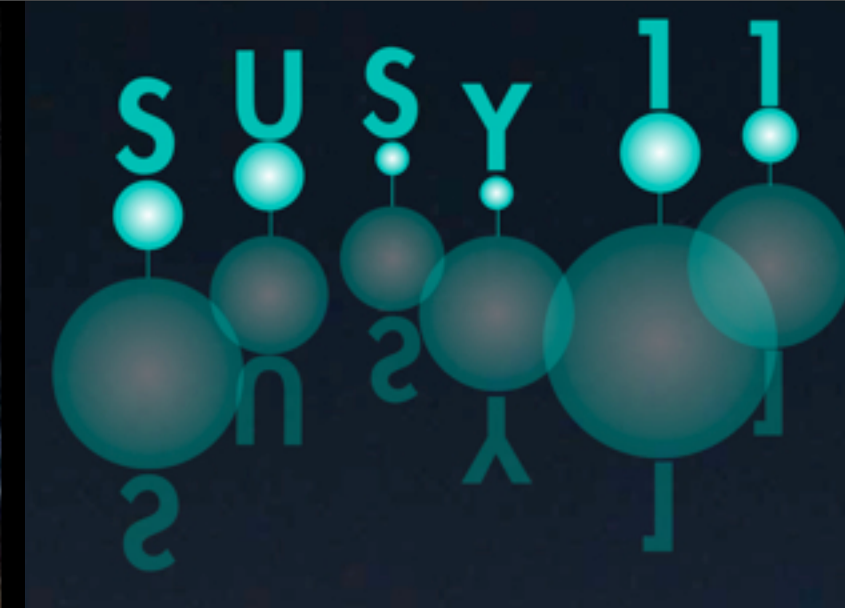


PARTICLE PHYSICS 2011

Luis Anchordoqui

Mandelstam Variables

Cross sections and decay rates can be written using kinematic variables that are relativistic invariants

For any two particle to two particle process we have at our disposal 4-momenta associated with each particle invariant variables are scalar products $p_A \cdot p_B, p_A \cdot p_C, p_A \cdot p_D$

conventional to use related (Mandelstam) variables

$$
\begin{aligned}
s &= (p_A + p_B)^2 = (p_C + p_D)^2 \\
t &= (p_A - p_C)^2 = (p_B - p_D)^2 \\
u &= (p_A - p_D)^2 = (p_B - p_C)^2
\end{aligned}
$$

Because $p_i^2 = m_i^2$ (with $i = A, B, C, D$) and $p_A + p_B = p_C + p_D$ due to energy momentum conservation

$$
\begin{aligned}
s + t + u &= \sum_i m_i^2 + 2p_A^2 + 2p_A.(p_B - p_C - p_D) \\
&= \sum_i m_i^2
\end{aligned}
$$

i.e. only two of the three variables are independent

Crossing

$A + B \to C + D$

s-channel process
- s ☛ is positive (square cm energy)
- t, u ☛ are negatives

From this process we can form another process $A\overline{C} \to \overline{B} + D$

antiparticle of C to left-hand side ☚ by taking ☛ antiparticle of B to right-hand side

Antiparticles have momenta which are negatives of particle momenta

$$p_B \to -p_B \quad \& \quad p_C \to -p_C$$

relative to s-channel reaction

$$s = (p_A - p_B)^2, \; t = (p_A + p_C)^2 \quad \& \quad u = (p_A - p_D)^2$$

This is called t-channel process

t ☛ is positive and represents square cm energy

$s \leq 0 \qquad u \leq 0$ ☛ are square of momentum transfers

Crossing

From $A + B \to C + D$ we form another process $A + \overline{D} \to \overline{B} + C$

antiparticle of B to left-hand side ☚ by taking ☛ antiparticle of D to right-hand side

$s = (p_A - p_B)^2, t = (p_A - p_C)^2$ & $u = (p_A + p_D)^2$

u-channel process $\to$

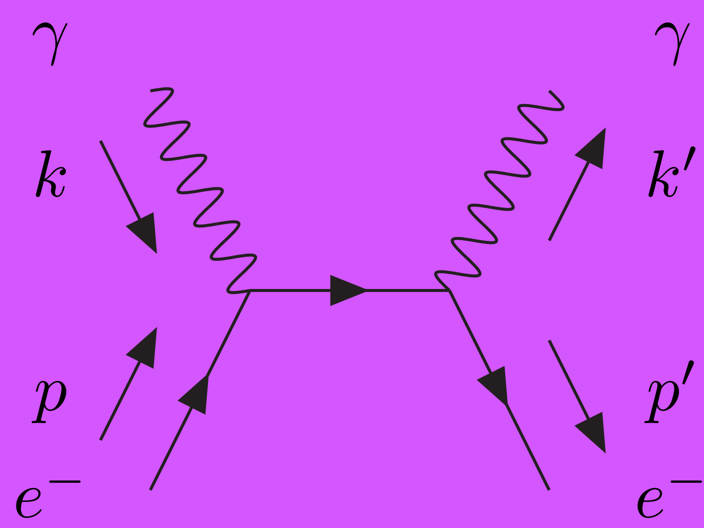
u ☛ is positive (square cm energy of $A\overline{D}$ system)

$s \leq 0$ and $t \leq 0$ ☛ are square of momentum transfers

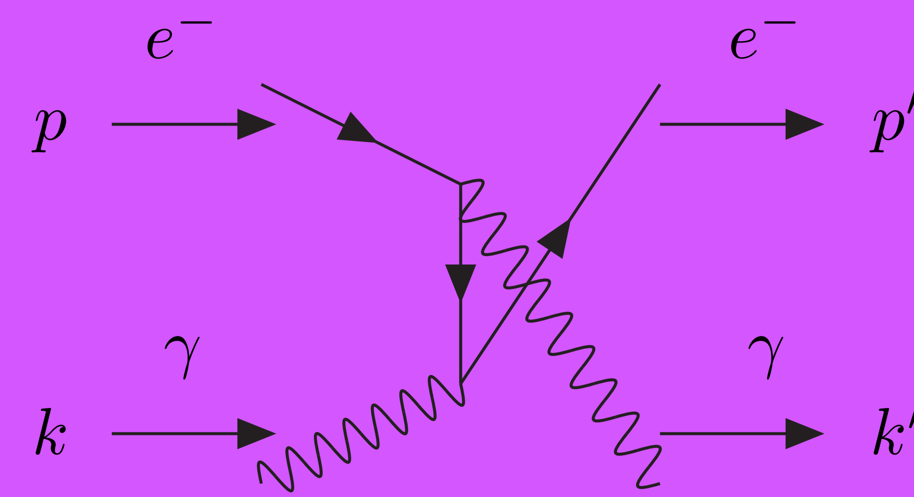
EXAMPLE

Amplitude for pair annihilation by crossing amplitude for Compton scattering

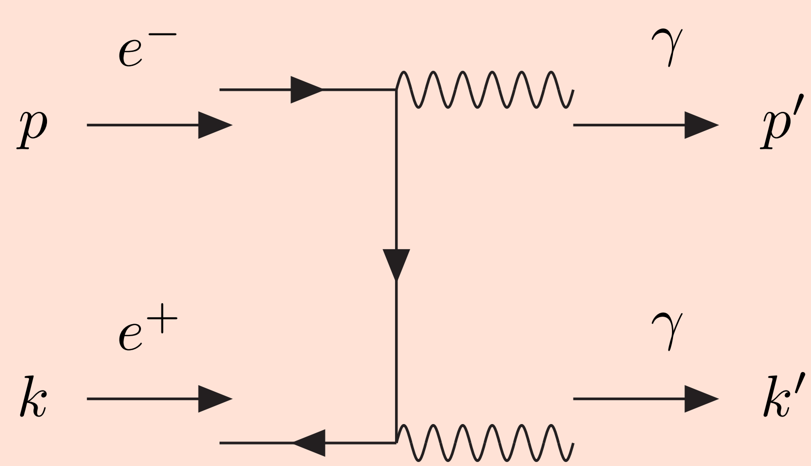
$e^+e^- \to \gamma\gamma$ $\qquad$ $e^-\gamma \to e^-\gamma$



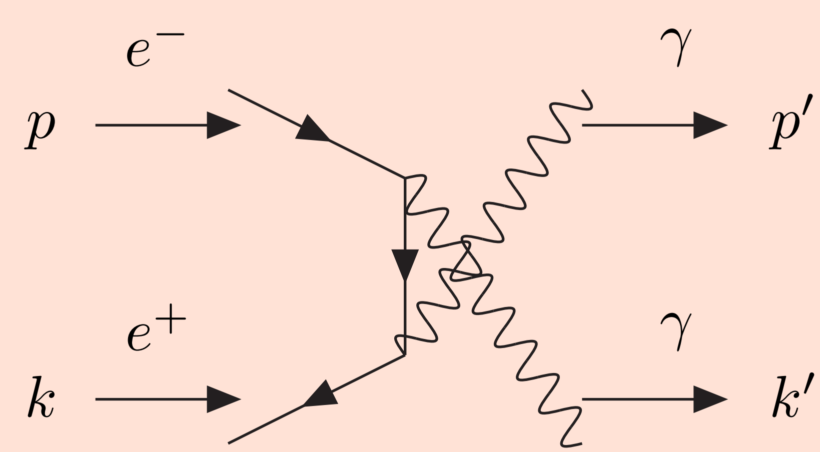
s-channel



u-channel



t-channel



u-channel

Leading order contributions of some QED processes

Process	$\overline{|\mathfrak{M}|^2}/2e^4$
$\underbrace{\text{Møller scattering}}_{e^-e^-\to e^-e^-}$	$\underbrace{\frac{s^2+u^2}{t^2}}_{\text{forward}} + \underbrace{\frac{2s^2}{tu}}_{\text{interference}} + \underbrace{\frac{s^2+t^2}{u^2}}_{\text{backward}}$
(Crossing $s \leftrightarrow u$)	($u \leftrightarrow t$ symmetric)
$\underbrace{\text{Bhabha scattering}}_{e^-e^+\to e^-e^+}$	$\underbrace{\frac{s^2+u^2}{t^2}}_{\text{forward}} + \underbrace{\frac{2u^2}{ts}}_{\text{interference}} + \underbrace{\frac{u^2+t^2}{s^2}}_{\text{timelike}}$
$e^-\mu^- \to e^-\mu^-$	$\underbrace{\frac{s^2+u^2}{t^2}}_{\text{forward}}$
$e^-e^+ \to \mu^-\mu^+$	$\underbrace{\frac{u^2+t^2}{s^2}}_{\text{timelike}}$
$\underbrace{\text{Compton scattering}}_{e^-\gamma\to e^-\gamma}$	$-\underbrace{\frac{u}{s}}_{\text{timelike}} - \underbrace{\frac{s}{u}}_{\text{backward}}$
$\underbrace{\text{pair annihilation}}_{e^+e^-\to\gamma\gamma}$	$\underbrace{\frac{u}{t}}_{\text{forward}} + \underbrace{\frac{t}{u}}_{\text{backward}}$

Motivation for Feynman Rules

Nonrelativistic perturbation expansion of transition amplitude is

$$T_{fi} = -i2\pi\delta(E_f - E_i)\left[\langle f|V|i\rangle + \sum_{n\neq 1}\langle f|V|n\rangle \frac{1}{E_i - E_n}\langle n|V|i\rangle + \dots\right]$$ ♨

we have associated factors of $\langle f|V|n\rangle$ with vertices and identified $1/(E_i - E_n)$ as propagator

State vectors are eigenstates of Hamiltonian in absence of V

$$H_0|n\rangle = E_n|n\rangle$$

Using completness relation $|n\rangle\langle n| = 1$ we rewrite ♨ as

$$T_{fi} = 2\pi\delta(E_f - E_i)\langle f|(-iV) + (-iV)\frac{i}{E_i - H_0}(-iV) + \dots|i\rangle$$ ⌘

Propagator for spinless particles

It is natural to take $(-iV)$ rather than V as perturbation parameter

Vertex factor is $(-iV)$ and propagator

may be regarded as i times inverse of Schrödinger operator

$$-i(E_i - H_0)\psi = -iV\psi$$ ⊡

acting on intermediate state

We can now apply same technique to various relativistic wave eqs. to deduce form of propagators for corresponding particles

Form of Klein-Gordon equation corresponding to ⊡ is

$$i(\Box^2 + m^2)\phi = -iV\phi$$ ⌫

see ☛ $(\partial_\mu \partial^\mu + m^2)\phi = -V\phi$

Guided by relativistic generalization of ⌘ we expect propagator for spinless particle to be inverse of operator on left-hand side of ⌫

For intermediate state of momentum p this gives

$$\frac{1}{i(-p^2 + m^2)} = \frac{i}{p^2 - m^2}$$

Propagator for spin-1/2 particles

In a similar fashion, an electron in an electromagnetic field satisfies

$$(\not{p} - m_e)\psi = e\gamma^\mu A_\mu \psi \quad \text{✡}$$

As before, we must multiply by $-i$

Hence, vertex factor is is $-ie\gamma^\mu$

Electron propagator is therefore inverse of $-i$ times left-hand side of ✡

$$\frac{1}{-i(\not{p} - m_e)} = \frac{i}{\not{p} - m_e} = \frac{i(\not{p} + m_e)}{p^2 - m_e^2} = \frac{i}{p^2 - m_e^2} \sum_s u\bar{u}$$

we have used $\not{p}\not{p} = p^2$ and completeness relation

$$\begin{aligned}
(\Lambda_+)_{\alpha\beta} &\equiv \frac{1}{2m} \sum_{r=1}^{2} u_\alpha^{(r)}(p)\ \bar{u}_\beta^{(r)}(p) \\
&= \frac{1}{2m\,(m+E)} \left[\sum_r (\not{p} + m) u^{(r)}(0)\, \bar{u}^{(r)}(0)(\not{p} + m) \right]_{\alpha\beta} \\
&= \frac{1}{2m\,(m+E)} \left[(m + \not{p})\ \frac{1+\gamma^0}{2}\ (m + \not{p}) \right]_{\alpha\beta} \\
&= \frac{1}{2m\,(m+E)} \left\{ m(\not{p} + m) + \frac{1}{2}(\not{p} + m)[(m - \not{p})\gamma^0 + 2E] \right\}_{\alpha\beta} \\
&= \frac{1}{2m}\,(\not{p} + m)_{\alpha\beta}
\end{aligned}$$

Numerator contains sum over spin states of virtual electron

SUMMARY

- General form of propagator of virtual particle of mass m is

$$\frac{i}{p^2 - m^2} \sum_{\text{spins}}$$

- Spin sum is completeness relation ☛ we include all possible spin states of propagating particle

- Also integrate over different momentum states that propagate

-- for diagrams we have considered so far this momentum is fixed by momenta of external particles --

Gauge freedom in photon propagator

Propagator for photon is not unique
on account of freedom in choice of A^μ

Recall that physics is unchanged
by transformation that is associated with invariance of QED
under gauge transformations of wavefunctions of charged particles

$$A_\mu \to A'_\mu = A_\mu + \partial_\mu \chi$$ ☎

χ is any function that satisfies

$$\Box^2 \chi = 0$$ ⌛

Wave equation for a photon $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \qquad \partial_\mu F^{\mu\nu} = e\, j^\nu$

can be written as ☛

$$(g^{\nu\lambda}\Box^2 - \partial^\nu \partial^\lambda) A_\lambda = j^\nu$$ ♣

photon propagator cannot exist until we remove gauge freedom of A_λ

Photon Propagator

So far we worked in Lorentz class of gauges with $\partial_\lambda A^\lambda = 0$

wavefunction A^μ for a free photon satisfies equation

$$\Box^2 A^\mu = 0$$

which has solutions

$$A^\mu = \epsilon^\mu(q) e^{-iq\,\cdot\,x} \quad \Psi$$

where four vector ϵ^μ is polarization vector of photon

With this in mind wave equation ♣ simplifies to $g^{\nu\lambda}\Box^2 A_\lambda = j^\nu$

since $g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu$ propagator is

$$i\frac{-g_{\mu\nu}}{q^2}$$

(inverse of momentum space operator multiply by $-i$)

PROPAGATOR OF SPIN-1 PARTICLES

Wave equation for a spin-1 particle of mass M

obtained from that for photon by replacement $\Box^2 \to \Box^2 + M^2$

From ♣ we see that wavefunction B_λ for a free particle satisfies

$$\left[g^{\nu\lambda}(\Box^2 + M^2) - \partial^\nu \partial^\lambda\right] B_\lambda = 0$$ ☕

Proceeding exactly as before

we determine inverse of momentum space operator by solving

$$\left[g^{\nu\lambda}(-p^2 + M^2) - p^\nu p^\lambda)\right]^{-1} = \delta^\mu_\lambda (A g_{\mu\nu} + B p_\mu p_\nu)$$ ☫

for A and B

Propagator ☛ quantity in brackets on right-hand side of ☫ times i

is found to be

$$\frac{i(-g^{\mu\nu} + p^\mu p^\nu / M^2)}{p^2 - M^2}$$

Polarization vectors

- Numerator is sum over three spin states of massive particle when taken on-shell $p^2 = M^2$
- We first take divergence ∂_ν of ☕
- Two terms cancel and we find ☛ $M^2 \partial^\lambda B_\lambda = 0$ ☢
- Hence for a massive vector particle we have no choice but to take $\partial^\lambda B_\lambda = 0$
  - ➢ it is not a gauge condition
- As a consequence wave equation reduces to

$$(\Box^2 + M^2) B_\mu = 0$$

with free particle solutions

$$B_\mu = \epsilon_\mu \, e^{-ip \,.\, x}$$

- condition ☢ demands $p^\mu \,.\, \epsilon_\mu = 0$
  reducing independent polarization vectors from 4 to 3

Photon poarization vectors

Lorentz condition for photons $\partial_\mu A^\mu = 0$ gives $q_\mu \cdot \epsilon^\mu = 0$

reducing number of independent components of ϵ^μ to 3

We can explore consequences of additional gauge freedom ☏

Choose a gauge parameter

$$\chi = iae^{-iq.x}$$

with a constant so that ⌛ is satisfied

Substituting this together with Ψ into ☏

show that physics is unchanged by replacement

$$\epsilon_\mu \to \epsilon'_\mu = \epsilon_\mu + aq_\mu$$

Feynmanology

2 polarization vectors $(\epsilon_\mu, \epsilon'_\mu)$

which differ by a multiple of q_μ describe same photon

Use this freedom to ensure that time component of ϵ^μ vanishes

$\epsilon^0 \equiv 0$ and Lorentz condition reduces to $\vec{\epsilon}\,.\,\vec{q} = 0$

This (noncovariant) choice of gauge is known as Coulomb gauge

This means there are only two independent polarization vectors

and they are both transverse to three-momentum of photon

e.g. for a photon traveling along z-axis we may take

$$\epsilon_1 = (1,0,0), \quad \epsilon_2 = (0,1,0)$$

A free photon is thus described by its momentum q

and a polarization vector $\vec{\epsilon}$

CONCLUSION: we can obtain invariant amplitude $\mathfrak{M}$

by drawing all Feynman diagrams for process

(topologically distinct and connected)

and assigning multiplicative factors with various elements of each diagram

Feynman rules for $-i\mathfrak{M}$

• **External Lines**	Multiplicative factor
spin-0 boson (or antiboson)	1
spin-$\frac{1}{2}$ fermion (in, out)	$u,\ \overline{u}$
spin-$\frac{1}{2}$ antifermion (in, out)	$\overline{v},\ v$
spin-1 photon (in, out)	$\epsilon_\mu,\ \epsilon^*_\mu$

• **Internal Lines – Propagators**

spin-0 boson	$\frac{i}{p^2-m^2}$
spin-$\frac{1}{2}$ fermion	$\frac{i(\not{p}+m)}{p^2-m^2}$
massive spin-1 boson	$\frac{-i(g_{\mu\nu}-p_\mu p_\nu/M^2)}{p^2-M^2}$
massless spin-1 boson (Feynman gauge)	$\frac{-ig_{\mu\nu}}{p^2}$

• **Vertex Factors**

photon–spin-0 (charge e)	$-ie(p+p')^\mu$
photon–spin-$\frac{1}{2}$ (charge e)	$-ie\gamma^\mu$

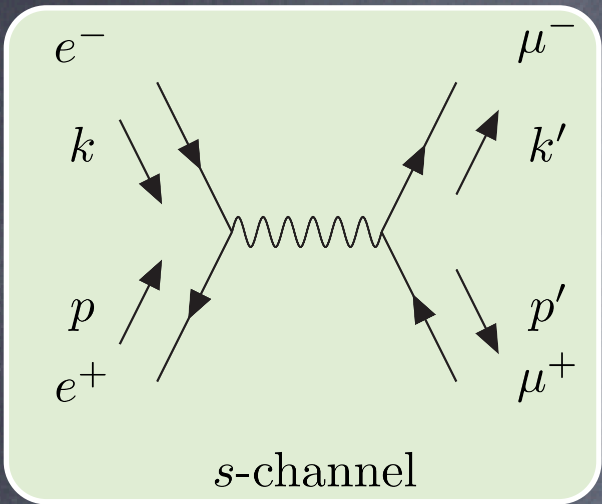
Loops: $\int d^4k/(2\pi)^4$ over loop momentum; include -1 if fermion loop and take trace of associated γ-matrices

Identical fermions: -1 between diagrams which differ only in $e^- \leftrightarrow e^-$ or initial $e^- \leftrightarrow$ final e^+

Beyond the Trees

Bulk of hadrons produced in $e^- e^+$ annihilations are fragments of q and antiq produced by $e^- e^+ \to q\bar{q}$

QED cross section for this process is readily obtained from



$$\sigma_{e^+e^- \to \mu^+\mu^-} = \frac{4\pi\alpha^2}{3Q^2}$$

center-of-mass energy squared is

$$s = Q^2 = 4E_{\text{beam}}$$

Required cross section is
$$\sigma_{e^+e^- \to q\bar{q}} = 3\, e_q^2\, \sigma_{e^+e^- \to \mu^-\mu^+}$$

we have taken account of fractional charge of quark e_q

extra factor of 3 arises because we have diagram for each quark color and cross sections have to be added

To obtain cross section for producing all types of hadrons must sum over all quark flavors $q = u, d, s, \ldots$ and hence

$$\sigma_{e^+e^- \to \text{hadrons}} = \sum_q \sigma_{e^+e^- \to q\bar{q}} = 3 \sum_q e_q^2\, \sigma_{e^+e^- \to \mu^-\mu^+}$$

R

That simple calculation leads to dramatic prediction

$$R \equiv \frac{\sigma_{e^+e^-\to\text{hadrons}}}{\sigma_{e^+e^-\to\mu^-\mu^+}} = 3\sum_q e_q^2$$

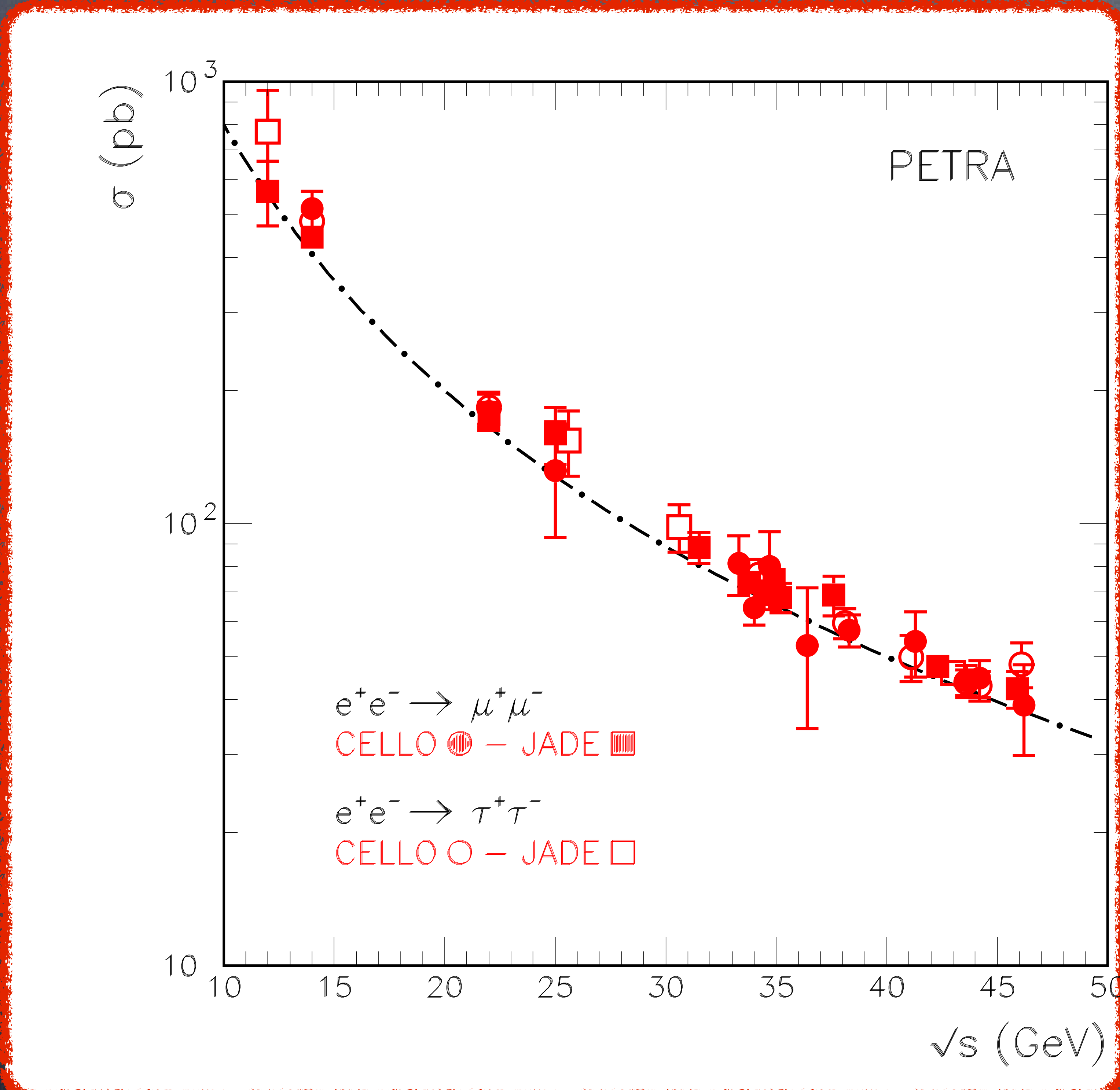
Because $\sigma_{e^+e^-\to\mu^-\mu^+}$ is well known (from last class)

measurement of total e^+e^- annihilation cross section into hadrons directly counts number of quarks, their flavors, and their colors

We have

$$\begin{aligned} R &= 3\left[\left(\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \left(\tfrac{1}{3}\right)^2\right] = 2 && \text{for } u,\ d,\ s, \\ &= 2 + 3\left(\tfrac{2}{3}\right)^2 = \frac{10}{3} && \text{for } u,\ d,\ s,\ c, \\ &= \frac{10}{3} + 3\left(\tfrac{1}{3}\right)^2 = \frac{11}{3} && \text{for } u,\ d,\ s,\ c,\ b \end{aligned}$$

Cross Section



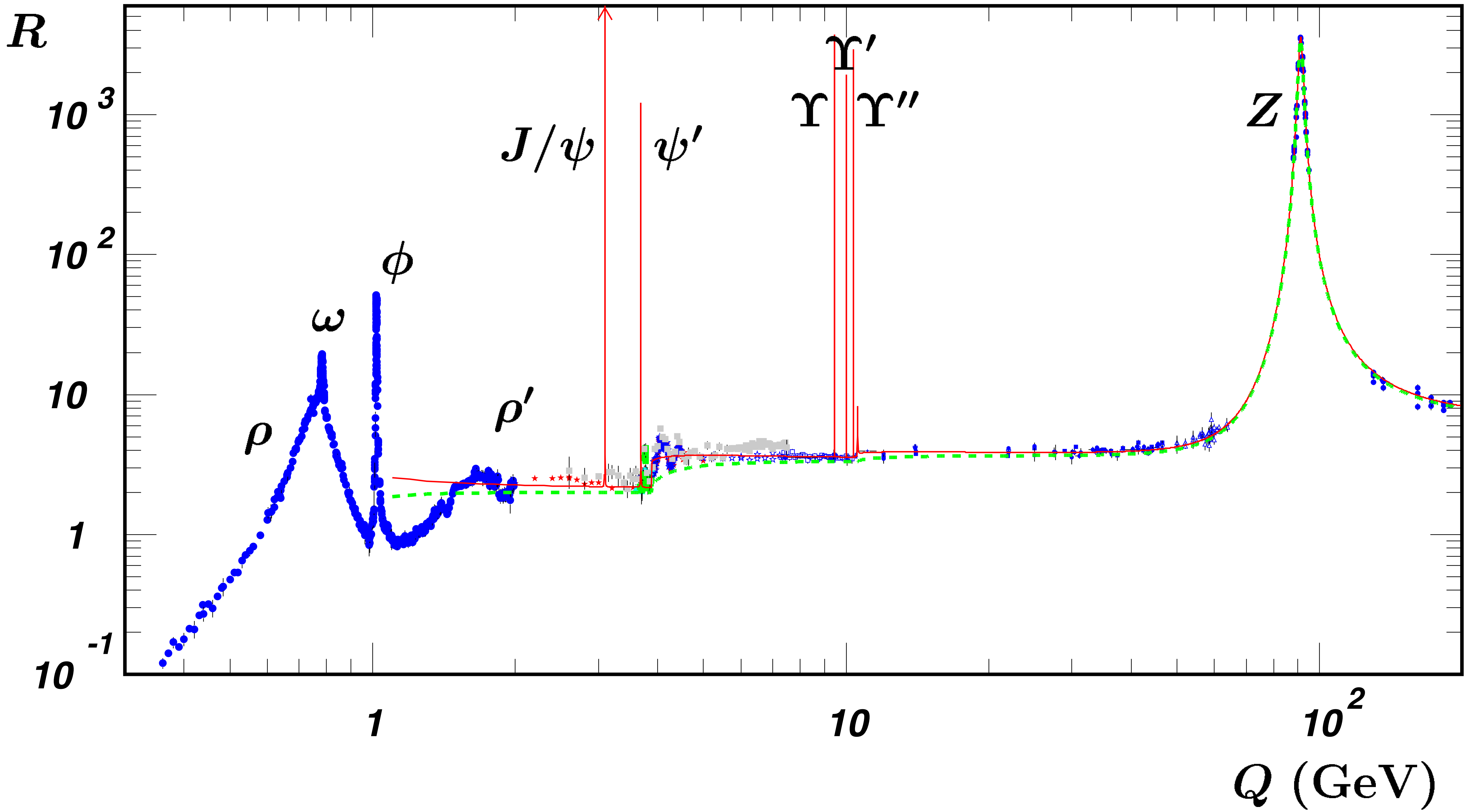
$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{20(\mathrm{nb})}{E^2_{\mathrm{beam}}/\mathrm{GeV}^2}$$

cross section measured at PETRA versus center-of-mass energy

solid (open) symbols $e^+e^- \to \mu^+\mu^-$ $(e^+e^- \to \tau^+\tau^-)$

–·– relativistic limit of lowest order QED prediction

R(Q)



Predictions compared to R measurements

Value of $R \simeq 2$ apparent below threshold for producing charmed at $Q = 2(m_c + m_u) \simeq 3.7$ GeV

Above threshold for all five quark flavors $(Q > 2m_b \simeq 10 \text{ GeV}), R = \frac{11}{3}$ as predicted

These measurements confirm that there are three colors of quark

$R = \dfrac{11}{3}$ would be reduced by a factor of 3 if there was only 1

Results for R will be modified when interpreted in context of QCD

Previous study is based on (leading order) process $e^+e^- \to q\bar{q}$

We should also include diagrams where q and/or antiq radiate g

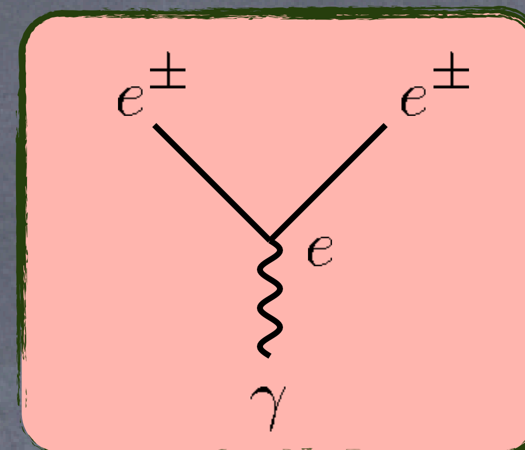
R(alpha, s)

In general

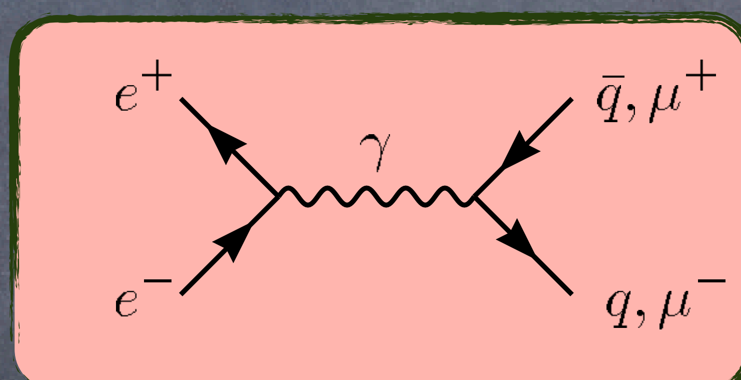
$$R(\alpha, s) = \frac{\sigma_{e^+e^- \to q\bar{q}}}{\sigma_{e^+e^- \to \mu^+\mu^-}}$$

is a function of electromagnetic coupling α

$$\alpha = \frac{e^2}{4\pi}$$



and annihilation energy $s = 4E_{\text{beam}}^2$



In antiparticles are drawn using only particle (e^-, μ^-, q) lines

BUT

we omit arrow lines indicating time direction of antip 4-momenta

We'll adopt this simplified notation

-- whenever there is no danger of confusion --

UV divergences

When annihilation energy far exceeds light masses m of q and ℓ
we must expect that for dimensionless observable R

$$R(\alpha, s) \underset{s \gg m^2}{\longrightarrow} \text{constant}$$

as there is no intrinsic scale in theories with massless exchange bosons

This prediction disagrees with experiment
and is not true in renormalizable QFT
Exchange of a massless photon is ultraviolet divergent
requiring introduction of a cutoff Λ
A scale is introduced into calculation a
and dimensionless observable $R(\alpha, s, \Lambda^2)$ is of form

$$R = R\left(\alpha, \frac{s}{\Lambda^2}\right)$$

This seems ugly ☛ it is not:
Λ appears order by order in perturbative series but not in final answer

Therefore ☛

$$\Lambda^2 \frac{dR}{d\Lambda^2} = 0$$

Renormalization Group Equation

Renormalization group equation $\Lambda^2 \frac{dR}{d\Lambda^2} = 0$

can be written more explicitly

$$\Lambda^2 \frac{\partial R}{\partial \Lambda^2} + \Lambda^2 \frac{\partial \alpha}{\partial \Lambda^2} \frac{\partial R}{\partial \alpha} = 0$$

which exhibits that R can depend on Λ directly or via coupling α

- can be rewritten in variable $t \equiv \ln(s/\Lambda^2)$

Using $\Lambda^2 \partial/(\partial \Lambda^2) = -\partial/[\partial \ln(s/\Lambda^2)]$ we obtain

$$\left(-\frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \alpha} \right) R\left(\alpha(s), \frac{s}{\Lambda^2} \right) = 0$$

where

$$\beta = \Lambda^2 \frac{\partial \alpha}{\partial \Lambda^2} = \frac{\partial \alpha}{\partial t}$$

Beta Function

With identification $\Lambda^2 = s$ ☛ renormalization group equation has very simple solution

$$R\Big(\alpha(s),\,1\Big)$$ ♻

in which observable depends on s only via coupling

Because $\alpha(s)$ is dimensionless ☛ dimensional analysis requires

$$\alpha(s) = F\left(\alpha(\Lambda^2),\,\frac{s}{\Lambda^2}\right)$$

which is consistent with

$$\beta = \Lambda^2\frac{\partial\alpha}{\partial\Lambda^2} = \frac{\partial\alpha}{\partial t}$$

$$\Lambda^2\frac{d\alpha(s)}{d\Lambda^2} = \left[\frac{\partial F}{\partial z}(\alpha(s), z)\right]_{z=1} = \beta(\alpha)$$

Solution is

$$t = \ln(s/\Lambda) = \int_{\alpha(\Lambda)}^{\alpha(s)}\frac{dx}{\beta(x)}$$

Hints of calculations

replacing

$$R\Big(\alpha(s),\,1\Big)$$

in

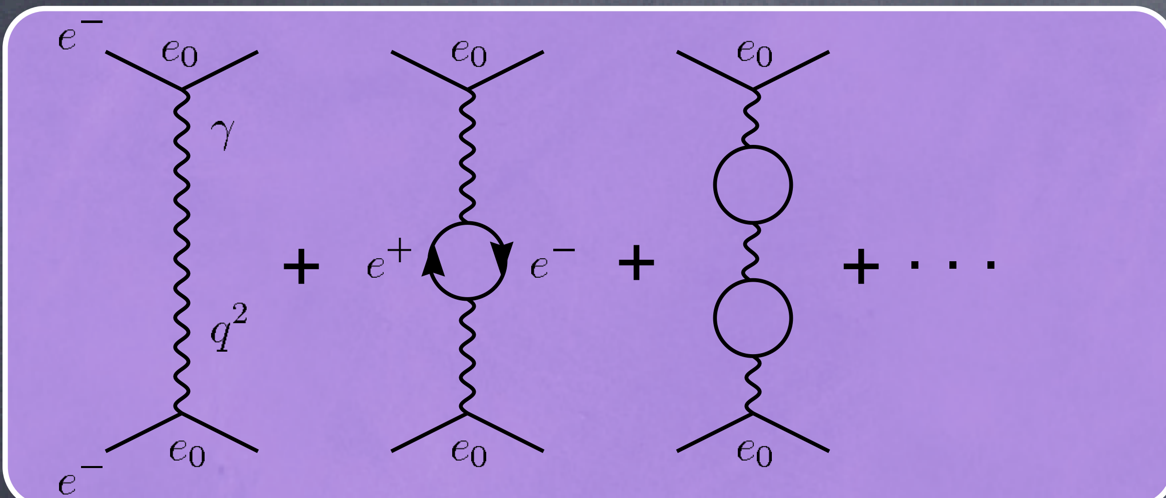
$$\left(-\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial \alpha}\right)R\left(\alpha(s),\,\frac{s}{\Lambda^2}\right)=0$$

we obtain

$$\left(-\frac{\partial \alpha}{\partial t}\frac{\partial R}{\partial \alpha}+\frac{\partial \alpha}{\partial t}\frac{\partial R}{\partial \alpha}\right)$$

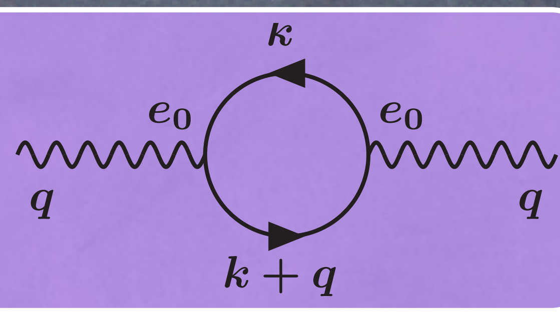
Running couplings

Running of coupling is described by β-function
which can be computed perturbatively
In QFT interaction of 2 electrons by exchange of virtual photon
is described by a perturbative series



$$= e_0^2 - e_0^2\Pi(q^2) + e_0^2\Pi^2(q^2) - \ldots$$
$$= \frac{e_0^2}{1+\Pi(q^2)}$$

with $\Pi(q^2) =$



Note negative sign
associated with fermion loop
which is made explicit in
to introduce summation

$\Pi(q^2)$ is ultraviolet divergent
as $k \to \infty$ (ask Maddie!!!)
explicit calculation confirms this
and we therefore write $\Pi(q^2)$
in terms of
divergent and finite part

$$\Pi(q^2) = \frac{e_0^2}{12\pi^2}\int_{m_e^2}^{\Lambda^2}\frac{dk^2}{k^2} - \frac{e_0^2}{12\pi^2}\ln\frac{-q^2}{m_e^2}$$
$$= \frac{e_0^2}{12\pi^2}\ln\left(\frac{\Lambda^2}{-q^2}\right)$$

Renormalizable charge

The trick is to introduce a new charge e which is finite

$$e^2 \quad = \quad e_0^2 \left[1 - \Pi(-q^2 = \mu^2) + \cdots\right]$$ ☆

or

$$e \quad = \quad e_0 \left[1 - \frac{1}{2}\Pi(-q^2 = \mu^2) + \cdots\right]$$ ö

e_0 is infinitesimal and combines with divergent loop Π

to yield finite physical charge e

This operation is performed at some reference momentum μ

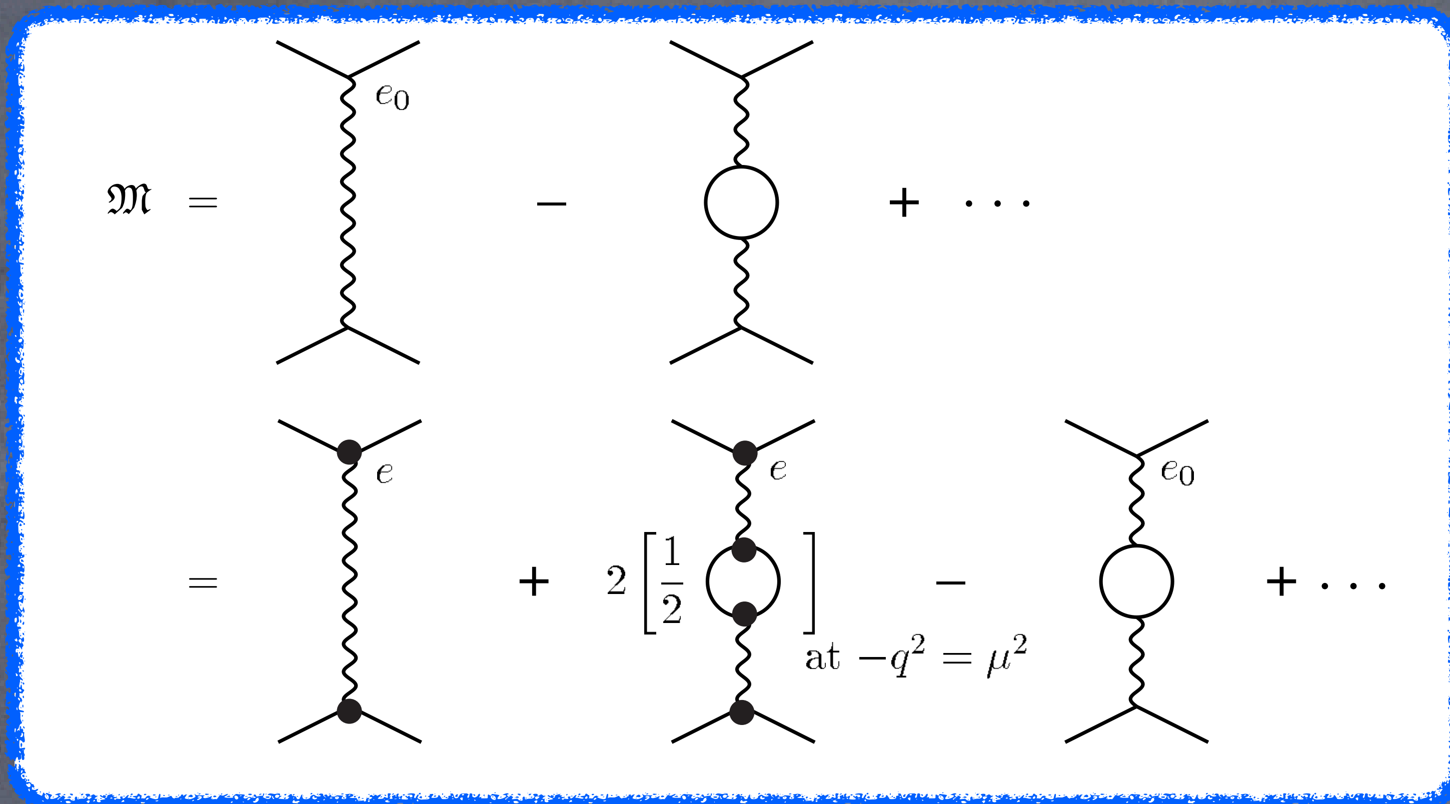
e.g. $e(\mu = 0)$ is Thomson charge with

$$\alpha = e^2(\mu = 0)/(4\pi) = 1/137.035999679(94)$$

Removing UV divergences

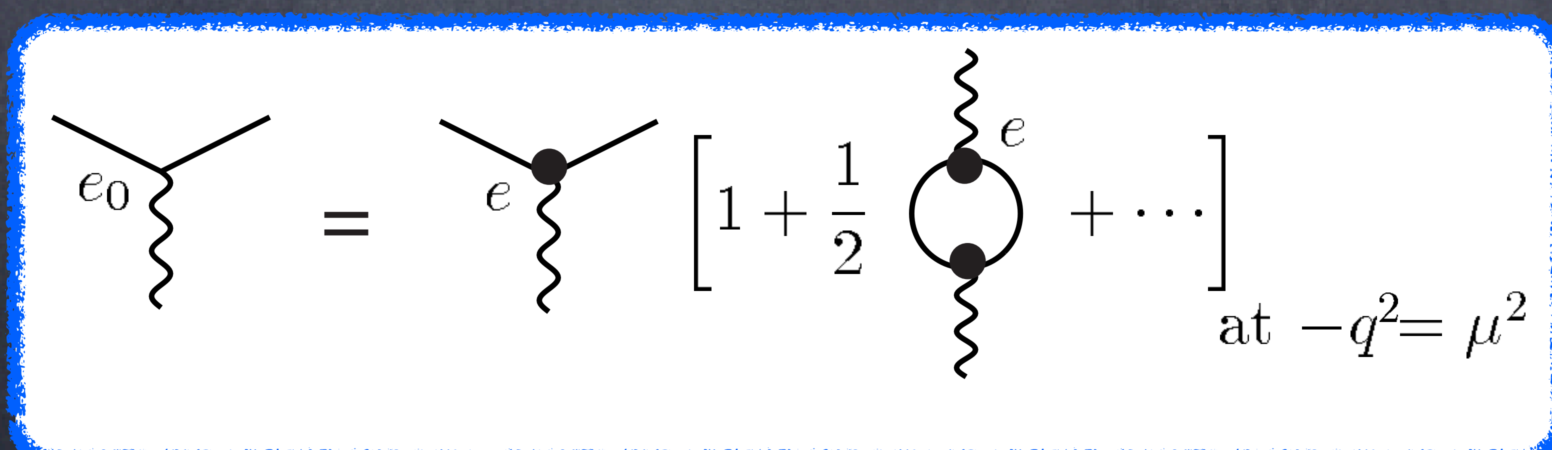
To illustrate how this works we calculate $e^- e^-$ scattering

Amplitude is (ignoring identical particle effects)



where ★ has been obtained

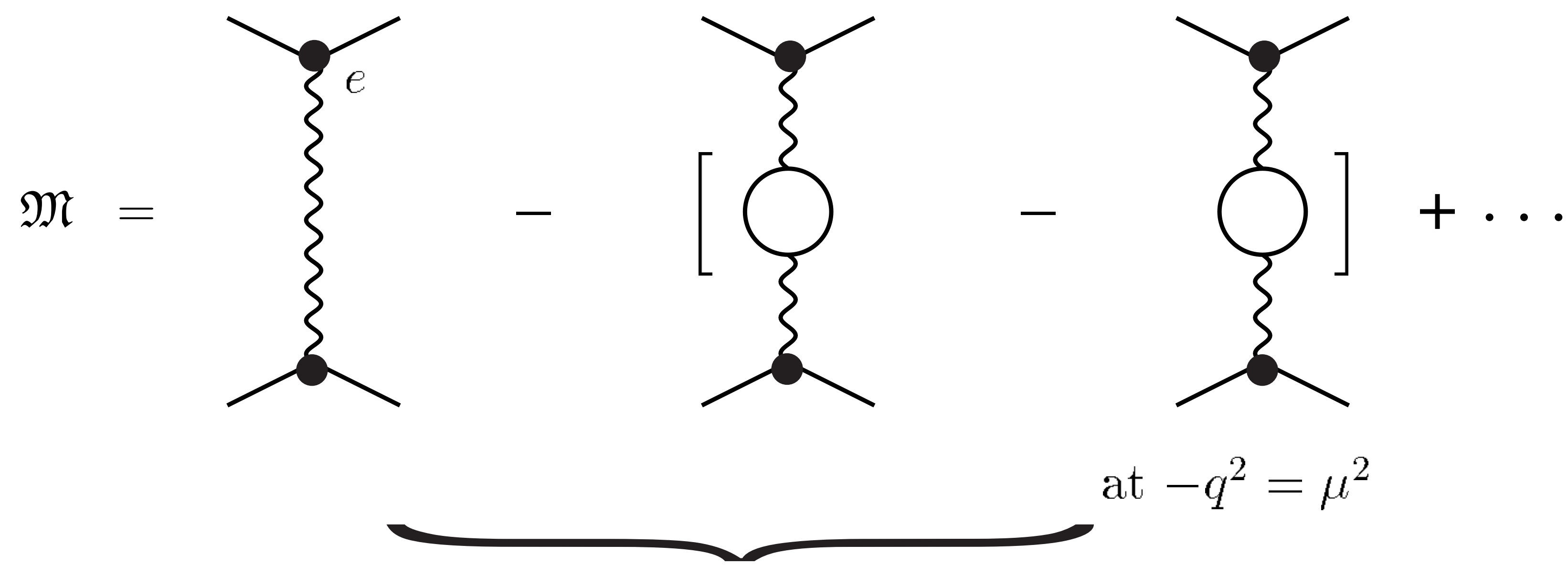
by substituting renormalized charge e for bare charge using ö



Removing UV divergences (cont'd)

In last term of ★ we can just replace e_0 by e
as additional terms associated with substitution ♜
only appear in higher order

Therefore ★ can be rewritten as:



$$\frac{\alpha}{3\pi}\ln\left(\frac{\Lambda^2}{-q^2}\right) - \frac{\alpha}{3\pi}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$$

$$= \frac{\alpha}{3\pi}\ln\left(\frac{\mu^2}{-q^2}\right) = \text{finite!}$$

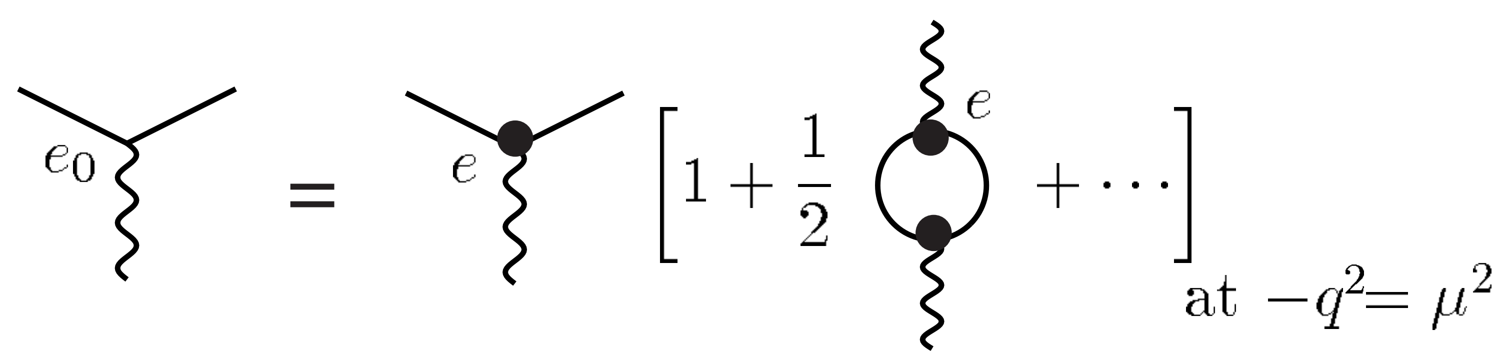
Hints of calculations

$$e^2 = \frac{e_0^2}{1+\Pi(q^2)} \qquad\qquad e_0^2 = e^2(1+\Pi(q^2))$$

Using Taylor series for $f(x) = \sqrt{1+x}$ around $x=0$

$$e_0 = e\left[1 + \frac{1}{2}\Pi(q^2) - \frac{1}{8}\Pi^2(q^2) + \frac{1}{16}\Pi^3(q^3) - \dots\right]$$

we get correct sign to remove infinity in loop diagram of photon propagator



Running charge

Divergent parts cancel and we obtain a finite result to $\mathcal{O}(\alpha^2)$

In a renormalizable theory this cancellation happens at every order of perturbation theory

Price we have paid is introduction of a parameter $\alpha(\mu^2)$ which is fixed by experiment

Electron charge (unfortunately) cannot be calculated

In summary ☛ by using substitution ✮
perturbation series using infinitesimal charges e_0 and infinite loops $\prod$ has been reshuffled order by order to obtain finite observables

Running charge can be written as

$$\alpha = \alpha_0 \left[1 - \Pi(q^2) + \cdots\right] = \frac{\alpha_0}{1 + \Pi(q^2)}$$

QED Beta Function

For QED result ❥

$$\alpha(Q^2 = -q^2) = \frac{\alpha_0}{1 - b\alpha_0 \ln \frac{Q^2}{\Lambda^2}}$$ ❀

with $b = 1/3\pi$

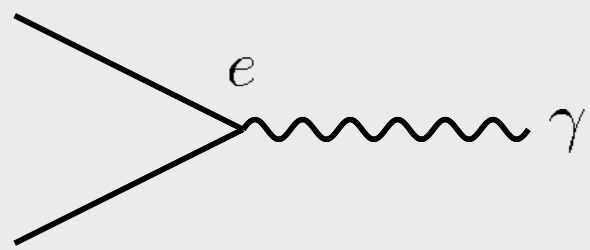
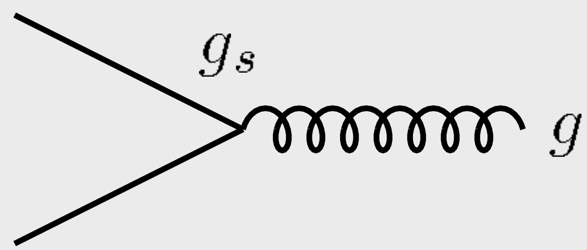
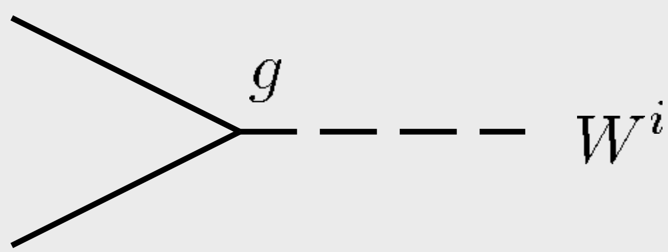
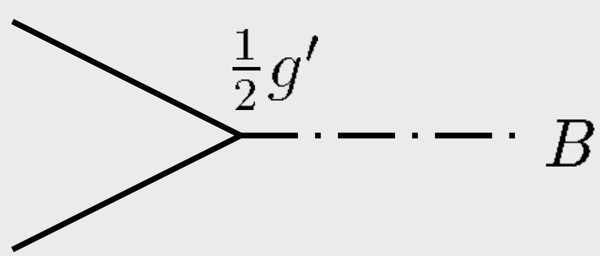
Ultraviolet cutoff is eliminated
by renormalizing charge to some measured value at $Q^2 = \mu^2$

$$\frac{1}{\alpha(Q^2)} - \frac{1}{\alpha(\mu^2)} = -b \ln \frac{Q^2}{\mu^2}$$ ♞

One also notices that b determines β-function
to leading order in perturbation theory
We obtain indeed from ♬ and ❀ that

$$\beta(\alpha) = \frac{\partial \alpha(Q^2)}{\partial t} = b\alpha^2 + \mathcal{O}(\alpha^3)$$

b-values for running of coupling constants

coupling $\alpha \equiv \frac{g^2}{4\pi}$	b-value
	$\frac{1}{3\pi}$
	$\frac{2n_q - 33}{12\pi}$
	$\frac{4n_g + \frac{1}{2}n_d - 22}{12\pi}$
	$\frac{-\frac{20}{3}n_g + \frac{1}{2}n_d}{12\pi}$

n_q : number of quarks (2–6)

n_g : number of generations (3)

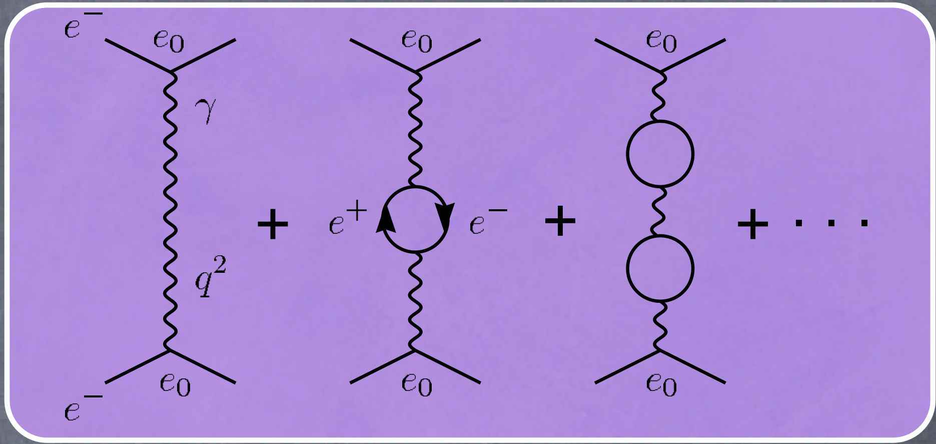
n_d : number of Higgs doublets (1)

Charge Screening

From it is clear that much of structure of gauge theory is dictated by identifying momentum dependence of couplings

Formal arguments have revealed screening of electric charge

There is physics associated with



In QFT a charge is surrounded by virtual e^+e^- pairs which screen charge more efficiently at large than at small distances

Therefore

$$\alpha^{-1}(\mu^2 = 0) \simeq 137$$

is smaller than short-distance value

$$\alpha^{-1}(\mu^2 = m_Z^2) = 127.925 \pm 0.016$$

We note that qualitatively

$$\frac{1}{\alpha(0)} - \frac{1}{\alpha(m_Z^2)} \simeq 9 \simeq \frac{1}{3\pi} \ln\left(\frac{m_Z^2}{m_e^2}\right)$$

Asymptotic Freedom

For 3 generations of quarks b -value for QCD is negative

While $q\bar{q}$ pairs screen color charge

just like e^+e^- pairs screen electric charge ($2n_f/12\pi$ term in b)

gluon loops reverse that effect with larger negative b-value $\dfrac{-33}{12\,\pi}$

Color charge grows with distance yielding asymptotic freedom!!!

property: $\alpha_s \to 0$ as $Q \to \infty$

On the other hand ☛ theory becomes strongly coupled at $Q^2 \sim \Lambda^2_{\mathrm{QCD}}$

-- infrared slavery --

presumably leading to confinement of quarks and gluons

QCD Beta Function

$$Q^2 \frac{\partial \alpha_s(Q^2)}{\partial Q^2} = \beta\left(\alpha_s(Q^2)\right)$$

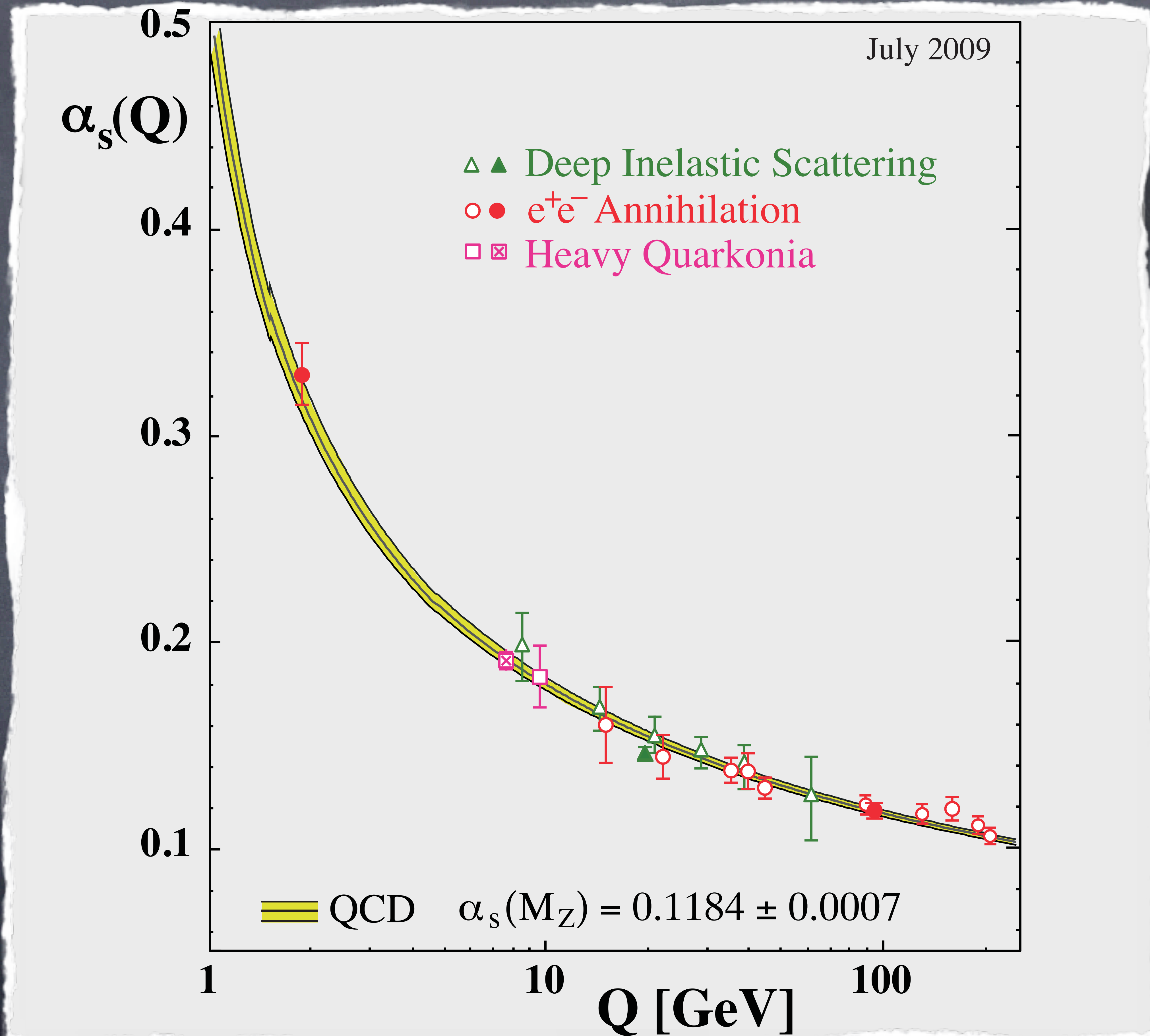
PERTURBATIVE EXPANSION OF β-FUNCTION CALCULATED TO COMPLETE 4-LOOP APPROXIMATION

$$\begin{aligned}\beta(\alpha_s(Q^2)) = & - \beta_0 \alpha_s^2(Q^2) - \beta_1 \alpha_s^3(Q^2) \\ & - \beta_2 \alpha_s^4(Q^2) - \beta_3 \alpha_s^5(Q^2) + \mathcal{O}(\alpha_s^6)\end{aligned}$$

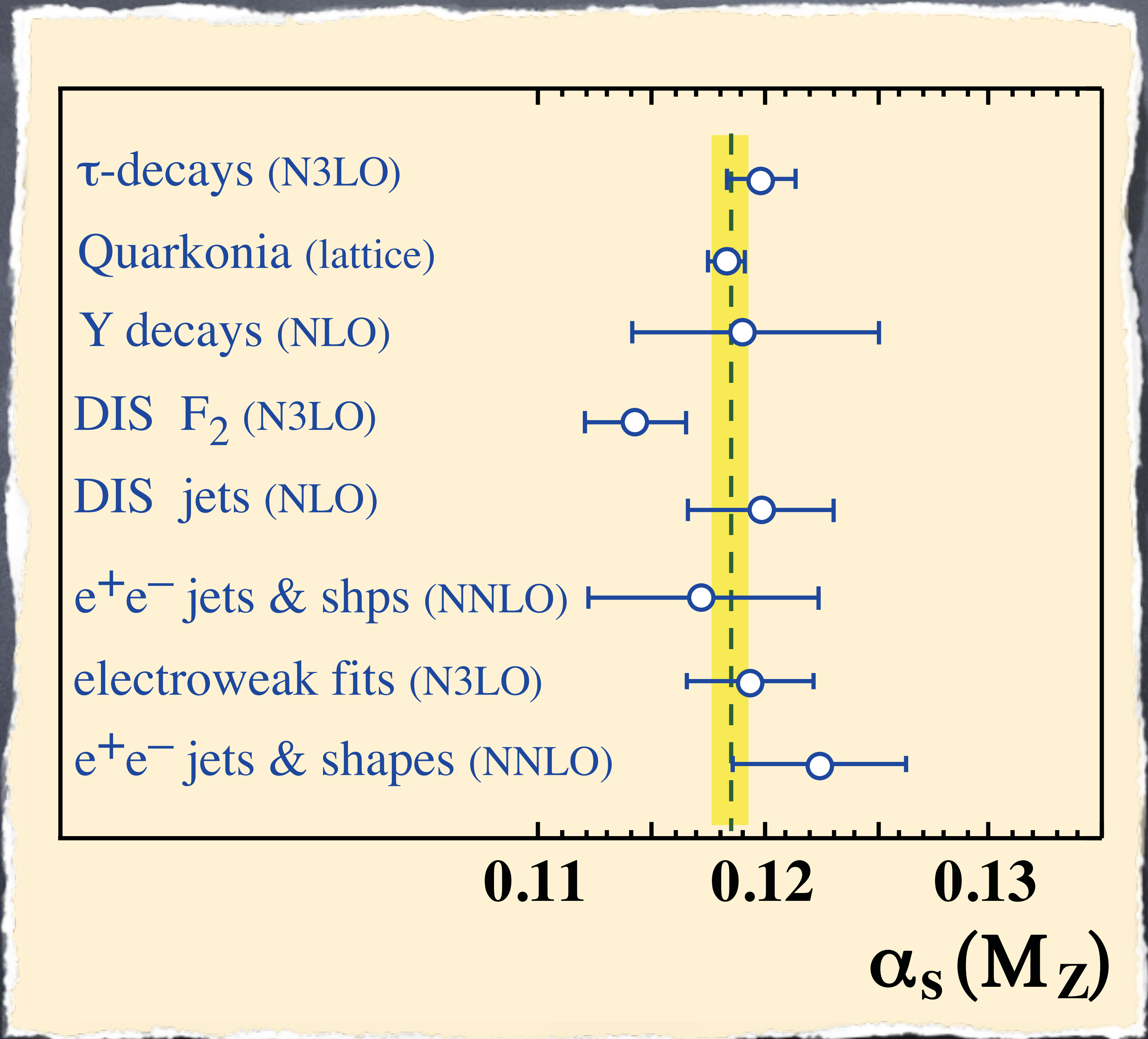
$$\begin{aligned}\beta_0 &= \frac{33 - 2N_f}{12\pi}, \\ \beta_1 &= \frac{153 - 19N_f}{24\pi^2}, \\ \beta_2 &= \frac{77139 - 15099N_f + 325N_f^2}{3456\pi^3}, \\ \beta_3 &\approx \frac{29243 - 6946.3N_f + 405.089N_f^2 + 1.49931N_f^3}{256\pi^4}\end{aligned}$$

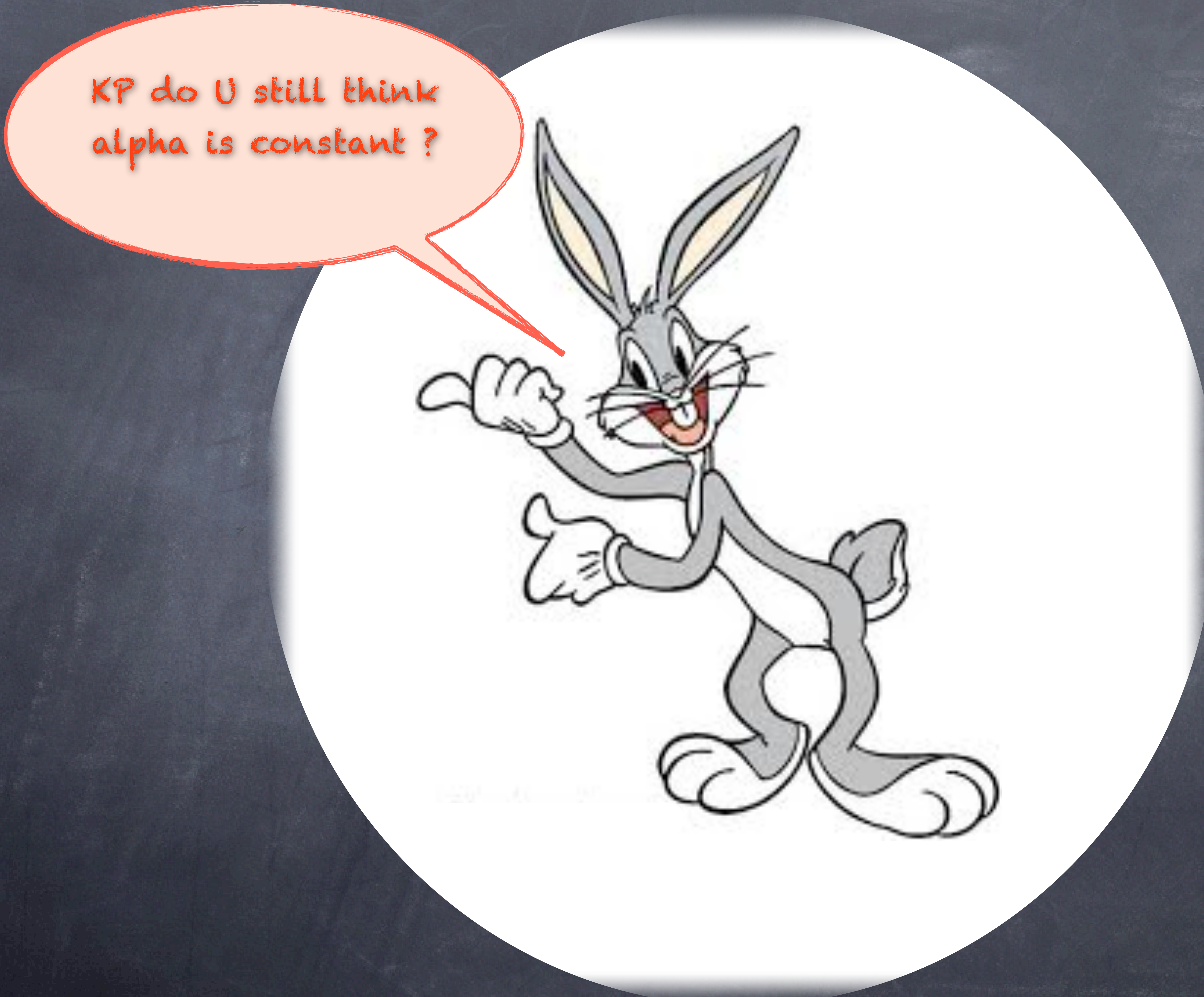
N_f NUMBER OF ACTIVE QUARK FLAVOURS AT ENERGY SCALE Q

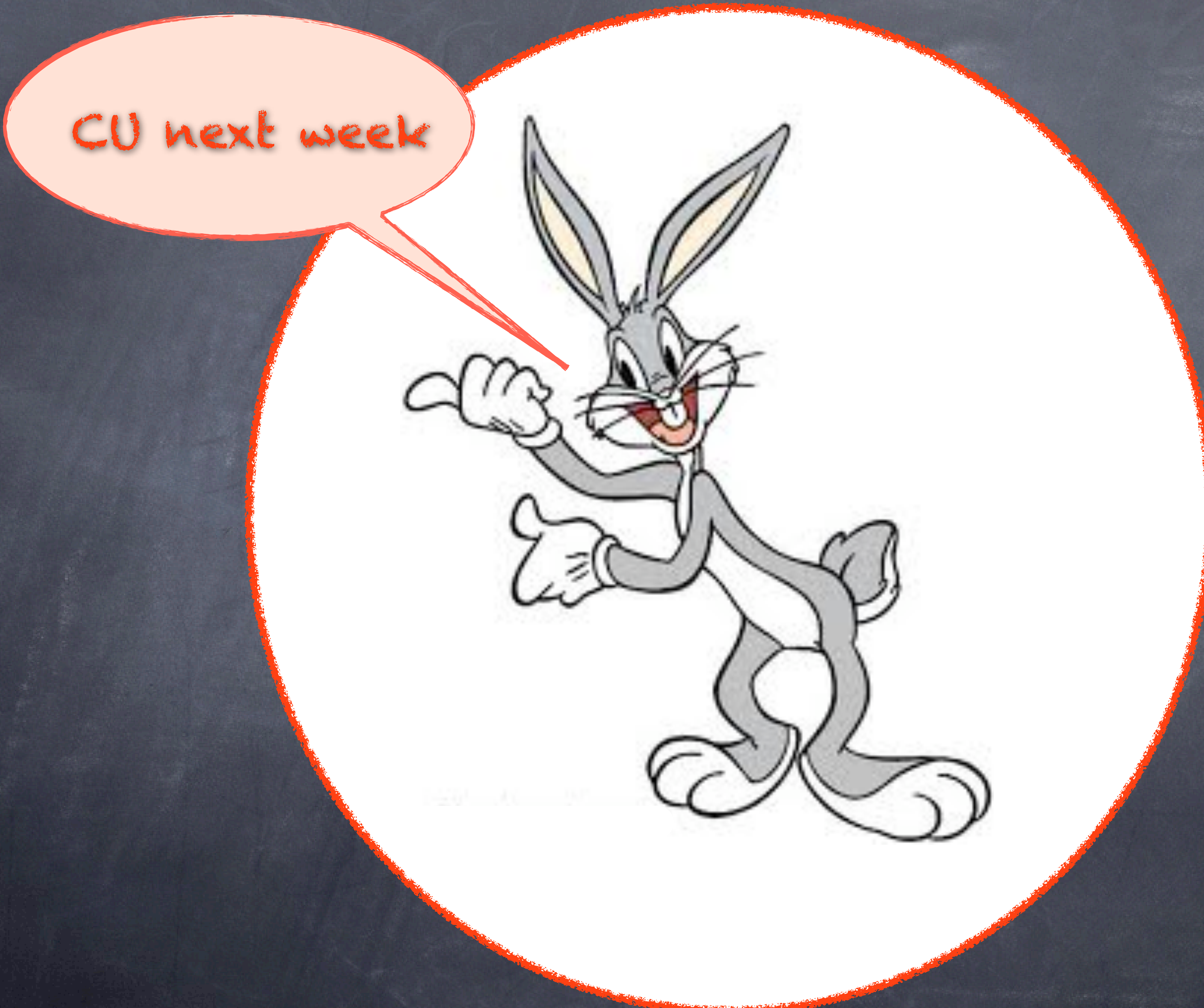
2009 World Average of Alpha-Strong



Summary of measurements of $\alpha_s(M_Z)$




KP do U still think alpha is constant ?


CU next week