Classical Mechanics

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Reference text books

- **J. R. Taylor**
  Classical Mechanics
  University Science Books
- **J. B. Marion and S. T. Thorton**
  Classical Dynamics of Particles and Systems
  Saunders College Publishing
- **V. D. Barger and M. G. Olsson**
  Classical Mechanics: A Modern Perspective
  McGraw-Hill

Syllabus

- **Fundamental aspects of Newton's theory of motion**

  ✍ Newton's Laws ✍ Projectiles ✍ Conservation Laws ✍ Rockets
- **Oscillations**
- **Motion in Non-Inertial Reference Frames**
- **Newtonian Gravity**

  ✍ Newton's Law ✍ Ocean Tides ✍ Poisson's Equation
- **Motions of the planets in the Solar System**
- **midterm exam** (Thursday, October 18, 11:00 - 12:15 am)
- **Lagrangian Mechanics**

  ✍ Calculus of Variations ✍ Lagrange Equations ✍ Holonomic Systems
- **Dynamics of a System of Particles**

  ✍ Center of Mass ✍ Elastic and Inelastic Collisions ✍ Cross Sections
- **Motion of rigid bodies**
- **midterm exam** (Thursday, November 29, 11:00 - 12:15 am)
- **Hamiltonian Mechanics**
- **final exam** (Tuesday, December 18, 7:30 - 9:30 am)

http://www.gravity.phys.uwm.edu/∼doqui/

Historical Overview

- Greeks first to think seriously about mechanics (more than 2000 yr ago)
- Development of mechanics as we know today began

$\Downarrow$

Galileo (1564 - 1642) and Newton (1642 -1727)

- Development of two completely equivalent formulations of mechanics

(late XVIII and early XIX centuries)

$\Downarrow$

Lagrange (1564 - 1642) and Hamilton (1805 - 1865)

$\Downarrow$

provide dramatically simpler solutions to many complicated problems

- XX $\Leftrightarrow$ became clear classical mechanics does not correctly describe
  - motion of objects moving close to the speed of light
  - motion of microscopic particles inside atoms

$\Downarrow$

Development of two completely new forms of mechanics

Relativistic Mechanics Quatum Mechanics

- Last few decades $\Rightarrow$ with the advent of chaos theory research in classical mechanics has been rekindled!

Scalars and Vectors

☺ **Physical quantities ⇒ represented by 2 distinct classes of objects**

☞ **Scalars**: quantities that are invariant under coordinate transformations
(denoted by real numbers)

☞ **Vectors**: defined in terms of transformation properties
(represented by directed line elements in space e.g. $\overrightarrow{PQ}$)

☞ Line elements (and, therefore, vectors) are movable
do not carry intrinsic position information

☞ Vectors possess a magnitude and a direction
Scalars possess a magnitude but no direction

There are two approaches to vector analysis

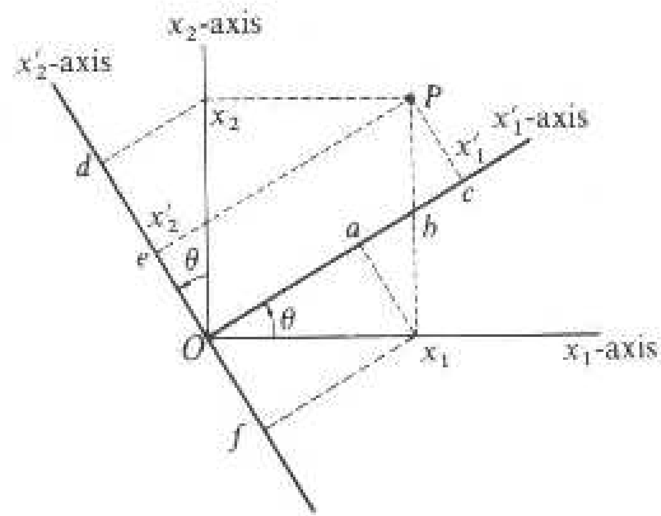
☞ Geometric approach ⇒ based on line elements in space

☞ Coordinate approach ⇒ assumes that space is defined by Cartesian coordinates and uses these to characterize vectors

In physics ☚ we adopt second approach

Coordinate Transformations

- Consider a point P with coordinates (x_1, x_2)
- Consider a new coordinate system generated via rotation of angle θ



which are the P coordinates in the prime system?

☞ $x'_1 \Rightarrow$ projection of x_1 onto x'_1-axis + projection of x_2 onto x'_1-axis

$$\begin{aligned} x'_1 &= x_1 \cos\theta + x_2 \sin\theta \\ &= x_1 \cos\theta + x_2 \cos(\pi/2 - \theta) \end{aligned}$$

☞ $x'_2 \Rightarrow$ projection of x_2 onto x'_2-axis + projection of x_2 onto x'_2-axis

$$\begin{aligned} x'_2 &= -x_1 \sin\theta + x_2 \cos\theta \\ &= x_1 \cos(\pi/2 + \theta) + x_2 \cos\theta \end{aligned}$$

Coordinate Transformations (cont'd)

NOTATION

angle between x'_i -axis and x_j-axis $\equiv (x'_i, x_j)$

$\Downarrow$

direction cosine of the x'_i-axis relative to x_j-axis ☞ $\lambda_{ij} \equiv \cos(x'_i, x_j)$

$\Downarrow$

$$
\begin{aligned}
\lambda_{11} &= \cos(x'_1, x_1) = \cos\theta \\
\lambda_{12} &= \cos(x'_1, x_2) = \cos(\pi/2 - \theta) = \sin\theta \\
\lambda_{21} &= \cos(x'_2, x_1) = \cos(\pi/2 + \theta) = -\sin\theta \\
\lambda_{22} &= \cos(x'_2, x_2) = \cos\theta
\end{aligned}
$$

EQUATIONS OF TRANSFORMATION BECOME

$$
\begin{aligned}
x'_1 &= x_1 \cos(x'_1, x_1) + x_2 \cos(x'_1, x_2) \\
&= \lambda_{11} x_1 + \lambda_{12} x_2
\end{aligned}
$$

$$
\begin{aligned}
x'_2 &= x_1 \cos(x'_2, x_1) + x_2 \cos(x'_2, x_2) \\
&= \lambda_{21} x_1 + \lambda_{22} x_2
\end{aligned}
$$

Coordinate Transformations (cont'd)

GENERALIZATION FOR 3 DIMENSIONS

$$x'_1 = \lambda_{11}\, x_1 + \lambda_{12}\, x_2 + \lambda_{13}\, x_3$$

$$x'_2 = \lambda_{21}\, x_1 + \lambda_{22}\, x_2 + \lambda_{23}\, x_3$$

$$x'_3 = \lambda_{31}\, x_1 + \lambda_{32}\, x_2 + \lambda_{33}\, x_3$$

SUMMATION NOTATION

$$x'_i = \sum_{j=1}^{3} \lambda_{ij}\, x_j \qquad i = 1,\ 2,\ 3$$

INVERSE TRANSFORMATION

$$x_i = \sum_{j=1}^{3} \lambda_{ji}\, x_j \qquad i = 1,\ 2,\ 3$$

ROTATION MATRIX

$$\mathbf{\Lambda} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix}$$

Orthogonality of Rotation Matrices

Euler's Rotation Theorem:

✍ **Any rotation can be given as a composition of rotations about 3 axis and hence can be represented by a 3×3 matrix operating on a vector**

$$\begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} \\ \lambda_{31} & \lambda_{32} & \lambda_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

In a rotation a vector must keep its original length ✎ $x'_i x'_i = x_i x_i$

Transformation Equation

$$\begin{aligned} x_i x_i &= (\lambda_{ij} x_j)(\lambda_{ik} x_k) \\ &= \lambda_{ij}(x_j \lambda_{ik}) x_k \\ &= \lambda_{ij}(\lambda_{ik} x_j) x_k \\ &= \lambda_{ij}\lambda_{ik} x_j x_k \\ &= x_i x_i \Leftrightarrow \lambda_{ij}\lambda_{ik} = \delta_{jk} \qquad (i,\ j,\ k = 1,\ 2,\ 3) \end{aligned}$$

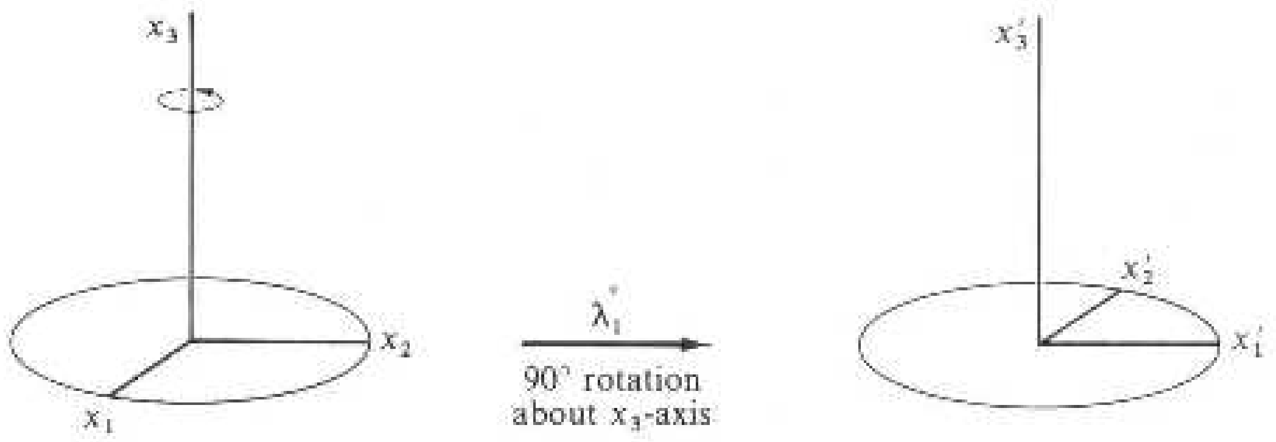
☞ **Kronecker delta**

$$\delta_{ij} = \begin{cases} 0, & \text{if } i \neq k \quad \text{coordinate axes in each of the systems} \\ 1, & \text{if } i = k \quad \text{are mutually perpendicular} \end{cases}$$

Orthogonality condition guarantees ✎ $\Lambda^{-1} = \Lambda^T$ **and** $\Lambda^T \Lambda = 1$

Geometrical Significance of Transformation Matrices

- Consider coordinate axis rotated counterclockwise 90° about the x_3-axis



In such a rotation ☞ $x'_1 = x_2,\ x'_2 = -x_1,\ x'_3 = x_3$

The only nonvanishing cosines are:

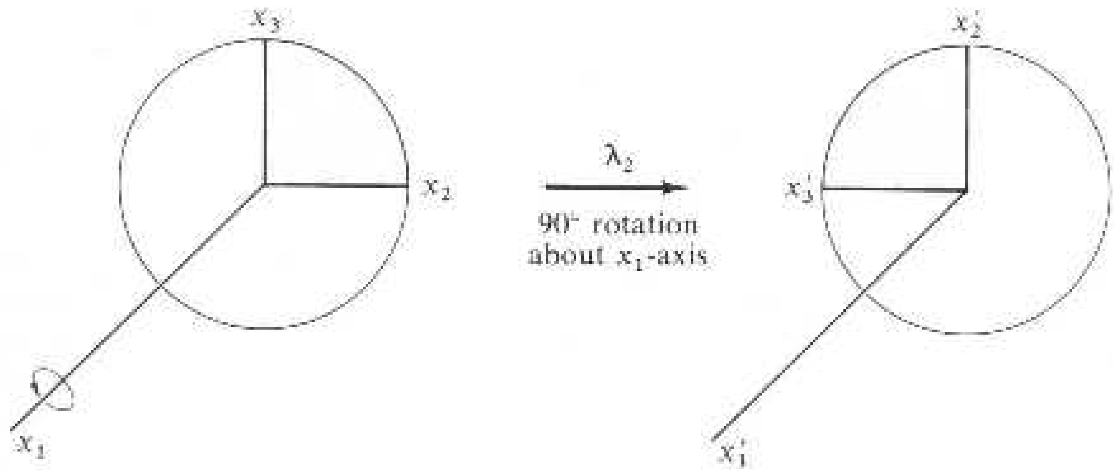
$$\begin{aligned} \cos(x'_1, x_2) &= 1 = \lambda_{12} \\ \cos(x'_2, x_1) &= -1 = \lambda_{21} \\ \cos(x'_3, x_3) &= 1 = \lambda_{33} \end{aligned}$$

The transformation matrix looks like:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Geometrical Significance of Transformation Matrices (cont'd)

- Next consider the counterclockwise rotation through $90°$ about the x_1-axis



In such a rotation ☞ $x_1' = x_1$, $x_2' = x_3$, $x_3' = -x_2$, **with**

$$\boldsymbol{\lambda_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

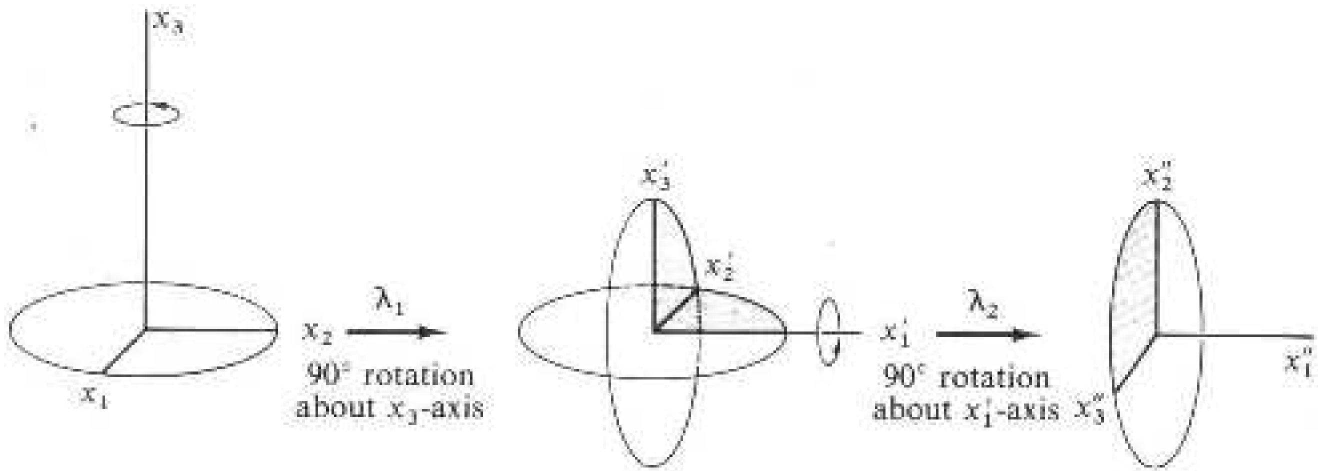
The rotation about the x_3-axis is defined by $\vec{x}' = \boldsymbol{\lambda_1}\vec{x}$

The rotation about the new x'-axis is defined by $\vec{x}'' = \boldsymbol{\lambda_2}\,\vec{x}'$

The transformation matrix for the combined transformation is

$$\vec{x}'' = \boldsymbol{\lambda_2}\boldsymbol{\lambda_1}\vec{x}$$

Geometrical Significance of Transformation Matrices (cont'd)



$$
\begin{pmatrix} x''_1 \\ x''_2 \\ x''_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

$$
= \begin{pmatrix} x_2 \\ x_3 \\ x_1 \end{pmatrix}
$$

Geometrical Significance of Transformation Matrices (cont'd)

The two rotations may be represented by a single matrix

$$\boldsymbol{\lambda_3} = \boldsymbol{\lambda_2}\boldsymbol{\lambda_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

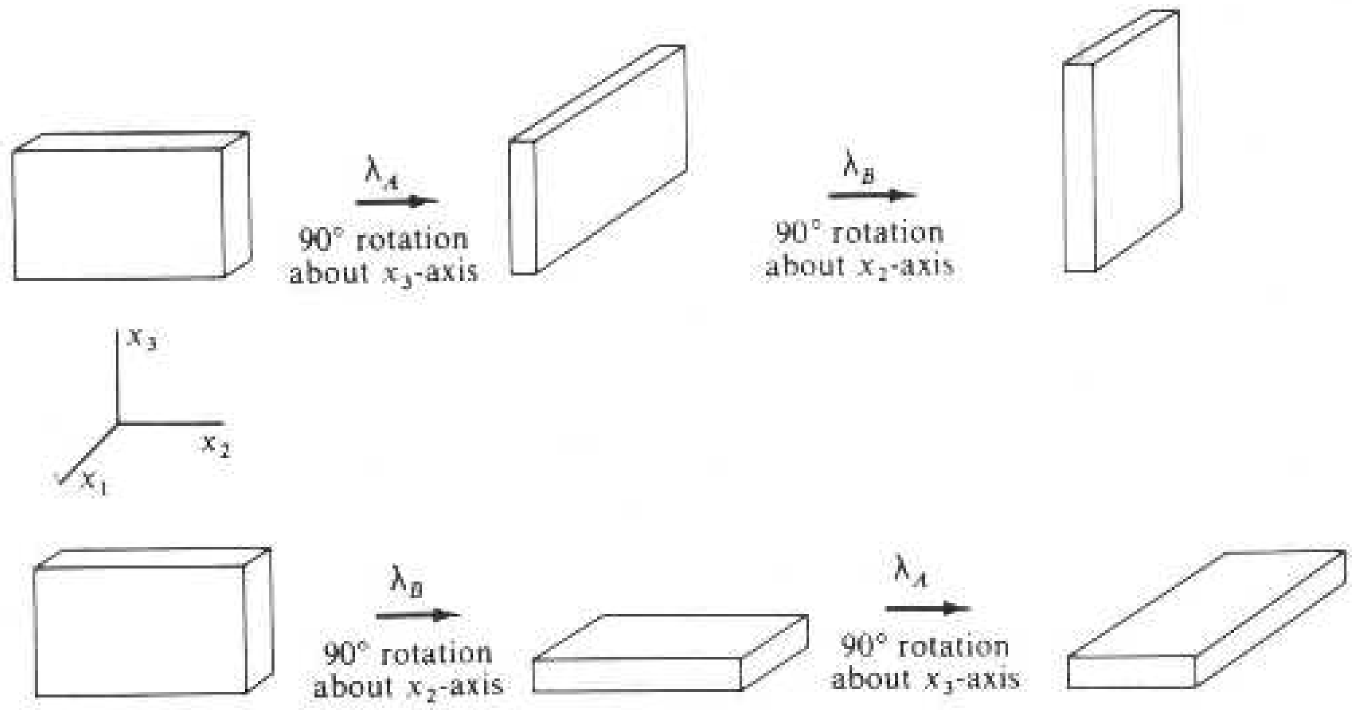
The order in which the transformation matrices operate on $\vec{x}$ is

IMPORTANT

$$\begin{aligned} \boldsymbol{\lambda_4} &= \boldsymbol{\lambda_1}\boldsymbol{\lambda_2} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \\ &\neq \boldsymbol{\lambda_3} \end{aligned}$$

Geometrical Significance of Transformation Matrices (cont'd)

PRODUCT OF MATRICES IS NOT COMMUTATIVE



- **Different final orientations of the parallelepiped that undergoes rotations corresponding to two rotation matrices $\lambda_A\,\lambda_B$ when successive rotations are made in different order**

The Scalar Product

- **A scalar quantity is invariant under all possible rotational transformations**
- **Individual components of a vector are not scalars**

$$\Downarrow$$

they change under rotational transformation

- **Can we form a scalar combining one or more vector components?**

Suppose that we were to define the "ampersand" product

$$\Downarrow$$

$$\vec{x} \,\&\, \vec{y} = x_1\, y_2 + x_2\, y_3 + x_3\, y_1 = \mathcal{S}$$

Is $\vec{x}\&\vec{y}$ invariant under transformation as must be the case if $\mathcal{S}$ is a scalar number?

- Take $\vec{x}(1,0,0,)$ and $\vec{y}(0,1,0) \Rightarrow \vec{x}\,\&\,\vec{y} = 1$
- rotate the basis through 45° degrees about the z-axis

$$\Downarrow$$

$\vec{x}\,' = (1/\sqrt{2},\ -1/\sqrt{2},\ 0)$ and $\vec{y}\,' = (1/\sqrt{2},\ 1/\sqrt{2},\ 0) \Rightarrow \vec{x}\,' \,\&\, \vec{y}\,' = 1/2$

- Not invariant under rotational transformation

The Scalar Product: 2^{nd} trial

Consider now the dot product ☞ $\vec{x}.\vec{y} = \sum_i x_i y_i$

✍ Take $x'_i = \sum_j \lambda_{ij} x_j$ and $y'_i = \sum_k \lambda_{ik} y_k$

$$\begin{aligned}
\vec{x}\,'.\vec{y}\,' &= \sum_i x'_i y'_i \\
&= \sum_i \left(\sum_j \lambda_{ij} x_j \right) \left(\sum_k \lambda_{ik} y_k \right) \\
&= \sum_{jk} \left(\sum_i \lambda_{ij} \lambda_{ik} \right) x_j y_k \\
&= \sum_j \left(\sum_k \delta_{jk} x_j y_k \right) \\
&= \sum_j x_j y_j \\
&= \vec{x}.\vec{y}
\end{aligned}$$

Invariant under rotational transformations !!!

Unit Vectors

✜ To describe a vector in terms of the component along the three coordinate axis together with convinient specification of the axis

$$\Downarrow$$

Introduce unit vectors ☚ vectors having a length equal to the unit of length used along the particular coordinate axis

The following ways of expressing the vector $\vec{x}$ are equivalent:

$$\begin{aligned}
\vec{x} &= (x_1,\ x_2,\ x_3) \\
&= x_1\boldsymbol{e_1} + x_2\boldsymbol{e_2} + x_3\boldsymbol{e_3} \\
&= \sum_{i=1}^{3} x_i\boldsymbol{e_i} \\
&= x_1\hat{\imath} + x_2\hat{\jmath} + x_3\hat{k}
\end{aligned}$$

If any two unit vectors are orthogonal

$$\Downarrow$$

$$\boldsymbol{e_i} \,.\, \boldsymbol{e_j} = \delta_{ij}$$

The Vector Product

$$\vec{c} = \vec{a} \times \vec{b}$$

Components of $\vec{c}$ defined by the relation

$$c_i \equiv \sum_{jk} \epsilon_{ijk} a_j b_k$$

Levi-Civita density

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any index is equal to any other index} \\ +1, & \text{if } i,\ j,\ k \text{ form an } \textit{even} \text{ permutation of } 1,\ 2,\ 3 \\ -1, & \text{if } i,\ j,\ k \text{ form an } \textit{odd} \text{ permutation of } 1,\ 2,\ 3 \end{cases}$$

$$\begin{aligned} c_1 &= \sum_{jk} \epsilon_{1jk}\ a_j b_k \\ &= \epsilon_{123}\ a_2 b_3 + \epsilon_{132}\ a_3 b_2 \\ &= a_2 b_3 - a_3 b_2 \end{aligned}$$

$$c_2 = a_3 b_1 - a_1 b_3$$

$$c_3 = a_1 b_2 - a_2 b_1$$

Vector Operations

HOMEWORK

Scalar Product

$$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\,\cos\theta$$

Vector Product

$$|\vec{a} \times \vec{b}| = |\vec{a}|\,|\vec{b}|\,\sin\theta$$

$\theta \equiv$ angle between $\vec{a}$ and $\vec{b}$

Homework Clues

Scalar Product

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \sum_i \frac{a_i}{|\vec{a}|}\frac{b_i}{|\vec{b}|}$$

$\frac{a_i}{|a|} \equiv$ direction cosines ☛ Sum of direction cosines is the cosine of θ

Vector Product

$$\begin{aligned}
(|\vec{a}|\,|\vec{b}|\,\sin\theta)^2 &= |\vec{a}|^2\,|\vec{b}|^2 - |\vec{a}|^2\,|\vec{b}|^2\,\cos^2\theta \\
&= \left(\sum_i a_i^2\right)\left(\sum_i b_i^2\right) - \left(\sum_i a_i b_i\right)^2 \\
&= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2
\end{aligned}$$

If we take the positive square root of both sides

$$|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$$

Differentiation of a scalar with respect to a scalar

- If a scalar function $\phi(s)$ is differentiated with respect to a scalar variable s, because neither part of the derivative can change under coordinate transformation, the derivative itself cannot change and must be a scalar

Because of scalar identity ☚ in the x_i and x_i' coordinate systems

$$\Downarrow$$

$$\phi = \phi' \text{ and } s = s'$$

$$\Downarrow$$

$$d\phi = d\phi' \,\text{and}\, ds = ds'$$

$$\Downarrow$$

$$\frac{d\phi}{ds} = \frac{d\phi'}{ds'} = \left(\frac{d\phi}{ds}\right)'$$

Differentiation of a vector with respect to a scalar

The components of a vector $\vec{a}$ transform according to

$$a'_i = \sum_j \lambda_{ij}\, a_j$$

$$\Downarrow$$

$$\frac{da'_i}{ds'} = \frac{d}{ds'} \sum_j \lambda_{ij} a_j$$
$$= \sum_j \lambda_{ij} \frac{da_j}{ds'}$$

but of course $s = s'$ ☛ $\frac{da'_i}{ds'} = \left(\frac{da_i}{ds}\right)' = \sum_j \lambda_{ij} \left(\frac{da_j}{ds}\right)$

$$\Downarrow$$

$d\vec{a}/ds$ is a vector

Useful Formulae

Sum Rules

$$\frac{d}{ds}(\vec{a}+\vec{b}) = \frac{d\vec{a}}{ds} + \frac{d\vec{b}}{ds}$$

$$\frac{d}{ds}(\vec{a} \cdot \vec{b}) = \vec{a} \cdot \frac{d\vec{b}}{ds} + \frac{d\vec{a}}{ds} \cdot \vec{b}$$

$$\frac{d}{ds}(\vec{a} \times \vec{b}) = \vec{a} \times \frac{d\vec{b}}{ds} + \frac{d\vec{a}}{ds} \times \vec{b}$$

$$\frac{d}{ds}(\phi\vec{a}) = \phi\frac{d\vec{a}}{ds} + \frac{d\phi}{ds}\vec{a}$$

Physical Examples of derivatives

$$\vec{r} = \sum x_i \boldsymbol{e_i} \qquad \text{position}$$

$$\vec{v} = \dot{\vec{r}} = \sum_i \frac{dx_i}{dt} \boldsymbol{e_i} \qquad \text{velocity}$$

$$\vec{a} = \dot{\vec{v}} = \ddot{\vec{r}} = \sum_i \frac{d^2x_i}{dt^2} \boldsymbol{e_i} \qquad \text{acceleration}$$

Useful Formulae (cont'd)

Gradient operator

$$\text{grad} \equiv \vec{\nabla} = \sum_i e_i \frac{\partial}{\partial x_i}$$

The gradient operator can:

(a) operate directly on a scalar function

$$\vec{\nabla}\phi$$

(b) be used in a scalar product with a vector function $\rightarrow$ *the divergence*

$$\vec{\nabla}.\vec{a}$$

(c) be used in a vector product with a vector function $\rightarrow$ *the curl*

$$\vec{\nabla} \times \vec{a}$$

Integration of vectors

$$\int_V \vec{a} \;\; dv = \left(\int_V a_1 \;\; dv, \; \int_V a_2 \;\; dv, \; \int_V a_3 \;\; dv \right)$$