad I_{zz}$$

These latter quantities are referred to as the principal moments of inertia
Note that the products of inertia are all zero in the new reference frame

In this frame the moment of inertia tensor takes the form of a diagonal matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

Principal Axes of Rotation (cont'd)

HOMEWORK

(I) verify that:

- $\mathbf{e}_x$
- $\mathbf{e}_y$
- $\mathbf{e}_z$

are indeed the eigenvectors of the matrix

$$\mathbf{I} = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

with the eigenvalues

- I_{xx}
- I_{yy}
- I_{zz}

(II) verify that:

$\mathbf{L} = \mathbf{I}\,\vec{\omega}$ is indeed parallel to ω
whenever ω is directed along $\mathbf{e}_x$, $\mathbf{e}_y$, or $\mathbf{e}_z$

Principal Axes of Rotation (cont'd)

When expressed in the new coordinate system, Eq. $(*)$ yields

$$\mathbf{L} = (I_{xx}\,\omega_x,\, I_{yy}\,\omega_y, I_{zz}\,\omega_z)$$

$$\Downarrow$$

the rotational kinetic energy becomes

$$T = \frac{1}{2}\left(I_{xx}\,\omega_x^2 + I_{yy}\,\omega_y^2 + I_{zz}\,\omega_z^2\right)$$

CONCLUSION

There are many great simplifications to be had by choosing a coordinate system whose axes coincide with the principal axes of rotation of the rigid body

But how do we determine the directions of the principal axes in practice?

Principal Axes of Rotation (cont'd)

In general ☞ we have to solve the eigenvalue equation

$$\mathbf{I}\,\hat{\boldsymbol{\omega}} = \lambda\,\hat{\boldsymbol{\omega}}$$

or

$$\begin{pmatrix} I_{xx}-\lambda & I_{xy} & I_{xz} \\ I_{yx} & I_{yy}-\lambda & I_{yz} \\ I_{zx} & I_{zy} & I_{zz}-\lambda \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

✐ $\hat{\boldsymbol{\omega}} = (\cos\alpha,\ \cos\,\beta,\ \cos\gamma)$

➡ $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

✎ α is the angle the unit eigenvector subtends with the x-axis

β the angle it subtends with the y-axis

γ the angle it subtends with the z-axis

Unfortunately

the analytic solution of the above matrix equation is generally quite difficult

Fortunately

sometimes the rigid body under investigation possesses some kind of symmetry so that at least one principal axis can be found by inspection

$\Downarrow$

the other two principal axes can be determined as follows

Principal Axes of Rotation (cont'd)

If the z-axis is known to be a principal axes in some coordinate system

$\Downarrow$

the two products of inertia I_{xz} and I_{yz} are zero
otherwise ☞ $(0, 0, 1)$ would not be an eigenvector

The other two principal axes must lie in the x-y plane ☞ $\cos\gamma = 0$

$\Downarrow$

$\cos\beta = \sin\alpha$, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

The first two rows in the matrix equation thus reduce to

$$(I_{xx} - \lambda)\cos\alpha + I_{xy}\sin\alpha = 0 \qquad (\spadesuit)$$

$$I_{xy}\cos\alpha + (I_{yy} - \lambda)\sin\alpha = 0$$

Eliminating λ between the above two equations, we obtain

$$I_{xy}\,(1 - \tan^2\alpha) = (I_{xx} - I_{yy})\tan\alpha \qquad (\clubsuit)$$

Principal Axes of Rotation (cont'd)

Now ☞ $\tan(2\,\alpha) \equiv 2\,\tan\alpha/(1-\tan^2\alpha)$

$\Downarrow$

Eq. (♣) yields

$$\tan(2\,\alpha) = \frac{2\,I_{xy}}{I_{xx}-I_{yy}} \qquad (\heartsuit)$$

There are two values of α which satisfy the above equation
(lying between $-\pi/2$ and $\pi/2$)

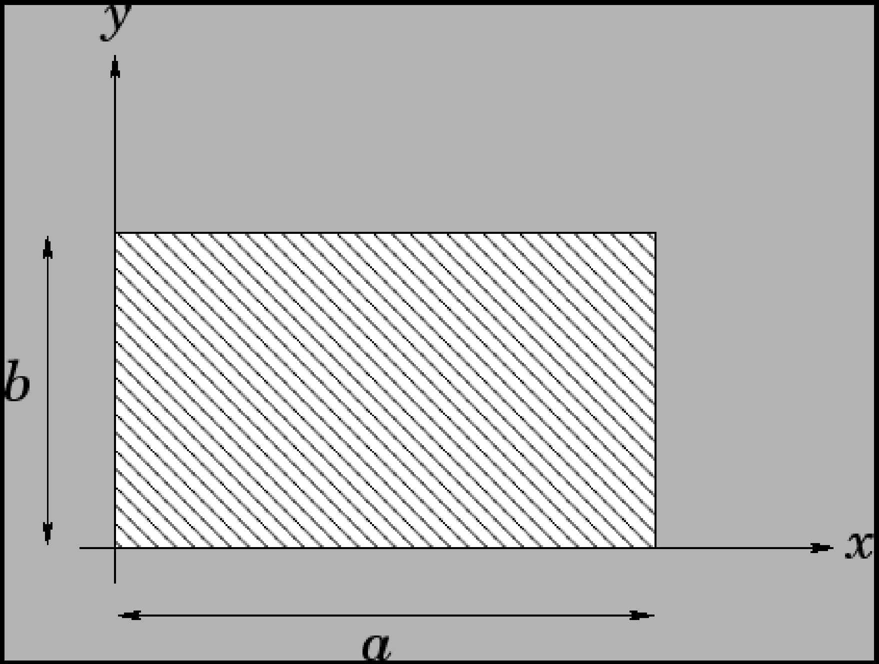
These specify the angles α
which the two mutually orthogonal principal axes in the x-y plane
make with the x-axis

$\Downarrow$

we have now determined the directions of all three principal axes

Incidentally ☞ once we have determined the orientation angle α of a principal axis
we can substitute back into Eq. (♠) to obtain
the corresponding principal moment of inertia λ

Principal Axes of Rotation (cont'd)



Consider a uniform rectangular lamina of mass m and sides a and b which lies in the x-y plane

Suppose that the axis of rotation passes through the origin
(i.e., through a corner of the lamina)

Since $z = 0$ throughout the lamina ☞ $I_{xz} = I_{yz} = 0$

$$\Downarrow$$

the z-axis is a principal axis

Principal Axes of Rotation (cont'd)

After some straightforward integration

$$I_{xx} = \frac{1}{3}\, m\, b^2 \qquad I_{yy} = \frac{1}{3}\, m\, a^2 \qquad I_{xy} = -\frac{1}{4}\, m\, a\, b$$

$$\Downarrow$$

from Eq. (♡)

$$\Downarrow$$

$$\alpha = \frac{1}{2} \tan^{-1}\left(\frac{3}{2}\,\frac{a\, b}{a^2 - b^2}\right)$$

which specifies the orientation of the two principal axes which lie in the x-y plane

For the special case where $a = b$

$$\Downarrow$$

$$\alpha = \pi/4 \text{ and } \alpha = 3\pi/4$$

i.e., the two in-plane principal axes of a square lamina
(at a corner)
are parallel to the two diagonals of the lamina

Steiner's Theorem

For the kinetic energy to be separable into translational and rotational portions

$\Downarrow$

choose a body coordinate system whose origin is the c.m. of the body

For certain geometrical shapes it may not be always convinient to compute the elements of the inertia tensor using such a coordinate system

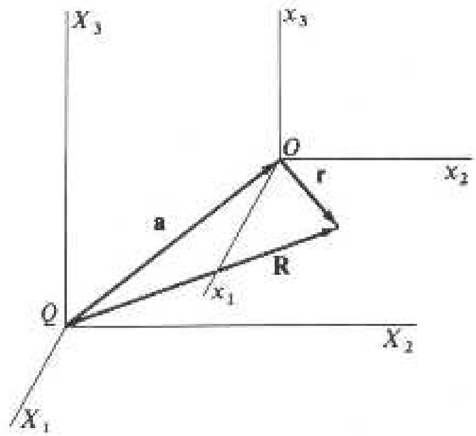
$\Downarrow$

Consider some other set of coordinate axis X_i
(also fixed with respect to the body)
having the same orientation that x_i-axes
but with origin Q that does not corresponds with the c.m. origin O
(origin Q may be located either outside or within the body)

The elements of the inertia tensor relative to X_i-axes can be written as

$$J_{ij} = \sum_\alpha m_\alpha \left(\delta_{ij} \sum_k X_{\alpha,k}^2 - X_{\alpha_i} X_{\alpha,j} \right)$$

Steiner's Theorem (cont'd)



If the vector connecting Q and O is $\vec{a}$

$$\Downarrow$$

$$\vec{R} = \vec{a} + \vec{r}$$

with components

$$X_i = a_i + x_i$$

The tensor element J_{ij} then becomes

$$J_{ij} = \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k (x_{\alpha,k} + a_k)^2 - (x_{\alpha,j} + a_i)(x_{\alpha,i} + a_j) \right]$$

Steiner's Theorem (cont'd)

$$
\begin{aligned}
J_{ij} &= \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k x_{\alpha,k}^2 - x_{\alpha,i} x_{\alpha,j}\right] \\
&+ \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k (2x_{\alpha,k} a_k + a_k^2) - (a_i x_{\alpha,j} + a_j x_{\alpha,i} + a_i a_j)\right]
\end{aligned}
$$

Identifying the first summation as I_{ij} and regrouping

$$
\begin{aligned}
J_{ij} &= I_{ij} + \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k a_k^2 - a_i a_j\right] \\
&+ \sum_\alpha m_\alpha \left[2\delta_{ij} \sum_k (x_{\alpha,k} a_k - a_i x_{\alpha,j} - a_j x_{\alpha,i})\right]
\end{aligned}
$$

Each term in the last sumation involves a term of the form $\sum_\alpha m_\alpha x_{\alpha,k}$
but because O is located at the c.m. ☞ $\sum_\alpha m_\alpha r_\alpha = 0$
or for the kth component ☞ $\sum_\alpha m_\alpha x_{\alpha,k} = 0$

$$\Downarrow$$

all such terms vanish

Steiner's Theorem (cont'd)

$$J_{ij} = I_{ij} + \sum_\alpha m_\alpha \left[\delta_{ij} \sum_k a_k^2 - a_i a_j \right]$$

✎ $\sum_\alpha m_\alpha = M$
✎ $\sum_k a_k^2 \equiv a^2$

$\Downarrow$

solving for I_{ij} we have the result

$$I_{ij} = J_{ij} - M(a^2 \delta_{ij} - a_i a_j)$$

which allows the calculation of the elements I_{ij} of the desired inertia tensor (with origin at c.m.)
once those with respect to X_i-axes are known

The second term in the right hand side is the inertia tensor referred to the origin Q for a point mass M

The above equation is the general form of Steiner's parallel-axis theorem (1796-1863)

Euler's Equations for a Rigid Body

The fundamental equation of motion of a rotating body

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

is only valid in an inertial frame

However ☞ we have seen that $\vec{L}$ is most simply expressed in a frame of reference whose axes are aligned along the principal axes of rotation of the body

Such a frame of reference rotates with the body and is therefore non-inertial

$\Downarrow$

it is helpful to define two Cartesian coordinate systems with the same origins

coordinates x, y, z denote the fixed inertial frame

coordinates x', y', z' co-rotates with the body in such a manner that the x'-, y'-, and z'-axes are always pointing along its principal axes of rotation

Since this body frame co-rotates with the body its instantaneous angular velocity is the same as that of the body

$\Downarrow$

$$\frac{d\vec{L}}{dt} = \frac{d\vec{L}}{dt'} + \vec{\omega} \times \vec{L}$$

Euler's Equations for a Rigid Body (cont'd)

Recall that

- d/dt is the time derivative in the fixed frame
- d/dt' the time derivative in the body frame

The equation of motion of a rotating body can be re-written as

$$\vec{\tau} = \frac{d\vec{L}}{dt'} + \vec{\omega} \times \vec{L} \qquad (\aleph)$$

In the body frame let

$\vec{\tau} = (\tau_{x'}, \tau_{y'}, \tau_{z'})$ and $\vec{\omega} = (\omega_{x'}, \omega_{y'}, \omega_{z'})$

$$\Downarrow$$

$$\vec{L} = (I_{x'x'}\,\omega_{x'}, I_{y'y'}\,\omega_{y'}, I_{z'z'}\,\omega_{z'})$$

- $I_{x'x'}$
- $I_{y'y'}$ $\rightarrow$ are the principal moments of inertia
- $I_{z'z'}$

Euler's Equations for a Rigid Body (cont'd)

In the body frame the components of Eq. (ℵ) yield

$$\tau_{x'} = I_{x'x'}\,\dot{\omega}_{x'} - (I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}$$

$$\tau_{y'} = I_{y'y'}\,\dot{\omega}_{y'} - (I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}$$

$$\tau_{z'} = I_{z'z'}\,\dot{\omega}_{z'} - (I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}$$

$$\Rightarrow \;\dot{} = d/dt$$

We have made use of the fact that the moments of inertia of a rigid body are constant in time in the co-rotating body frame

The above equations are known as Euler's equations

Euler's Equations for a Rigid Body (cont'd)

Consider a rigid body which is constrained to rotate about a fixed axis with constant angular velocity

$$\Downarrow$$

$$\dot{\omega}_{x'} = \dot{\omega}_{y'} = \dot{\omega}_{z'} = 0$$

$$\Downarrow$$

Euler's equations reduce to

$$\tau_{x'} = -(I_{y'y'} - I_{z'z'})\,\omega_{y'}\,\omega_{z'}$$

$$\tau_{y'} = -(I_{z'z'} - I_{x'x'})\,\omega_{z'}\,\omega_{x'}$$

$$\tau_{z'} = -(I_{x'x'} - I_{y'y'})\,\omega_{x'}\,\omega_{y'}$$

These equations specify the components of the steady torque exerted on the body by the constraining supports (in the body frame)

The steady angular momentum is written (in the body frame)

$$\vec{L} = (I_{x'x'}\,\omega_{x'},\, I_{y'y'}\,\omega_{y'},\, I_{z'z'}\,\omega_{z'})$$

Euler's Equations for a Rigid Body (cont'd)

It is easily seen that $\vec{\tau} = \vec{\omega} \times \vec{L}$

The torque is perpendicular to
- ☞ the angular velocity vector
- ☞ the angular momentum vector

If the axis of rotation is a principal axis

$\Downarrow$

2 of the 3 components of ω are zero (in the body frame)

$\Downarrow$

all 3 components of the torque are zero

To make the body rotate steadily about a principal axis

$\Downarrow$

zero external torque is required

Euler's Equations for a Rigid Body (cont'd)

Suppose that the body is freely rotating ☞ there are no external torques

Let the body be also rotationally symmetric about the z'-axis

$$\Downarrow$$

$$I_{x'x'} = I_{y'y'} = I_\perp \qquad \text{and} \qquad I_{z'z'} = I_\parallel$$

(of course in general $I_\perp \neq I_\parallel$)

Euler's equations yield

$$I_\perp \frac{d\omega_{x'}}{dt} + (I_\parallel - I_\perp)\,\omega_{z'}\,\omega_{y'} = 0 \qquad (\spadesuit)$$

$$I_\perp \frac{d\omega_{y'}}{dt} - (I_\parallel - I_\perp)\,\omega_{z'}\,\omega_{x'} = 0 \qquad (\clubsuit)$$

$$\frac{d\omega_{z'}}{dt} = 0 \longrightarrow \omega_{z'} \text{is a constant of motion}$$

Equation ($\spadesuit$) and ($\clubsuit$) can be written

$$\frac{d\omega_{x'}}{dt} + \Omega\,\omega_{y'} = 0 \qquad \text{and} \qquad \frac{d\omega_{y'}}{dt} - \Omega\,\omega_{x'} = 0$$

$$\Rightarrow \Omega = (I_\parallel / I_\perp - 1)\,\omega_{z'}$$

Euler's Equations for a Rigid Body (cont'd)

It is easily seen that the solution to the (♠) and (♣) system of equations is

$$\omega_{x'} = \omega_\perp \cos(\Omega\, t)$$

$$\omega_{y'} = \omega_\perp \sin(\Omega\, t)$$

➨ $\omega_\perp$ ☞ is a constant

The projection of $\vec{\omega}$ onto the x'-y' plane has the fixed length $\omega_\perp$ and rotates steadily about the z'-axis with angular velocity Ω

$\Downarrow$

the length of the angular velocity vector

$$\omega = (\omega_{x'}^2 + \omega_{y'}^2 + \omega_{z'}^2)^{1/2}$$

is a constant of the motion

The angular velocity vector makes some constant angle α with the z'-axis which implies that

$$\omega_{z'} = \omega \cos\alpha \qquad \text{and} \qquad \omega_\perp = \omega \sin\alpha$$

Euler's Equations for a Rigid Body (cont'd)

The components of the angular velocity vector are

$$\omega_{x'} = \omega \sin\alpha \cos(\Omega\, t)$$

$$\omega_{y'} = \omega \sin\alpha \sin(\Omega\, t)$$

$$\omega_{z'} = \omega \cos\alpha$$

where

$$\Omega = \omega \cos\alpha \left(\frac{I_{\|}}{I_{\perp}} - 1 \right) \qquad (\dashv)$$

In the body frame the angular velocity vector precesses about the symmetry axis (i.e., the z'-axis) with the angular frequency Ω

The components of the angular momentum vector are

$$L_{x'} = I_{\perp}\, \omega \sin\alpha \cos(\Omega\, t)$$

$$L_{y'} = I_{\perp}\, \omega \sin\alpha \sin(\Omega\, t)$$

$$L_{z'} = I_{\|}\, \omega \cos\alpha$$

In the body frame the angular momentum vector is also of constant length and precesses about the symmetry axis with the angular frequency Ω

Euler's Equations for a Rigid Body (cont'd)

The angular momentum vector makes a constant angle θ with the symmetry axis

$$\tan\theta = \frac{I_\perp}{I_\parallel}\tan\alpha \qquad (\eth)$$

Note that

- the angular momentum vector
- the angular velocity vector ☞ all lie in the same plane
- the symmetry axis

In other words

$$\mathbf{e}_{z'} \cdot \vec{L} \times \vec{\omega} = 0$$

The angular momentum vector
lies between the angular velocity vector and the symmetry axis
(i.e., $\theta < \alpha$) for a flattened (or oblate) body (i.e., $I_\perp < I_\parallel$)

The angular velocity vector
lies between the angular momentum vector and the symmetry axis
(i.e., $\theta > \alpha$) for an elongated (or prolate) body (i.e., $I_\perp > I_\parallel$)

Eulerian Angles

We have seen how we can solve Euler's equations to determine
the properties of a rotating body in the co-rotating body frame

NEXT

$\Downarrow$

we investigate how to determine the same properties in the inertial fixed frame

The fixed frame and the body frame share the same origin

$\Downarrow$

we can transform from one to the other
by means of an appropriate rotation of our vector space

If we restrict ourselves to rotations about 1 of the Cartesian coordinate axes
3 successive rotations are required to transform the fixed into the body frame

There are many different ways to combined 3 successive rotations

Next ☞ we describe the most widely used method which is due to Euler

Eulerian Angles (cont'd)

We start in the fixed frame

which has coordinates $\begin{cases} x \\ y \\ z \end{cases}$ and unit vectors $\begin{cases} \mathbf{e}_x \\ \mathbf{e}_y \\ \mathbf{e}_z \end{cases}$

Our first rotation is counterclockwise through an angle ϕ about the z-axis (looking down the axis)

End in the new frame

which has coordinates $\begin{cases} x'' \\ y'' \\ z'' \end{cases}$ and unit vectors $\begin{cases} \mathbf{e}_{x''} \\ \mathbf{e}_{y''} \\ \mathbf{e}_{z''} \end{cases}$

The transformation of coordinates can be represented by

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

The angular velocity vector associated with ϕ has the magnitude $\dot{\phi}$ and is directed along $\mathbf{e}_z$ ☞ along the axis of rotation $\Rightarrow \vec{\omega}_\phi = \dot{\phi}\,\mathbf{e}_z$

$\dot{\phi}$ is the precession rate about the $\mathbf{e}_z$ axis as seen in the fixed frame

Eulerian Angles (cont'd)

The second rotation is counterclockwise through an angle θ about the x''-axis (looking down the axis)

the new frame

will have coordinates ✐ x''' ✏ y''' ✎ z''' and unit vectors ✐ $\mathbf{e}_{x'''}$ ✏ $\mathbf{e}_{y'''}$ ✎ $\mathbf{e}_{z'''}$

The transformation of coordinates can be represented by

$$\begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix}$$

The angular velocity vector associated with θ has the magnitude $\dot{\theta}$ and is directed along $\mathbf{e}_{x''}$ ☞ along the axis of rotation

$$\Downarrow$$

$$\vec{\omega}_\theta = \dot{\theta}\,\mathbf{e}_{x''}$$

Eulerian Angles (cont'd)

The third rotation is counterclockwise through an angle ψ about the z'''-axis (looking down the axis)

the new frame is the body frame

which has coordinates $\nearrow x'$, $\rightarrow y'$, $\searrow z'$ and unit vectors $\nearrow \mathbf{e}_{x'}$, $\rightarrow \mathbf{e}_{y'}$, $\searrow \mathbf{e}_{z'}$

The transformation of coordinates can be represented by

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x''' \\ y''' \\ z''' \end{pmatrix}$$

The angular velocity vector associated with ψ has the magnitude $\dot{\psi}$ and is directed along $\mathbf{e}_{z''}$ ☞ along the axis of rotation

$\mathbf{e}_{z'''} = \mathbf{e}_{z'}$ ☞ because the third rotation is about $\mathbf{e}_{z'''}$

$$\Downarrow$$

$$\vec{\omega}_\psi = \dot{\psi}\,\mathbf{e}_{z'}$$

$\dot{\psi}$ is minus the precession rate about the $\mathbf{e}_{z'}$ axis as seen in the body frame

Eulerian Angles (cont'd)

The full transformation between the fixed frame and the body frame is complicated

HOWEVER

the following results can easily be verified

$$\mathbf{e}_z = \sin\psi\,\sin\theta\,\mathbf{e}_{x'} + \cos\psi\,\sin\theta\,\mathbf{e}_{y'} + \cos\theta\,\mathbf{e}_{z'} \qquad (\star)$$

$$\mathbf{e}_{x''} = \cos\psi\,\mathbf{e}_{x'} - \sin\psi\,\mathbf{e}_{y'}$$

It follows from Eq. $(\star)$ that

$$\mathbf{e}_z\cdot\mathbf{e}_{z'} = \cos\theta$$

$$\Downarrow$$

θ is the angle of inclination between the z- and z'-axes

The total angular velocity is found to be

$$\vec{\omega} = \vec{\omega}_\phi + \vec{\omega}_\theta + \vec{\omega}_\psi$$

$$\Downarrow$$

$$\omega_{x'} = \sin\psi\,\sin\theta\,\dot{\phi} + \cos\psi\,\dot{\theta} \qquad (\text{ב})$$

$$\omega_{y'} = \cos\psi\,\sin\theta\,\dot{\phi} - \sin\psi\,\dot{\theta} \qquad (\text{ד})$$

$$\omega_{z'} = \cos\theta\,\dot{\phi} + \dot{\psi} \qquad (\text{ג})$$

Eulerian Angles (cont'd)

The angles ϕ, θ, ψ **are termed Eulerian angles**

Each has a clear physical interpretation

- ϕ is the angle of precession about the $\mathbf{e}_z$ axis in the fixed frame
- ψ is minus the angle of precession about the $\mathbf{e}_{z'}$ axis in the body frame
- θ is the angle of inclination between the $\mathbf{e}_z$ and $\mathbf{e}_{z'}$ axes

Using Eqs. (ב), (ד), (ג)

$$\Downarrow$$

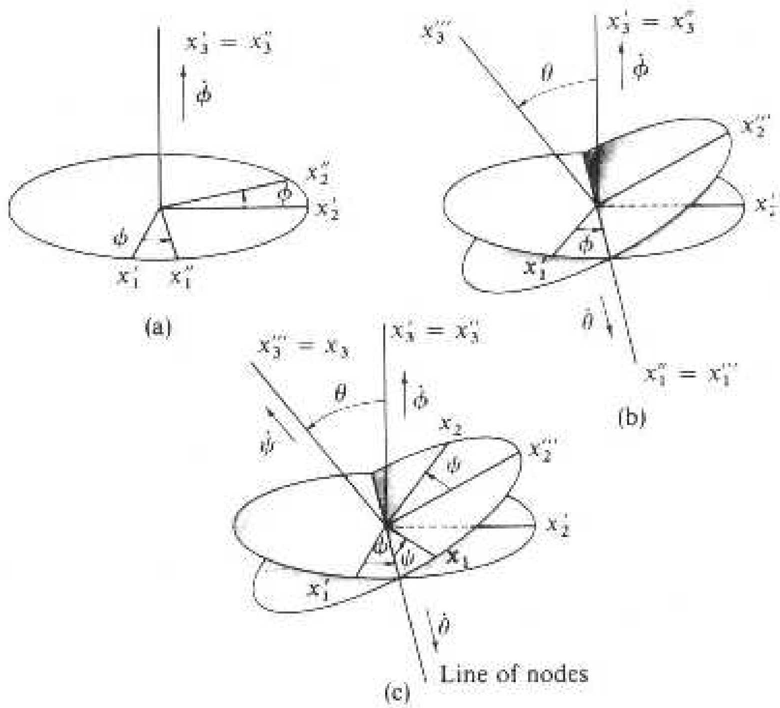
We can express the components of the angular velocity vector ω in the body frame entirely in terms of the Eulerian angles and their time derivatives

Warning!

The designation of the Euler angles is not universally agreed upon
(even the manner in which they are generated)

$\Downarrow$

some care must be taken in comparing results from different sources
If we identify the fixed system with $\vec{x}'$ and the body system with $\vec{x}$



Eulerian Angles (cont'd)

Consider a freely rotating body which is rotationally symmetric about one axis

In the absence of an external torque ☞ $\vec{L}$ is a constant of motion

Let $\vec{L}$ point along the z-axis

We saw that:

$\vec{L}$ subtends a constant angle θ with the axis of symmetry
(i.e., with the z'-axis)

$\Downarrow$

the time derivative of the Eulerian angle θ is zero

We also saw that:

- the angular momentum vector
- the axis of symmetry ☞ are co-planar
- the angular velocity vector

Consider an instant in time at which all of these vectors lie in the y'-z' plane

this implies that $\omega_{x'} = 0$

$\vec{\omega}$ subtends a constant angle α with the symmetry axis

$\Downarrow$

$$\omega_{y'} = \omega \sin\alpha \qquad \text{and} \qquad \omega_{z'} = \omega \cos\alpha$$

Eulerian Angles (cont'd)

Eq. ($\beth$) yields $\psi = 0 \Rightarrow$ Eq. ($\daleth$) yields

$$\omega \, \sin\alpha = \sin\theta \, \dot{\phi} \qquad (\varkappa)$$

This can be combined with Eq. ($\eth$) to give

$$\dot{\phi} = \omega \left[1 + \left(\frac{I_{\parallel}^2}{I_{\perp}^2} - 1 \right) \cos^2\alpha \right]^{1/2} \qquad (\digamma)$$

Finally ☞ Eqs. ($\gimel$) ($\eth$) and ($\varkappa$) yields

$$\dot{\psi} = \omega \, \cos\alpha - \cos\theta \, \dot{\phi} = \omega \cos\alpha \left(1 - \frac{\tan\alpha}{\tan\theta} \right) = \omega \, \cos\alpha \left(1 - \frac{I_{\parallel}}{I_{\perp}} \right)$$

A comparison of the above equation with Eq. ($\Finv$) gives

$$\dot{\psi} = -\Omega$$

✐ $\dot{\psi}$ is minus the precession rate in the body frame
(of the angular momentum and angular velocity vectors)
✎ $\dot{\phi}$ is the precession rate in the fixed frame
(of the angular velocity vector and the symmetry axis)

Eulerian Angles (cont'd)

It is known that the Earth's axis of rotation is slightly inclined to its symmetry axis
(which passes through the two poles)
the angle α is approximately 0.2 seconds of an arc

It is also known that the ratio of the moments of inertia is about $\frac{I_\parallel}{I_\perp} = 1.00327$
as determined from the Earth's oblateness

$$\Downarrow$$

from ($\dashv$) ☞ as viewed on Earth
the precession rate of the angular velocity vector about the symmetry axis is

$$\Omega = 0.00327\,\omega$$

giving a precession period of

$$T' = \frac{2\pi}{\Omega} = 305 \text{ days}$$

(of course ☛ $2\pi/\omega = 1$ day)
The observed period of precession is about 440 days

The disagreement between theory and observation
is attributed to the fact that the Earth is not perfectly rigid

Eulerian Angles (cont'd)

The (theroretical) precession rate of the Earth's symmetry axis is given by Eq. (F)
(as viewed from space)

$$\dot{\phi} = 1.00327\,\omega$$

The associated precession period is

$$T = \frac{2\pi}{\dot{\phi}} = 0.997 \text{ days}$$

The free precession of the Earth's symmetry axis in space
is superimposed on a much slower precession with a period of about 26,000 years
due to the small gravitational torque exerted on Earth by Sun and Moon
because of the Earth's slight oblateness