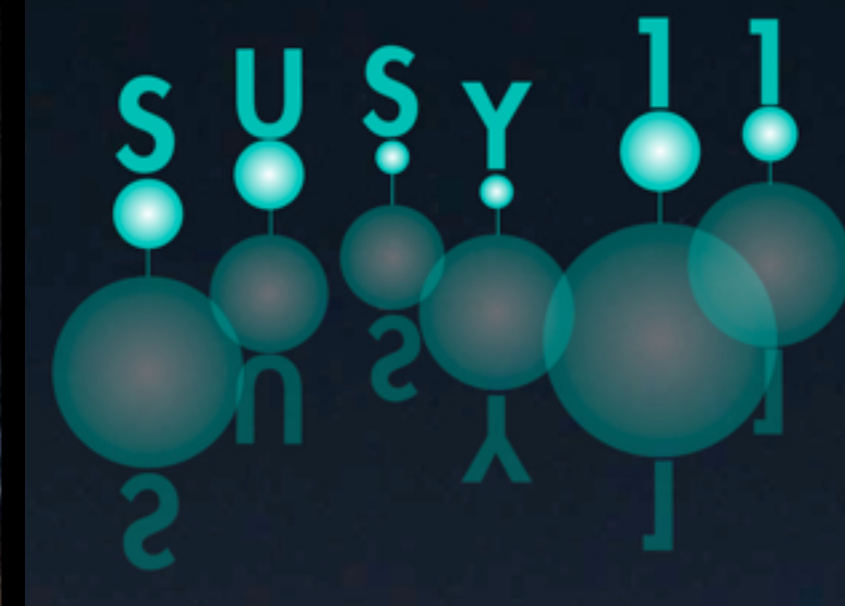

Particle Physics 2011

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Summary and Motivation

- Last class much importance has been given to symmetry principles
- Discussed:

  connection between exact symmetries and conservation laws

  local gauge invariance can serve as a dynamical principle

  to guide assembly of interacting field theories
- BUT in several areas we are still far from where we need to be

  E.G. gauge principle has lead us to theories

  in which all interactions are mediated by massless bosons

  and we know that carriers of weak force are massive
- There are many situations in physics

  where exact symmetry of interaction is hidden by circumstances

  Canonical example is that of a Heisenberg ferromagnet:

  infinite crystalline array of spin-1/2 magnetic dipoles

Below Curie temperature ground state is completely ordered configuration

all dipoles are aligned in some arbitrary direction

distorting rotation invariance of the underlying interaction

- It is thus of interest to learn how to deal with hidden symmetries

Paramagnetic Phase

Nearest-neighbor interaction between spins
(or magnetic dipole moments)
is invariant under group of spatial rotations $SO(3)$

In disordered paramagnetic phase
-- which exists above T_C --
medium displays exact symmetry in absence of external field

spontaneous magnetization of system is zero
and there is no preferred direction in space

$SO(3)$ invariance is manifest

Privileged direction may be selected by imposing external $\vec{B}$-field
which tends to align spins in material

$SO(3)$ symmetry is hence broken down to an axial $SO(2)$
--symmetry of rotations around the external field direction --

Full symmetry is restored when external field is turned off

Ferromagnetic Phase

✓ Below T_C situation is rather different

system is in ordered ferromagnetic phase

✓ In absence of an external field

lowest energy configuration has non-zero spontaneous magnetization
because nearest-neighbor force favors parallel alignment of spins

$SO(3)$ symmetry is said to be spontaneously broken down to $SO(2)$

✓ Vestiges of original symmetry $SO(3)$

direction of spontaneous magnetization is random

measurable properties of infinite ferromagnet

do not depend upon its orientation

✓ Ground state is thus infinitely degenerate

✓ Particular direction for spontaneous magnetization is chosen

by imposing external field which breaks $SO(3)$ symmetry explicitly

Higgs Mechanism

Contrasting paramagnetic case ☛ spontaneous magnetization does not return to zero when the external field is turned off

For rotational invariance to be broken spontaneously
it is crucial that ferromagnet be infinite in extent
so that rotation from one degenerate ground state to another
would require impossible task
of rotating infinite number of elementary dipoles

Spontaneous symmetry breaking can arise
when Lagrangian of a system possesses symmetries
which do not however hold for ground state of system

Higgs mechanism is a gauge theoretic realization
of such spontaneous symmetry breaking

$\lambda\phi^4$-potential

Consider simple world consisting just of scalar particles described by Lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi\right)\left(\partial^\mu \phi\right) - V(\phi)} \quad *$$

study how particle spectrum depends on effective potential $V(\phi)$

if potential is even functional of scalar field $V(\phi) = V(-\phi)$

Lagrangian is invariant under symmetry operation

which replaces ϕ by $-\phi$

consider an explicit potential

$$\boxed{V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4} \quad ♠$$

$\lambda > 0$ so that energy is bounded from below

Vacuum expectation value

Two qualitatively different cases
corresponding to manifest or spontaneously broken symmetry
may be distinguished depending on sign of coefficient μ^2

If $\mu^2 > 0$ potential has a unique minimum at $\phi = 0$

corresponding to ground state -- a.k.a. vacuum --

Identification most easily seen in Hamiltonian formalism

substituting ✲ into $\mathcal{H}(x) = \pi(x)\ \dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi)$

$$\boxed{\mathcal{H} = \frac{1}{2}\left[(\partial_0 \phi)^2 + \left(\vec{\nabla}\phi\right)^2\right] + V(\phi)}$$

state of lowest energy corresponds to $\phi = \langle \phi \rangle_0$

value of constant $\langle \phi \rangle_0$ is determined by dynamics of theory

it corresponds to absolute minimum (or minima) of potential $V(\phi)$

(We usually refer to $\langle \phi \rangle_0$ as vacuum expectation value of field ϕ)

Hints for the calculation

$$\begin{aligned}\mathcal{L} &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - V(\phi)\\ &= \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi)\end{aligned}$$

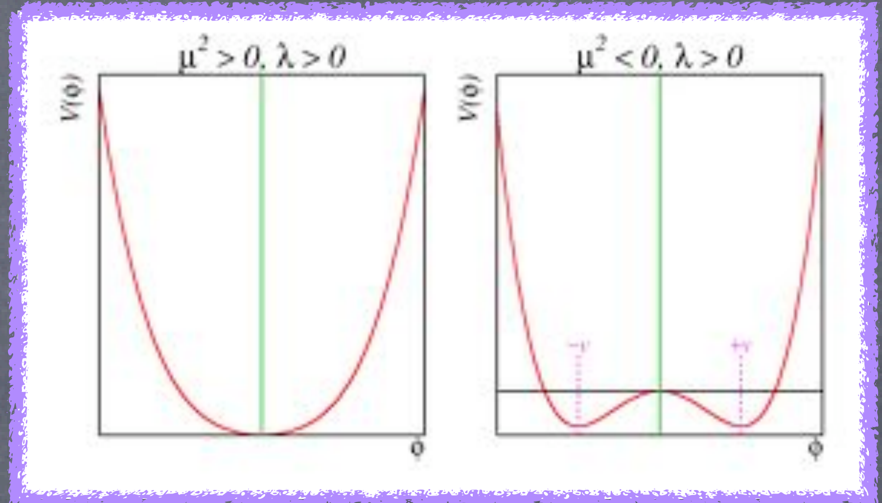
$$\pi(x) \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}$$

$$\begin{aligned}\mathcal{H} &= \pi(x)\dot{\phi} - \mathcal{L}\\ &= \frac{1}{2}[\dot{\phi}^2 + (\vec{\nabla}\phi)^2] + V(\phi)\end{aligned}$$

Lagrangian symmetries

For $\mu > 0$ ☛ vacuum obeys reflection symmetry of $\mathcal{L}$ with $\langle\phi\rangle_0 = 0$

To study small oscillations around this minimum consider $\mathcal{L}$ of a free particle with mass μ



$$\boxed{\mathcal{L} = \tfrac{1}{2}[(\partial_\mu\phi)(\partial^\mu\phi) - \mu^2\phi^2]}$$

ϕ^4 term shows that field is self-interacting because 4-particle vertex exists with coupling λ

For $\mu^2 < 0$ ☛ $\mathcal{L}$ has a mass term of wrong sign for field ϕ

potential has 2 minima satisfying $\phi(\mu^2 + \lambda\phi^2) = 0$

$$\langle\phi\rangle_0 = \pm v \quad \text{with} \quad v = \sqrt{-\mu^2/\lambda}$$

(Extremum $\phi = 0$ does not correspond to energy minimum)

Potential has two degenerate lowest energy states

either of which may be chosen to be vacuum

because of parity invariance of Lagrangian

physical consequences must be independent of this choice

Vacuum symmetries

Whatever is our choice ☛ symmetry of theory is spontaneously broken:

parity transformation $\phi \to -\phi$ is an invariant of Lagrangian but not of vacuum state

Take $\langle \phi \rangle_0 = +v$

Perturbative calculations involve expansions around classical minimum

$$\boxed{\phi(x) = v + \eta(x)} \quad ⊙$$

$\eta(x)$ represents quantum fluctuations about this minimum

substituting ⊙ into ✻ + ♠

$$\boxed{\mathcal{L}' = \tfrac{1}{2}(\partial_\mu \eta)(\partial^\mu \eta) - \lambda v^2 \eta^2 - \lambda v \eta^3 - \tfrac{1}{4}\lambda \eta^4 + \text{const}} \quad ⚑$$

field η has a mass term of correct sign

Identifying first two terms of $\mathcal{L}'$ with $\mathcal{L} = \dfrac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \dfrac{1}{2} m^2 \phi^2$

gives ☛ $m_\eta = \sqrt{2\lambda v^2} = \sqrt{-2\mu^2}$

Higher-order terms in η represent interaction of η field with itself

Hints for the calculation

$$(v+\eta)^2 = v^2 + 2v\eta + \eta^2$$

$$(v+\eta)^4 = v^4 + 4v^3\eta + 6v^2\eta^2 + 4v\eta^3 + \eta^4$$

$$\frac{1}{2}v^3\lambda + \mathbf{v^3}\lambda\eta + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{v^2}\eta^{\mathbf{2}}\lambda - \frac{1}{4}\lambda v^4 - \lambda\mathbf{v^3}\eta - \frac{\mathbf{3}}{\mathbf{2}}\lambda\mathbf{v^2}\eta^{\mathbf{2}} - \lambda v\eta^3 - \frac{\mathbf{1}}{\mathbf{4}}\lambda\eta^{\mathbf{4}}$$

Massive Scalar Particle

- Note that $\mathcal{L}$ and $\mathcal{L}'$ are completely equivalent
- A transformation of type ⊙ cannot change physics
- If we could solve two Lagrangians exactly
  they must yield identical physics
- In QFT we are not able to perform such a calculation
  we do perturbation theory
  and calculate fluctuations around minimum energy
- Using $\mathcal{L}$ we find out that perturbation series does not converge
  because we are trying to expand about unstable point $\phi = 0$
  correct way to proceed is to adopt $\mathcal{L}'$
  and expand in η around stable vacuum $\langle \phi \rangle_0 = +v$
- In perturbation theory $\mathcal{L}'$ provides correct physical framework
  whereas $\mathcal{L}$ does not
- Therefore ☛ scalar particle
  --described by in-principle-equivalent Lagrangians $\mathcal{L}$ and $\mathcal{L}'$ --
  is massive

Complex Scalar Field

To approach our destination of generating a mass for gauge bosons

duplicate procedure for a complex scalar field $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$

with Lagrangian density

$$\boxed{\mathcal{L} = (\partial_\mu \phi)^*(\partial^\mu \phi) - \mu^2 \phi^* \phi - \lambda(\phi^* \phi)^2} \quad ✣$$

which is invariant under transformation $\phi \to e^{i\alpha}\phi$

$\mathcal{L}$ possesses a $U(1)$ global gauge symmetry

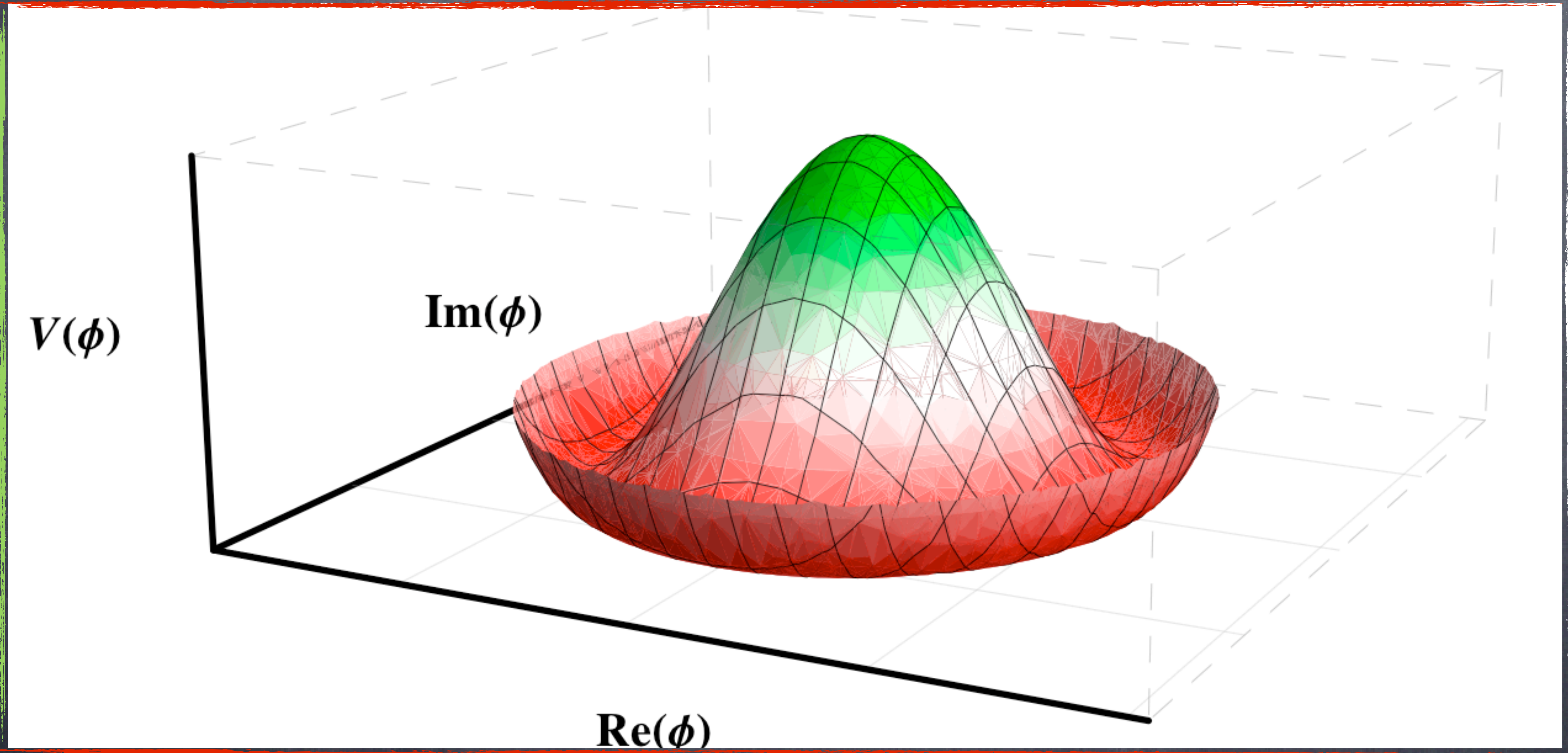
By considering $\lambda > 0$ and $\mu^2 < 0$ rewrite ✣ as

$$\boxed{\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)(\partial^\mu \phi_1) + \frac{1}{2}(\partial_\mu \phi_2)(\partial^\mu \phi_2) - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{1}{4}\lambda(\phi_1^2 + \phi_2^2)^2}$$

Mexican Hat

Circle of minima of potential $V(\phi)$ in ϕ_1 - ϕ_2 plane of radius v

$$\phi_1^2 + \phi_2^2 = v^2 \qquad \text{with} \qquad v^2 = -\mu^2/\lambda$$



we translate field ϕ to a minimum energy position

without loss of generality we may take $\phi_1 = v$ and $\phi_2 = 0$

Goldstone Boson

We expand $\mathcal{L}$ around vacuum in terms of fields η and ξ

by substituting $$\phi(x) = \sqrt{\tfrac{1}{2}}[v + \eta(x) + i\xi(x)]$$ ♜

into ✜

$$\mathcal{L}' = \tfrac{1}{2}(\partial_\mu \xi)^2 + \tfrac{1}{2}(\partial_\mu \eta)^2 + \mu^2 \eta^2 + \text{const.} + \mathcal{O}(\eta^3, \xi^3) + \mathcal{O}(\eta^4, \xi^4)$$ ♝

The third term has form of a mass term $(-\frac{1}{2}m_\eta^2 \eta^2)$ for η field

η -mass is again $m_\eta = \sqrt{-2\mu^2}$

First term in $\mathcal{L}'$ stands for kinetic energy of ξ

there is no corresponding mass term for ξ field

Theory contains a massless scalar ☛ so-called Goldstone boson

In attempting to generate massive gauge boson

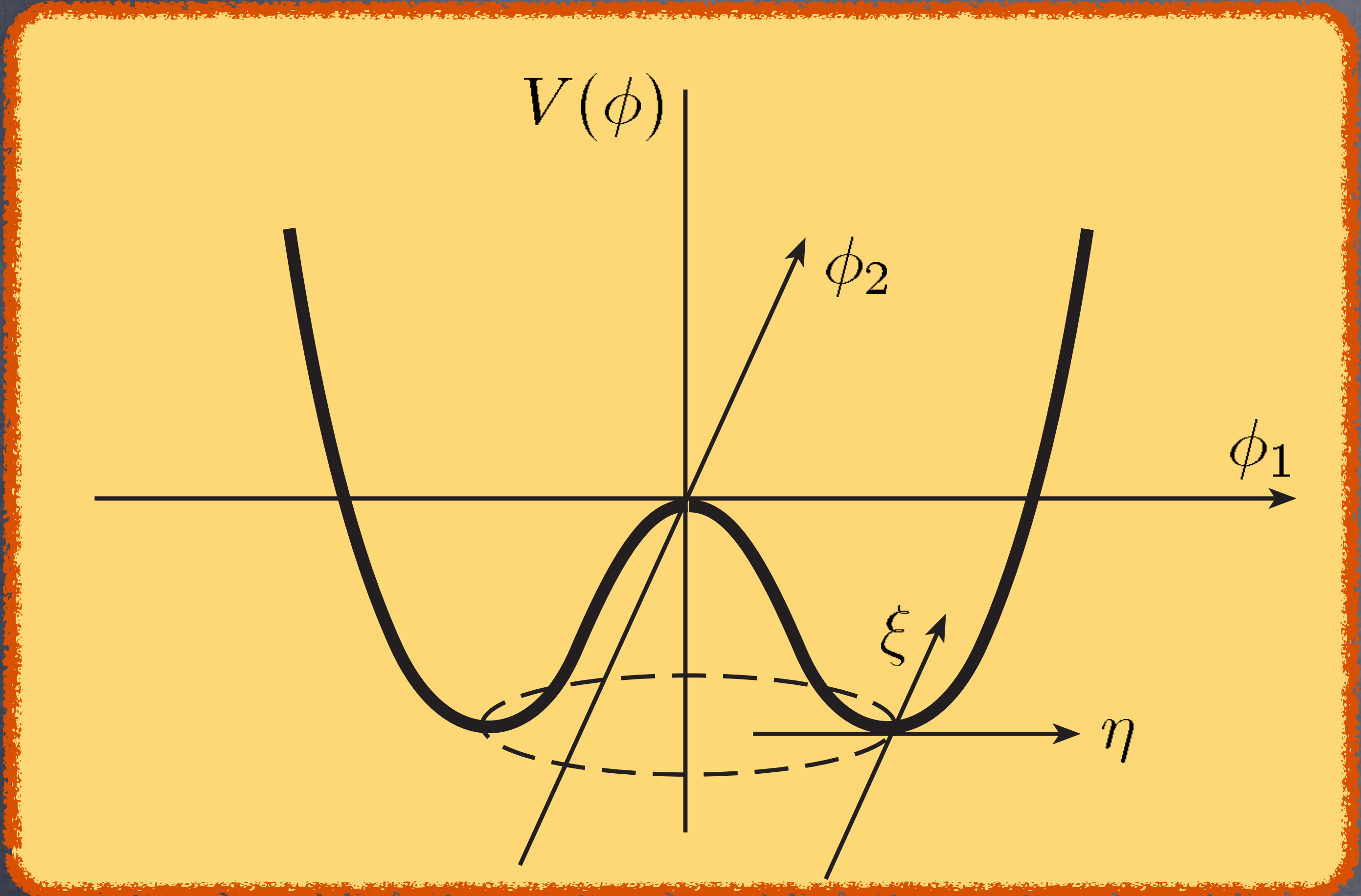
we have encountered a problem:

spontaneously broken gauge theory

seems to be plagued with its own massless scalar particle

Goldstone Theorem

Intuitively it is easily seen reason for presence of Goldstone boson
Potential in tangent ξ direction is flat ☛ implying massless mode
-- there is no resistance to excitations along ξ direction --



Potential $V(\phi)$

for a complex scalar field

for case $\mu^2 < 0$ and $\lambda > 0$

Lagrangian ♝ is a simple example of Goldstone Theorem:
In spontaneous symmetry breaking original symmetry is still present but nature manages to camouflage symmetry in such a way that its presence can be viewed only indirectly

In ferromagnet example ☛ analogue of our Goldstone boson is long-range spin waves which are oscillations of spin alignment

Spontaneous Symmetry Breaking of Local $U(1)$ Gauge Symmetry

Start with Lagrangian invariant under local $U(1)$ transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x)$$

accomplished replacing ∂_μ by covariant derivative $D_\mu = \partial_\mu + ieA_\mu$

Recall gauge field transforms as $A_\mu(x) \to A_\mu(x) - \partial_\mu \alpha(x)/e$

Gauge Invariant Lagrangian is then

$$\mathcal{L} = (\partial^\mu - ieA^\mu)\phi^*(\partial_\mu + ieA_\mu)\phi - \mu^2\phi^*\phi - \lambda(\phi^*\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Two cases depending upon parameters of effective potential

If $\mu^2 > 0$ -- aside from ϕ^4 self-interaction term --
this is just QED Lagrangian for charged scalar particle of mass μ

If $\mu^2 < 0$ ☛ spontaneously broken symmetry
this case demands a closer analysis

Counting Degrees of Freedom

Substituting $\phi(\eta(x), i\xi(x))$ (expression ♜) into Lagrangian ✪

$$\mathcal{L}' = \frac{1}{2}(\partial_\mu \xi)^2 + \frac{1}{2}(\partial_\mu \eta)^2 - v^2\lambda\eta^2 + \frac{1}{2}e^2v^2A_\mu A^\mu + evA_\mu\partial^\mu\xi - \frac{1}{4}F_{\mu\nu}^2 + \text{interaction terms} \quad ♥$$

Particle spectrum of $\mathcal{L}'$ appears to be:

massless Goldstone boson ξ massive scalar η massive vector A_μ

$$m_\xi = 0, m_\eta = \sqrt{2\lambda v^2} \text{ and } m_A = ev$$

We have generated mass for gauge field but still facing problem

occurrence of Goldstone boson

Because of presence of a term off-diagonal in fields $A_\mu \partial^\mu \xi$ care must be taken in interpreting Lagrangian ♥

Particle spectrum we assigned before to $\mathcal{L}'$ must be incorrect

Giving mass to A_μ we have raised polarization degrees of freedom (from 2 to 3) because it can now have longitudinal polarization

BUT translating field variables does not create degrees of freedom

We deduce fields in $\mathcal{L}'$ do not all correspond to distinct particles

Gauge Transformation

To find gauge transformation which eliminates field from $\mathcal{L}'$ we first note that to lowest order in ξ (♜)

can be rewritten as

$$\phi \simeq \sqrt{\tfrac{1}{2}}(v+\eta)\, e^{i\xi/v}$$

This suggests we should substitute different set of real fields

H, θ, A_μ

$$\phi \to \sqrt{\tfrac{1}{2}}\,[v+H(x)]e^{i\theta(x)/v}\,, \quad A_\mu \to A_\mu - \frac{1}{ev}\,\partial_\mu\theta$$

into original Lagrangian ✪

This is a particular choice of gauge with $\theta(x)$ chosen that H is real

We therefore anticipate that theory will be independent of θ

Higgs particle

$$\begin{aligned}\mathcal{L}\text{''} &= \tfrac{1}{2}(\partial_\mu H)^2 - \lambda v^2 H^2 + \tfrac{1}{2}e^2 v^2 A_\mu^2 - \lambda v H^3 - \tfrac{1}{4}\lambda H^4 \\ &+ \tfrac{1}{2}e^2 A_\mu^2 H^2 + v e^2 A_\mu^2 H - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\end{aligned}$$

Goldstone boson is not actually present in theory

Apparent extra degree of freedom is actually spurious
it corresponds only to freedom to make gauge transformation

Lagrangian describes just two interacting massive particles
a vector gauge boson A_μ and a massive scalar H
usually referred to as a Higgs particle

Unwanted massless Goldstone boson
has been turned into longitudinal polarization of A_μ

This is known as **Higgs mechanism**

Standard Model of Particle Physics

Standard model of weak, electromagnetic, and strong interactions is based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$

A single generation of quarks and leptons consists of five different representations of gauge group

$$Q_L(3,\ 2)_{1/6}, U_R(3,\ 1)_{2/3}, D_R(3,\ 1)_{-1/3}, L_L(1,\ 2)_{-1/2}, E_R(1,\ 1)_{-1}$$

-- sub-indices L and R indicate fermion chirality --

Notation:

left-handed lepton field L_L is a singlet of $SU(3)$ color group

doublet of $SU(2)$ weak isospin

and carries hypercharge $-1/2$ under $U(1)$ group

SM contains single higgs boson doublet $\phi(1,\ 2)_{1/2}$ whose VEV breaks gauge symmetry into $SU(3)_C \times U(1)_{\mathrm{EM}}$

Gauge Interactions

Interactions are mediated by:

8 $SU(3)$ color gluons $G^a_\mu(8,\ 1)_0$

3 $SU(2)$ left chiral gauge bosons $A^i_\mu(1,\ 3)_0$

and 1 $U(1)$ hypercharge gauge field $B_\mu(1,\ 1)_0$

Gauge interactions arise through covariant derivative

$$\mathbf{D}_\mu = \partial_\mu - \mathbf{i}\left[\mathbf{g_s}\sum_{\mathbf{a=1}}^{\mathbf{8}}\mathbf{G}^{\mathbf{a}}_\mu\,\mathbf{t}^{\mathbf{C}}_{\mathbf{a}} + \mathbf{g}\sum_{\mathbf{i=1}}^{\mathbf{3}}\mathbf{A}^{\mathbf{i}}_\mu\,\mathbf{t}^{\mathbf{L}}_{\mathbf{i}} + \tfrac{\mathbf{1}}{\mathbf{2}}\mathbf{g}'\mathbf{B}_\mu\right]$$

$\mathbf{t}^{\mathbf{a}}_{\mathbf{C}} = (\lambda^{\mathbf{a}}/\mathbf{2};\mathbf{0})$ for (quarks; lepton, Higgs)

$\mathbf{t}^{\mathbf{i}}_{\mathbf{L}} = (\tau^{\mathbf{i}}/\mathbf{2};\mathbf{0})$ for $SU(2)$ (doublets; singlets)

strength of interactions are described by their coupling constants

$g_s, g,$ and g'

Electroweak sector

First focus attention on electroweak sector

Anticipating a possible $SU(2)$ structure for weak currents we are led to construct an isospin triplet of weak currents

$$J^i_\mu(x) = \frac{1}{2}\,\bar{u}_L\,\gamma_\mu\tau_i\,u_L, \qquad \text{with}\, i = 1,\ 2,\ 3$$

for spinor operators

$$u_L = L_L = \begin{pmatrix} \nu_{eL} \\ e^-_L \end{pmatrix}, \qquad u_L = Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$$

whose corresponding charges $T^i = \int J^i_0(x)\,d^3x$

generate an $SU(2)_L$ algebra $[T_i,\ T_j] = i\epsilon_{ijk}T_k$

Higss Lagrangian

Presence of mass terms for A^i_μ destroy gauge invariance of $\mathcal{L}$

To approach goal of generating mass for gauge bosons
we entertain mechanism of spontaneous symmetry breaking

Consider complex scalar Higgs boson field
in spinor representation of $SU(2)_L$ and has charge $1/2$ under $U(1)$

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \sqrt{\tfrac{1}{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix}$$

Gauge invariant Lagrangian is thus

$$\mathcal{L}_\phi = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) - \mu^2\, \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

Repeat procedure of translating field ϕ to a true ground state

VEV is obtained by looking at stationary points of $\mathcal{L}_\phi$

$$\frac{\partial \mathcal{L}_\phi}{\partial(\phi^\dagger \phi)} = 0 \Rightarrow \phi^\dagger \phi \equiv \frac{1}{2}(\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2) = -\frac{\mu^2}{2\lambda}$$

Picking up the VEV

Values of $(\mathrm{Re}\ \phi^+,\ \mathrm{Im}\ \phi^+,\ \mathrm{Re}\ \phi^0,\ \mathrm{Im}\ \phi^0)$

range over surface of 4D sphere of radius v

such that $v^2 = -\mu^2/\lambda$ and $\phi^\dagger\phi = |\phi^+|^2 + |\phi^0|^2$

This implies that Lagrangian of ϕ

is invariant under rotations of this 4-dimensional sphere

invariant under group $SO(4)$ isomorphic to $SU(2)\times U(1)$

We must expand $\phi(x)$ about a particular minimum

Without loss of generality

define VEV of ϕ to be real parameter in ϕ^0 direction

$$\phi_1 = \phi_2 = \phi_4 = 0\ ,\ \phi_3^2 = -\mu^2/\lambda$$

We can now expand $\phi(x)$ about this particular vacuum

$$\boxed{\langle\phi\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}}$$

Electroweak Interactions

To introduce electroweak interactions with ϕ

we replace ∂_μ by covariant derivative D_μ (❖) in Lagrangian (☫)

and evaluate resulting kinetic term $(D_\mu\phi)^\dagger(D^\mu\phi)$ at VEV $\langle\phi\rangle$

Relevant terms are:

$$\Delta\mathcal{L} = \frac{1}{2}(0\ v)\left(\frac{1}{2}gA^j_\mu\tau_j + \frac{1}{2}g'B_\mu\right)\left(\frac{1}{2}gA^{k\mu}\tau_k + \frac{1}{2}g'B^\mu\right)\begin{pmatrix}0\\ v\end{pmatrix}$$

$$= \frac{1}{8}(0\ v)\begin{pmatrix} gA^3_\mu + g'B_\mu & g(A^1_\mu - iA^2_\mu) \\ g(A^1_\mu + iA^2_\mu) & -gA^3_\mu + g'B_\mu \end{pmatrix}^2 \begin{pmatrix}0\\ v\end{pmatrix}$$

$$= \frac{1}{8}v^2[g^2(A^1_\mu)^2 + g^2(A^2_\mu)^2 + (-gA^3_\mu + g'B_\mu)^2]$$

note that

$$\frac{1}{8}v^2[g^2(A^3_\mu)^2 - 2gg'A^3_\mu B^\mu + g'^2B^2_\mu] = \frac{1}{8}v^2[gA^3_\mu - g'B_\mu]^2 + 0[g'A^3_\mu + gB_\mu]^2$$

$$= \frac{1}{2}m_z^2Z^2_\mu + \frac{1}{2}m_A A^2_\mu$$

Gauge Bosons

There are three massive vector bosons:

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}(A^1_{\mu} \mp i A^2_{\mu})$$

$$Z^0_{\mu} = \frac{1}{\sqrt{g^2 + g'^2}}(g A^3_{\mu} - g' B_{\mu})$$

The fourth vector field orthogonal to Z^0_{μ}

$$A_{\mu} = \frac{1}{\sqrt{g^2 + g'^2}}(g' A^3_{\mu} + g B_{\mu})$$

remains massless

We identify this field with electromagnetic vector potential

ELECTROWEAK MASS SCALE

Gauge fields have eaten up Goldstone bosons and become massive

Scalar degrees of freedom

become longitudinal polarizations of massive vector bosons

Spontaneous symmetry breaking

rotates 4 $SU(2)_L \times U(1)_Y$ gauge bosons

to their mass eigenstates via gauge interaction term of Higgs fields

$$\{A^1_\mu,\ A^2_\mu\} \to \{\, W^+_\mu,\ W^-_\mu\,\} \text{ and } \{A^3_\mu,\ B_\mu\} \to A_\mu,\ Z^0_\mu$$

In terms of weak mixing angle θ_w (defined by $\tan\theta_w = g'/g$)

$$\boxed{\begin{pmatrix} Z^0_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_w & -\sin\theta_w \\ \sin\theta_w & \cos\theta_w \end{pmatrix} \begin{pmatrix} A^3_\mu \\ B_\mu \end{pmatrix}}$$

at lowest order in perturbation theory

$$\boxed{m_W = \frac{g\,v}{2} = \frac{g}{2\sqrt{2\,\lambda}} m_H \qquad \text{and} \qquad m_Z = \frac{m_W}{\cos\theta_w}}$$

Higgs mass m_H sets electroweak mass scale

Mass eigenstates

In terms of mass eigenstates covariant derivative becomes

$$\begin{aligned} D_\mu &= \partial_\mu - i\frac{g}{\sqrt{2}}(W^+_\mu T^+ + W^-_\mu T^-) - i\frac{1}{\sqrt{g^2+g'^2}}Z_\mu(g^2T^3 - g'^2Y) \\ &- i\frac{gg'}{\sqrt{g^2+g'^2}}A_\mu(T^3+Y) \end{aligned}$$

$$T^\pm = T^1 \pm iT^2$$

After identifying coefficient of electromagnetic interaction

$$e = \frac{gg'}{\sqrt{g^2+g'^2}} = g\sin\theta_w$$

with electron charge ☛ it becomes evident that:

electromagnetic interaction ($U(1)$ gauge symmetry with coupling e)

sits across weak isospin ($SU(2)_L$ symmetry with coupling g)

and weak hypercharge ($U(1)$ symmetry with coupling g')

Note that two couplings g and g' can be replaced by e and θ_w

θ_w is to be determined by experiment

Hints for the calculation

$$D_\mu = \partial_\mu - igA^a_\mu T^a - ig'YB_\mu$$

Hypercharge and weak Isospin charges

After identifying electric charge quantum number $Q = T^3 + Y$

we can rewrite covariant derivative

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}}(W_\mu^+ T^+ + W_\mu^- T^-) - i\frac{g}{\cos\theta_w} Z_\mu (T^3 - \sin^2\theta_w Q) - ieA_\mu Q$$

This uniquely determines coupling of $W^\pm$ and Z^0 to fermions once quantum numbers of fermion fields are specified

For right-handed fields ☛ $T^3 = 0$ and hence $Y = Q$

For the left-handed fields L_L and Q_L assignments

$$Y = -1/2 \quad Y = +1/6$$

combine with $T^3 = \pm 1/2$ to give electric charge assignments

Hints for the calculation

use following manipulation in the Z^0 coupling

$$g^2 T^3 - g'^2 Y = (g^2 + g'^2) T^3 - g'^2 Q$$

Weak isospin & hypercharge quantum numbers

Lepton	T	T^3	Q	Y	Quark	T	T^3	Q	Y
ν_e	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	u_L	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{1}{6}$
e_L^-	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}$	d_L	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	$\frac{1}{6}$
					u_R	0	0	$\frac{2}{3}$	$\frac{2}{3}$
e_R^-	0	0	-1	-1	d_R	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$

If we ignore fermion masses
Lagrangian for weak interactions of quarks and leptons follows directly from charge assignments given above

Fermionic kinetic energy terms are

$$\mathcal{L} = \bar{L}_L(i\not{D})L_L + \bar{E}_R(i\not{D})E_R + \bar{Q}_L(i\not{D})Q_L + \bar{U}_R(i\not{D})U_R + \bar{D}_R(i\not{D})D_R \quad ⌘$$

To work out physical consequences of fermion-vector boson couplings we should write ⌘ in terms of vector boson mass eigenstates

Electroweak Currents

Using last form of covariant derivative we rewrite Lagrangian as

$$\begin{aligned}\mathcal{L} &= \bar{L}_L(i\not\partial)L_L + \bar{E}_R(i\not\partial E_R + \bar{Q}_L(i\not\partial)Q_L + \bar{U}_R(i\not\partial)U_R + \bar{D}_R(i\not\partial)D_R \\ &+ g(W_\mu^+ J_W^{+\mu} + W_\mu^- J_W^{-\mu} + Z_\mu^0 J_Z^\mu) + eA_\mu j^\mu\end{aligned}$$

where

$$\begin{aligned}J_W^{+\mu} &= \frac{1}{\sqrt{2}}(\bar{\nu}_L\,\gamma^\mu e_L + \bar{u}_L\,\gamma^\mu d_L) \\ J_W^{-\mu} &= \frac{1}{\sqrt{2}}(\bar{e}_L\,\gamma^\mu \nu_L + \bar{d}_L\,\gamma^\mu u_L) \\ J_Z^\mu &= \left[\bar{\nu}_L\,\gamma^\mu\left(\frac{1}{2}\right)\nu_L + \bar{e}_L\,\gamma^\mu\left(-\frac{1}{2}+\sin^2\theta_w\right)e_L + \bar{e}_R\,\gamma^\mu\left(\sin^2\theta_w\right)e_R\right. \\ &+ \bar{u}_L\,\gamma^\mu\left(\frac{1}{2}-\frac{2}{3}\sin^2\theta_w\right)u_L + \bar{u}_R\,\gamma^\mu\left(-\frac{2}{3}\sin^2\theta_w\right)u_R \\ &+ \left.\bar{d}_L\,\gamma^\mu\left(-\frac{1}{2}+\frac{1}{3}\sin^2\theta_w\right)d_L + \bar{d}_R\,\gamma^\mu\left(\frac{1}{3}\sin^2\theta_w\right)d_R\right]\frac{1}{\cos\theta_w}, \\ j^\mu &= \bar{e}\,\gamma^\mu(-1)e + \bar{u}\,\gamma^\mu\left(+\frac{2}{3}\right)u + \bar{d}\,\gamma^\mu\left(-\frac{1}{3}\right)d\end{aligned}$$

and equivalent expressions hold for other two generations

QCD

Gauge invariant QCD Lagrangian
for interacting colored quarks q and vector gluons G_μ
-- with coupling specified by g_s --

obtained demanding invariance under local phase transformations to q

Using $\mathcal{L} = -\frac{1}{2}\mathrm{Tr}(\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu}) + \overline{\psi}(i\not{D} - m)\psi$ we obtain

$$\boxed{\mathcal{L}_{\mathrm{QCD}} = \bar{q}_j(i\gamma^\mu\partial_\mu - m)q_j + g_s(\bar{q}_j\gamma^\mu t_a q_j)G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a}$$

where q_1, q_2 and q_3 denote 3 color fields
and -- for simplicity -- we show just one quark flavor

Because we can arbitrarily vary phase of three quark color fields
it is not surprising that eight vector gluon fields are needed
to compensate all possible phase changes

Just as for photon ☛ local invariance requires gluons to be massless

Gluon self-interactions

Field strength tensor $G^a_{\mu\nu}$
has remarkable new property on account of $[\mathbf{G}_\mu, \mathbf{G}_\nu]$ term

Imposing gauge symmetry has required that:
kinetic energy term in $\mathcal{L}_{\mathrm{QCD}}$ is not purely kinetic
but includes an induced self-interaction between gauge bosons
This becomes clear if we rewrite $\mathcal{L}_{\mathrm{QCD}}$ in symbolic form

$$\mathcal{L}_{\mathrm{QCD}} = \text{"}\bar{q}q\text{"} + \text{"}G^2\text{"} + g_s\,\text{"}\bar{q}qG\text{"} + g_s\,\text{"}G^3\text{"} + \mathrm{g_s^2}\,\text{"}\mathrm{G}^4\text{"}$$

First 3 terms have QED analogues
They describe free propagation of q and G and $q-G$ interaction

Remaining 2 terms show presence of QCD 3- and 4-gluon vertices
and reflect fact that gluons themselves carry color charge

They have no analogue in QED
and arise on account of non-Abelian character of gauge group

Yukawa Lagrangian

Since explicit fermion mass terms violate the gauge symmetries masses of chiral fields arise from Yukawa interactions which couple a right-handed fermion with its left handed doublet and Higgs field after spontaneous symmetry breaking

For example to generate electron mass we include $SU(2)\times U(1)$ gauge invariant term in the Lagrangian

$$\boxed{\mathcal{L}_e^{\text{Yukawa}} = -Y_e\left[(\bar{\nu}_e,\bar{e})_L\begin{pmatrix}\phi^+\\ \phi_0\end{pmatrix}e_R + \bar{e}_R(\phi^-,\bar{\phi}^0)\begin{pmatrix}\nu_e\\ e\end{pmatrix}_L\right]} \quad ◎$$

Y_e is Yukawa coupling constant of electron

Higgs doublet has exactly required $SU(2)\times U(1)$ quantum numbers to couple to $\bar{e}_L e_R$

We spontaneously break symmetry and substitute

$$\boxed{\phi = \sqrt{\tfrac{1}{2}}\begin{pmatrix}0\\ v+H(x)\end{pmatrix}}$$

into ◎

Lepton masses

After spontaneous symmetry breaking has taken place
neutral Higgs field $H(x)$ is only remnant of Higgs doublet
The other three fields can be gauged away
On substitution of ϕ Lagrangian becomes

$$\mathcal{L}_e^{\text{Yukawa}} = -\frac{Y_e}{\sqrt{2}} v(\bar{e}_L e_R + \bar{e}_R e_L) - \frac{Y_e}{\sqrt{2}}(\bar{e}_L e_R + \bar{e}_R e_L) H$$

We choose Y_e so that

$$m_e = \frac{Y_e\, v}{\sqrt{2}}$$

and thus generate required electron mass

$$\mathcal{L}_e^{\text{Yukawa}} = -m_e \bar{e} e - \frac{m_e}{v} \bar{e} e H$$

Note however since Y_e is arbitrary
actual mass of electron is not predicted
Besides mass term ☛ Lagrangian contains an interaction term
-- coupling Higgs scalar to electron --

Quark masses

Quark masses are generated in similar way

To generate a mass for upper member of quark doublet we must construct complex conjugate of Higgs doublet

$$\bar{\phi} = i\tau^2 \phi^* = \begin{pmatrix} -\bar{\phi}^0 \\ \phi^- \end{pmatrix} \underbrace{\longrightarrow}_{\text{breaking}} \sqrt{\tfrac{1}{2}} \begin{pmatrix} v+H \\ 0 \end{pmatrix}$$

Because of special properties of $SU(2)$

$\bar{\phi}$ transforms identically to ϕ

but has opposite weak hypercharge to ϕ ☛ $Y = -1/2$

It can be used to construct a gauge invariant contribution to

$$\begin{aligned} \mathcal{L}_q^{\text{Yukawa}} &= -Y_d (\bar{u}, \bar{d})_L \begin{pmatrix} \phi^+ \\ \phi_0 \end{pmatrix} d_R + Y_u (\bar{u}, \bar{d})_L \begin{pmatrix} -\bar{\phi}^0 \\ \bar{\phi}^- \end{pmatrix} u_R + \text{h.c.} \\ &= -m_d \bar{d}d - m_u \bar{u}u - \frac{m_d}{v} \bar{d}dH - \frac{m_u}{v} \bar{u}uH \end{aligned}$$

Yukawa Lagrangian then takes the form

$$-\mathcal{L}^{\text{Yukawa}} = {Y_d}^{ij}\, \overline{Q_L i}\, \phi\, D_{Rj} + {Y_u}^{ij}\, \overline{Q_L i}\, \bar{\phi}\, U_{Rj} + {Y_e}^{ij}\, \overline{L_L i}\, \phi\, E_{Rj} + \text{h.c.}$$

ij are generation indices

Global Symmetries

- Standard Model also comprises an accidental global symmetry

$$U(1)_B \times U(1)_e \times U(1)_\mu \times U(1)_\tau$$

$U(1)_B$ is baryon number symmetry

$U(1)_{e,\mu,\tau}$ are three lepton flavor symmetries

with total lepton number given by $L = L_e + L_\mu + L_\tau$

- It is an accidental symmetry because we do not impose it

consequence of gauge symmetries and low energy particle content

- It is possible -- but not necessary --
  that effective interaction operators
  induced by high energy content of underlying theory
  may violate sectors of global symmetry

looking forward

- We have build SM from general group-theory considerations
  -- principles of symmetry and invariants --
- In real life
  model of nature is usually uncovered in a less pristine fashion
- To convey an impression of how theories developed
  and how SM has successfully confronted experiment
  we will describe a number of most important theoretical results
- We will start from most precisely tested theory in physics QED
  and carry on to QCD and electroweak theory