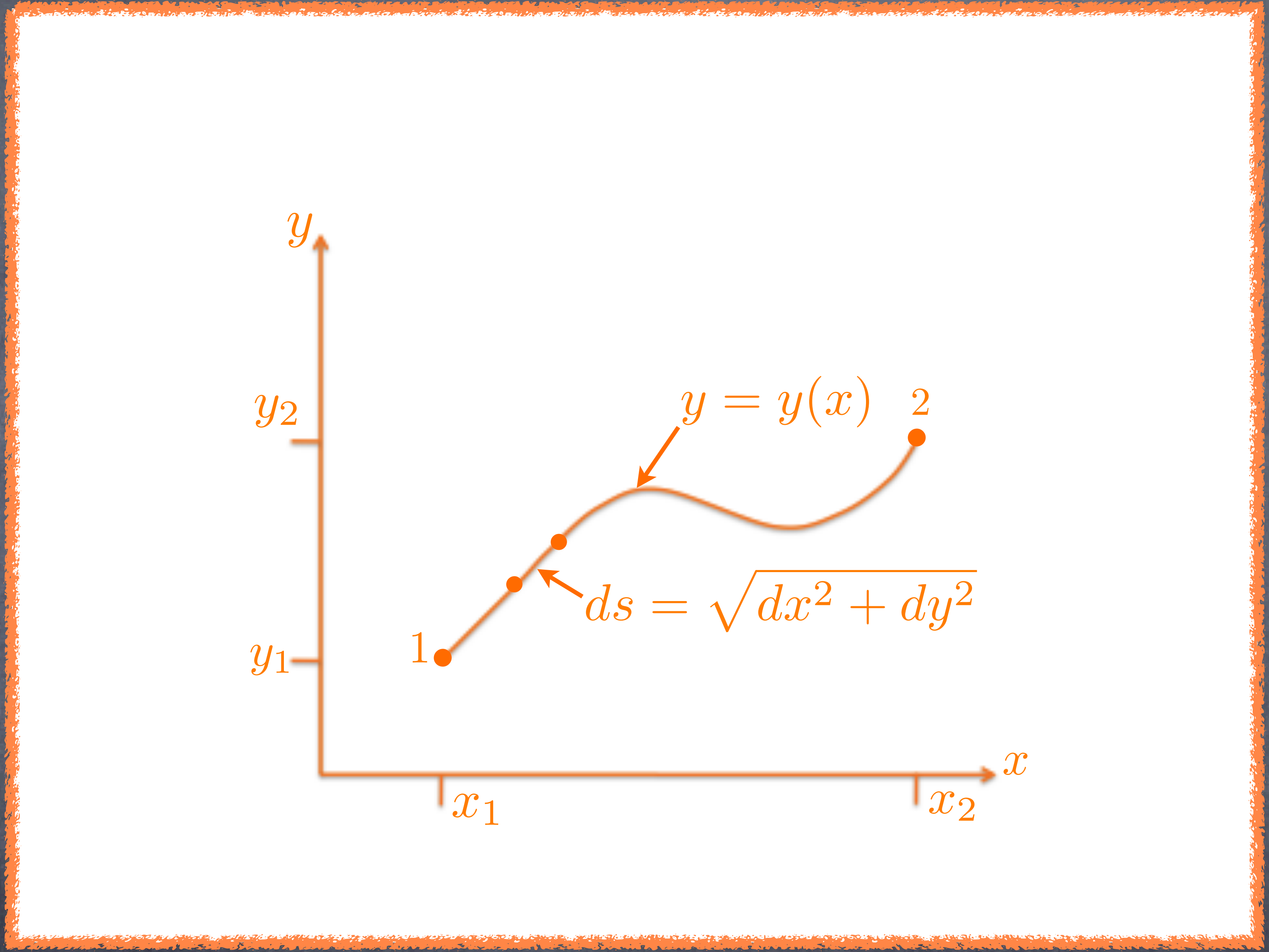
Particle Physics 2011

Luis Anchordoqui

Given two points in a plane



what is the shortest path between them?

Shortest Path between Two Points

The length of a short segment of the path is

$$ds = \sqrt{dx^2 + dy^2}$$

$$\Downarrow$$

$$dy = \frac{dy}{dx}dx \equiv y'(x)dx$$

$$\Downarrow$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + [y'(x)]^2}\,dx$$

The total length of the path between points 1 and 2 is

$$L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1 + [y'(x)]^2}\,dx$$

This equation puts our problem in a mathematical form

$$\Downarrow$$

find the function $y(x)$ for which the integral is minimum

Fermat's principle

What is the path that light follows between two points?

$$\Downarrow$$

Fermat (1601 - 1665)

$$\Downarrow$$

the path for which the time of travel of the light is minimum

The time for light to travel a short distance ds is ds/v

$v \equiv c/n$ speed of light in a medium with refractive index n

$$\text{time of travel} = \int_1^2 dt = \int_1^2 \frac{ds}{v} = \frac{1}{c}\int_1^2 n\, ds$$

In general ☛ refractive index can vary

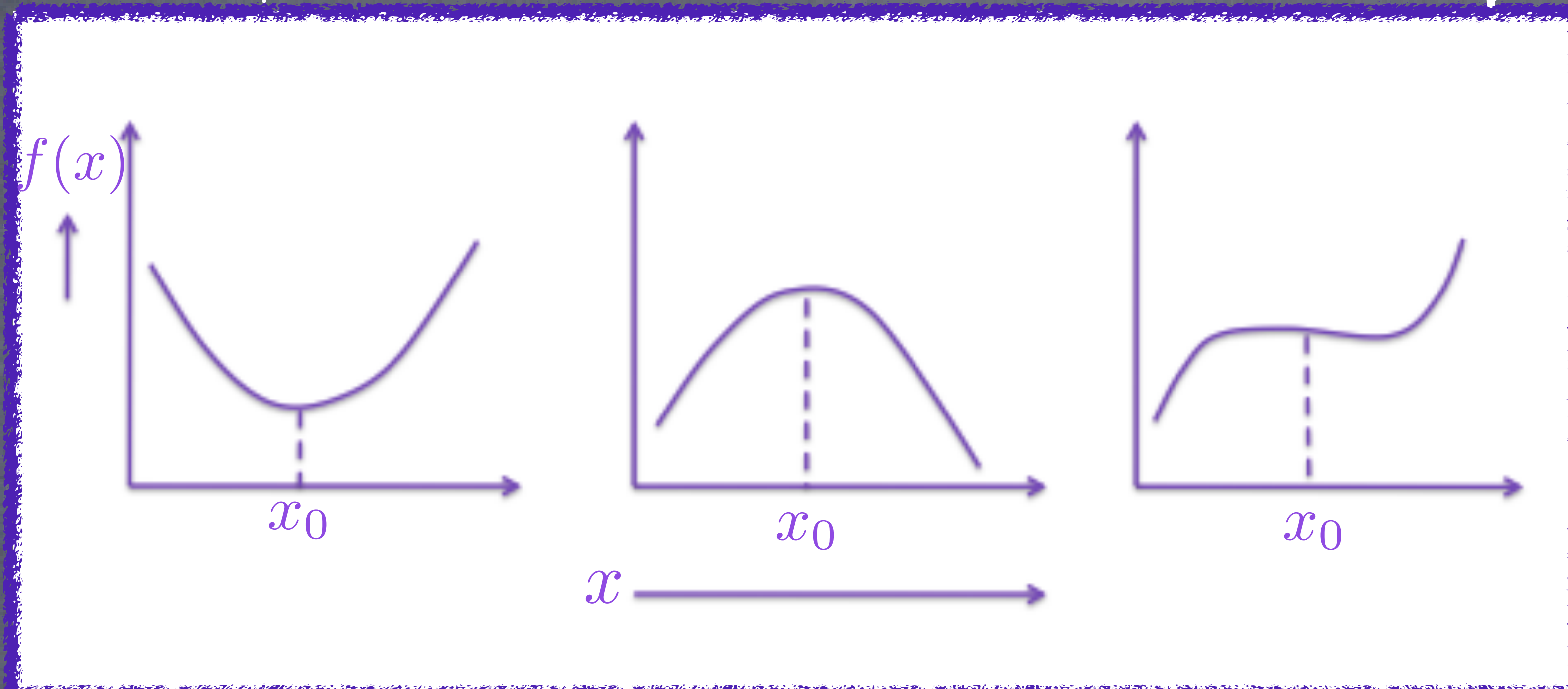
$$\boxed{\int_1^2 n(x,y)ds = \int_{x_1}^{x_2} n(x,y)\sqrt{1+[y'(x)]^2}\,dy}$$

Calculus of Variations

Standard minimization problem of elementary calculus

unknown value of the variable x at which a known function $f(x)$ has a minimum

❖ Recall that if $df/dx = 0$ at x_0 there are three possibilities



✓If $d^2f/dx^2 > 0 \Rightarrow f$ has a minimum

✓If $d^2f/dx^2 < 0 \Rightarrow f$ has a maximum

✓If $d^2f/dx^2 = 0 \Rightarrow f$ there may be a minimum, a maximum, or neither

New problem ☛ one step more complicated

Calculus of Variations

how infinitesimal variations of a path change an integral

Euler-Lagrange Equation

Consider an integral of the form

$$S = \int_{x_1}^{x_2} f[y(x),\, y'(x),\, x]\, dx$$

$y(x)$ ☛ unknown curve joining points (x_1, y_1) and (x_2, y_2)

$$\boxed{y(x_1) = y_1} \qquad \boxed{y(x_2) = y_2}$$

We have to find the curve that makes S a minimum

f ☛ function of 3 variables $f = f(y, y', x)$

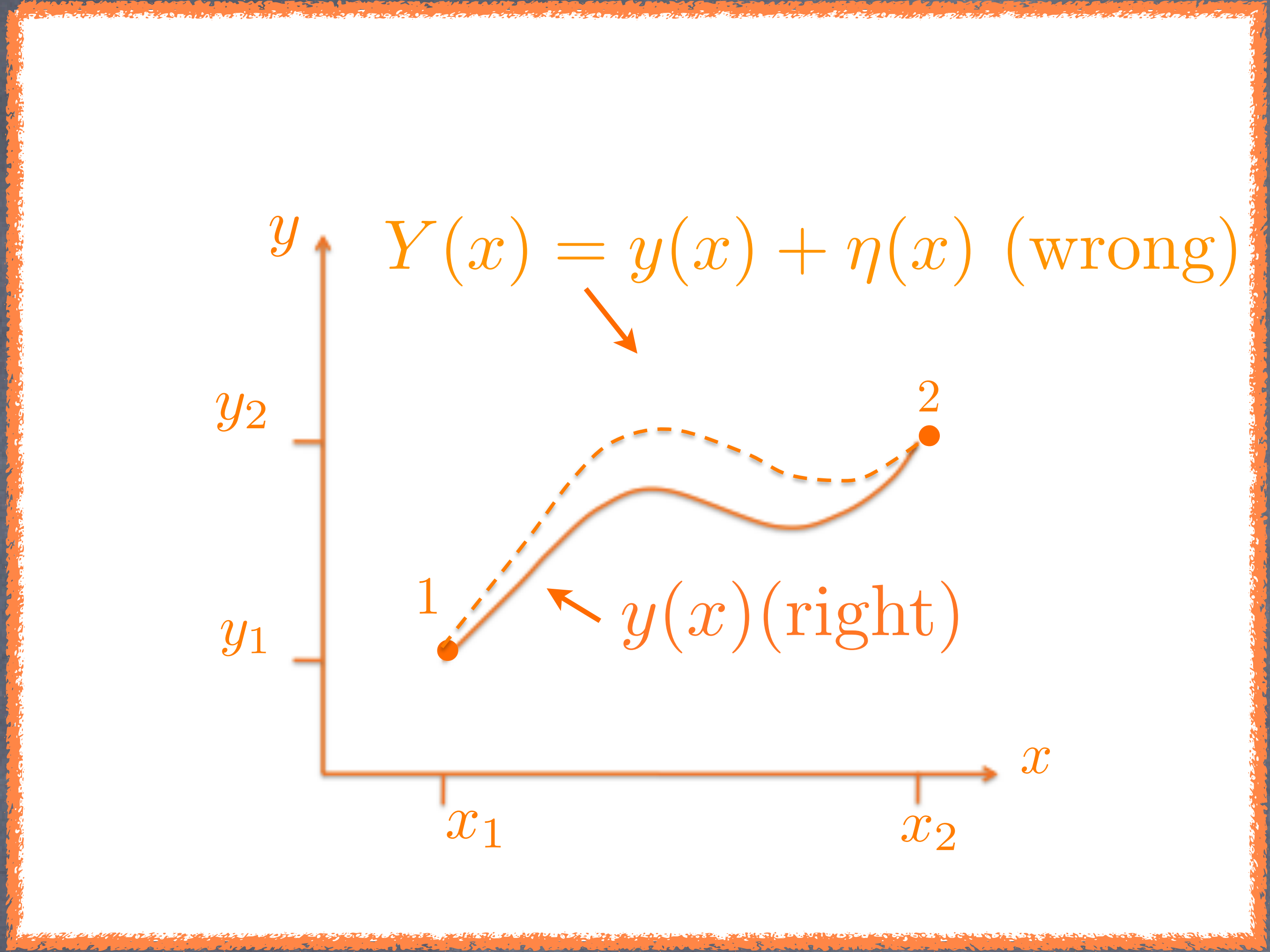
$$\Downarrow$$

but integral follows path $y = y(x)$

$$\Downarrow$$

integrand $f[y(x), y'(x), x]$ is actually a function of just one variable x

Euler-Lagrange Equation (cont'd)



If $y(x)$ ☛ right solution

$\Downarrow$

S evaluated for $y(x)$ is less than for any neighborhood curve $Y(x)$

convenient to write

$$Y(x) = y(x) + \eta(x)$$

since $Y(x)$ must pass through points 1 and 2

$\Downarrow$

$$\eta(x_1) = \eta(x_2) = 0$$

Euler-Lagrange Equation (cont'd)

The integral taken along the wrong curve $Y(x)$
must be larger than that along the right curve $y(x)$
no matter how close is the former to the latter
to express this requirement ☛ introduce parameter

$$\Downarrow$$

$$Y(x) = y(x) + \alpha\,\eta(x)$$

The integral S taken along the curve $Y(x)$ now depends on α

$$\Downarrow$$

$$S(\alpha)$$

The right curve $y(x)$ is obtained by setting $\alpha = 0$

$$\Downarrow$$

reduction to traditional problem from elementary calculus

$$\Downarrow$$

$dS/d\alpha = 0$ when $\alpha = 0$

Euler-Lagrange Equation (cont'd)

$$S(\alpha) = \int_{x_1}^{x_2} f(Y, Y', x)\, dx$$
$$= \int_{x_1}^{x_2} f(y + \alpha\eta, y' + \alpha\eta', x)\, dx$$

differentiate with respect to α

$$\Downarrow$$

$$\boxed{\frac{\partial f(y + \alpha\eta, y' + \alpha\eta', x)}{\partial\alpha} = \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}}$$

$$\Downarrow$$

$$\frac{dS}{d\alpha} = \int_{x_1}^{x_2} \frac{\partial f}{\partial\alpha}\, dx$$
$$= \int_{x_1}^{x_2} \left(\eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'}\right) dx$$
$$= 0$$

Euler-Lagrange Equation (cont'd)

Re-write second term on the right using integration by parts

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\, dx = \eta(x) \frac{\partial f}{\partial y'}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta(x)\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

endpoint term is zero

$$\Downarrow$$

$$\int_{x_1}^{x_2} \eta'(x) \frac{\partial f}{\partial y'}\, dx = - \int_{x_1}^{x_2} \eta(x)\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx$$

$$\Downarrow$$

$$\int_{x_1}^{x_2} \eta(x) \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) dx = 0$$

This condition must be satisfied for any choice of the function $\eta(x)$

We can conclude that ☛ $$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

$$\forall x \in x_1 \leq x \leq x_2$$

if all the functions concerned are continuous

Leonhard Euler (1707-1783) Joseph Lagrange (1736-1813)

Shortest path between two points in R^3

We saw that the length of a path between points 1 and 2 is

$$\boxed{L = \int_1^2 ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\,dx}$$

that has the standard form

$$\boxed{f(y, y', x) = \sqrt{1+y'^2}}$$

$$\frac{\partial f}{\partial y} = 0 \qquad\qquad \frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}}$$

$$\Downarrow$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \Rightarrow \frac{\partial f}{\partial y'} = C$$

$$\Downarrow$$

$$y'^2 = C^2(1+y'^2) \Rightarrow y'^2 = \widetilde{C} \Rightarrow y'(x) = m$$

integration leads to

$$\boxed{y(x) = mx + b}$$

Generalized Coordinates

Instead of Cartesian coordinates ☛ consider now

- spherical polar coordinates $(r,\ \theta,\ \phi)$
- cylindrical coordinates $(\rho,\ \phi,\ z)$
- or any set of generalized coordinates $(q_1,\ q_2,\ q_3)$

satisfying $q_i = q_i(\vec{r})$ for $i = 1,\ 2,\ 3$ and $r = r(q_1,\ q_2,\ q_3)$

Next re-write the Lagrangian in terms of these new variables

$$L = L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)$$

and the action integral

$$S = \int_{t_1}^{t_2} L(q_1, q_2, q_3, \dot{q}_1, \dot{q}_2, \dot{q}_3)\, dt$$

The value of the integral is unaltered by this change of variables

$$\Downarrow$$

The statement that S is stationary for variations of the path around the correct path must still be true in the new coordinates

Lagrangian Density

- We have seen that state of a physical system consisting of collection of N discrete point particles can be specified by a set of $3N$ generalized coordinates q_i
- Action of such a physical system ☛

$$S = \int L(q_i, \partial_t q_i) dt$$

is integral of so-called Lagrangian function from which system behavior is determined by principle of minimal action

- In local field theory ☛ Lagrangian can be written as spatial integral of Lagrangian density

$$S = \int \mathcal{L}(\phi, \partial_\mu \phi)\, d^4x$$

where field ϕ itself is a function of continuous parameters x^μ

Lagrangian Field Theory

- Minimization condition on δS yields

$$\begin{aligned} 0 &= \delta S \\ &= \int d^4x \, [\partial_\phi \mathcal{L} \;\; \delta\phi + \partial_{\partial_\mu \phi}\mathcal{L}\delta(\partial_\mu \phi)] \end{aligned}$$

where ☛ $\delta(\partial_\mu \phi) = \partial_\mu(\phi + \delta\phi) - \partial_\mu \phi = \partial_\mu(\delta\phi)$

- The second term in the integrand can be integrated by parts

$$\int d^4x \;\; \partial_{\partial_\mu \phi}\mathcal{L} \;\; \partial_\mu(\delta\phi) = \int d^4x \;\; \frac{\partial \mathcal{L}}{\partial_\mu \phi} \;\; \frac{\partial(\delta\phi)}{\partial x^\mu}$$

integration with respect to x^μ leads to

$$\begin{aligned} \int d^4x \;\; \frac{\partial \mathcal{L}}{\partial_\mu \phi} \;\; \frac{\partial(\delta\phi)}{\partial x^\mu} &= \partial_{\partial_\mu \phi}\mathcal{L} \quad \delta\phi \Big|_{\mho} - \int d^4x \, \delta\phi \;\; \partial_\mu(\partial_{\partial_\mu \phi}\mathcal{L}) \\ &= \int d^4x \, [\partial_\mu(\partial_{\partial_\mu \phi}\mathcal{L} \;\; \delta\phi) - \delta\phi \;\; \partial_\mu(\partial_{\partial_\mu \phi}\mathcal{L})] \end{aligned}$$

where $\mho$ denotes the boundary of the four dimensional spacetime region of integration

Lagrangian Field Theory

The summary from the previous slide is

♗

$$0 = \int d^4x [\partial_\phi \mathcal{L} \;\; \delta\phi - \partial_\mu(\partial_{\partial_\mu \phi}\mathcal{L}) \;\; \delta\phi + \partial_\mu(\partial_{\partial_\mu \phi}\mathcal{L} \;\; \delta\phi)]$$

Using Gauss theorem ☛ last term in ♗ can be written as a surface integral over the boundary of 4-dimensional spacetime region of integration

As in particle mechanics case ☛ initial and final configurations are assumed given and so $\delta\phi$ is zero at temporal beginning and end of this region

Hereafter we restrict our consideration to deformations $\delta\phi$ that also vanish on spatial boundary of integration region

Hence ☛ for arbitrary variations $\delta\phi$

♗ leads to Euler-Lagrange equation of motion for a field:

$$\partial_\mu(\partial_{\partial_\mu \phi}\mathcal{L}) - \partial_\phi \mathcal{L} = 0$$

Canonical Formalism

- The canonical momentum for the particle system is $p_i = \partial_{\dot{q}_i} L$

  the corresponding quantity for a field is $\pi(x) = \partial_{\dot{\phi}} \mathcal{L}$

  and is called momentum density conjugate to $\phi(x)$

- The Hamiltonian is defined by

$$\boxed{H = \sum_{i=1}^{3N} p_i \ \dot{q}_i - L(q_i, \dot{q}_i)}$$

and so we can write ☛

$$\boxed{H = \int d^3x \ \mathcal{H}(x)}$$

where ☛

$$\boxed{\mathcal{H}(x) = \pi(x) \ \dot{\phi}(x) - \mathcal{L}(\phi, \partial_\mu \phi)}$$

Canonical Quantization

- The Heisenberg commutation relations $[p_i, q_j] = -i\delta_{ij}$, $[p_i, p_j] = [q_i, q_j] = 0$ have as their field counterparts

$$\boxed{[\pi(\vec{x}, t), \phi(\vec{y}, t)] = -i\delta^{(3)}(\vec{x} - \vec{y})}$$

with all other pairs of operators commuting

- If there are various classical fields to be quantized

E.G. $\phi(x)$ and $\phi^\star(x)$ the equation $\partial_\mu[\partial_{\partial_\mu \phi^\star}\mathcal{L}] - \partial_{\phi^\star}\mathcal{L} = 0$ will too be satisfied and the field $\phi^\star$

will have its canonically conjugate momentum $\pi^\star = \partial_{\dot{\phi}^\star}\mathcal{L}$

- The Hamiltonian density will be

$$\boxed{\mathcal{H} = \pi(x)\ \dot{\phi} + \pi^\star(x)\ \dot{\phi}^\star - \mathcal{L}(\phi, \phi^\star, \partial_\mu\phi, \partial_\mu\phi^\star)}$$

and the additional commutation relation

$$\boxed{[\pi^\star(\vec{x}, t), \phi^\star(\vec{y}, t)] = -i\delta^{(3)}(\vec{x} - \vec{y})}$$

will be assumed to hold

Remarks on Quantization Procedure

note that commutation relations are only defined at equal times

Once commutators are given ☛ their values at different times are determined by the equations of motion

All commutators involving starred with unstared fields vanish at equal times since these are independent fields

In the commutation relations the times were set equal but not otherwise specified and therefore a change in the origin of time has no physical consequences

Lorentz Group

- One paramount prerequisite to be imposed on a theory describing the behavior of particles at high energies is that it be consistent with the special theory of relativity
- This can be achieved by demanding covariance of the equations under Lorentz-Poincare transformations
- A Lorentz-Poincare change of referencial is a real linear transformation of coordinates conserving norm of interval between different points of spacetime
- For such transformation ☛ new spacetime coordinates x'^{μ} are obtained from old ones x^{μ} according to $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} + a^{\mu}$ satisfying $x'_{\mu} x'^{\mu} = x_{\mu} x^{\mu}$
- Hereafter we treat the translation of spacetime axes separately and give the name of Lorentz transformation to the homogenous transformations with $a^{\mu} = 0$

Properties of Lorentz transformation

real transformation implies $(\Lambda_{\mu\nu})^* = \Lambda_{\mu\nu}$

and invariance of the norm yields

$$\boxed{g_{\mu\nu}\, x^\mu\, x^\nu = g_{\mu\nu}\, x'^\mu\, x'^\nu = g_{\mu\nu}\Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\, x^\alpha\, x^\beta}$$

i.e. $$\boxed{g_{\mu\nu}\, \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta = g_{\alpha\beta}}$$

$g_{\mu\nu} \equiv \mathrm{diag}(1,-1,-1,-1)$ ☛metric tensor

In addition there is a transformation law for the field $\phi(x)$

so that transformed fields $\phi'(x')$

satisfy same equations in new spacetime coordinates

Quantized theory will then also be Lorentz invariant if

--as indeed is the case--

commutation relations transform covariantly

Lorentz Invariance

in QFT ☛ it is possible to discuss Lorentz invariance
in a way divorced from specific form of equations of motion

To this end ☛ consider a system to be fixed
and some apparatus that serves to prepare a physical state $|\psi_A\rangle$

Consider now another apparatus which prepares physical state $|\psi_{A'}\rangle$
related to the first one by a Lorentz transformation

Apparatus A may be a black box that emits electrons
apparatus A' will be same source rotated through an angle θ
about some axis & moving with fixed velocity relative to apparatus A

Consider measuring apparatus M which is being used
to make measurements on state $|\psi_A\rangle$ & measuring apparatus M'
which differs from M only in that it is shifted relative to M
by same Lorentz transformation that connects A' with A

statement of relativistic invariance is that measurements made by
M on state $|\psi_A\rangle$ yield same results as those made by M' on state $|\psi_{A'}\rangle$

Lorentz Invariance - cont'd -

To obtain formal consequences of this statement ☛ we recall that in a quantum mechanical measurement we generally determine probability that physical system is in some state $|\phi\rangle$

E.g. we may ask for probability that electrons emitted have momentum p

The probability of that happening will be $|\langle\phi_p|\psi_A\rangle|^2$ where $|\phi_p\rangle$ describes state in which just this particular momentum is found for electron

For transform source and measuring apparatus corresponding probability is ☛ $|\langle\phi_{p'}|\psi_{A'}\rangle|^2$

$|\phi_{p'}\rangle$ ☛ state for which electron has momentum p' connected to p by same Lorentz transformation that connects A and A'

Because vector space of states contains all possible physical states $|\psi_A\rangle$ and $|\psi_{A'}\rangle$ must be related by some transformation $U(\Lambda)$ that depends on the Lorentz transformation Λ

Because measuring apparatus M and M' are connected by the same Lorentz transformation ☛ we must have both

$$|\psi_{A'}\rangle = U(\Lambda)\,|\psi_A\rangle \quad \text{and} \quad |\phi_{p'}\rangle = U(\Lambda)\,|\phi_p\rangle$$

Lorentz transformation of Scalar Field

The invariance requirement implies that $$|\langle\phi_{p'}|\psi_{A'}\rangle|^2 = |\langle\phi_p|\psi_A\rangle|^2$$

Deduce that $U(\Lambda)$ must be an unitary -or antiunitary- transformation

Time-reversal invariance is only symmetry requiring antiunitarity

here we take U to be unitary

* consider measurement of expectation value of scalar field $\phi(x)$

for a state $|\psi_A\rangle$ this will be $\langle\psi_A|\phi(x)|\psi_A\rangle$

for state $\psi_{A'}$ it will be measurement of expectation value of field

at transformed point i.e ➡ $\langle\psi_{A'}|\phi(x')|\psi_{A'}\rangle$

We thus have $$\langle\psi_A|\phi(x)|\psi_A\rangle = \langle\psi_A U^\dagger(\Lambda)|\phi(x')|U(\Lambda)\psi_A\rangle$$

scalar field in a Lorentz invariant theory would transform according to

$$\phi(x') = U(\Lambda)\phi(x)U^\dagger(\Lambda)$$ with $$x' = \Lambda\, x$$

Note 1: Hermitian adjoint

Adjoints of operators generalize conjugate transposes of square matrices to (possibly) infinite-dimensional spaces

Adjoint operator A is also called Hermitian conjugate (denoted by A^* or $A^\dagger$)

Consider a Hilbert space H with inner product $\langle .,. \rangle$ and a continuous linear operator $A : H \to H$ ☛ $A^* : H \to H$ is such that

$$\langle Ax, y \rangle = \langle x, A^* y \rangle \qquad \forall x, y \in H$$

properties

✓ $A^{**} = A$

✓ If A is invertible then so is A^* with $(A^*)^{-1} = (A^{-1})^*$

✓ $(A + B)^* = A^* + B^*$

✓ $(\lambda A)^* = \lambda^* A^*$ with λ^* ☛ complex conjugate of complex number λ

✓ $(AB)^* = B^* A^*$

Note 2: Unitary Operator

A unitary operator is a bounded linear operator $U : H \to H$ on a Hilbert space H

satisfying $U^*U = UU^* = \mathbb{I}$, where U^* is the adjoint of U

and $\mathbb{I} : H \to H$ is the identity operator.

U preserves the inner product $\langle , \rangle$ on the Hilbert space

i.e. for all vectors x and y in the Hilbert space $\langle Ux, Uy \rangle = \langle x, y \rangle$

Thus unitary operators are just automorphisms of Hilbert spaces

i.e. they preserve the structure (in this case, the linear space structure, the inner product, and hence the topology) of the space on which they act

The group of all unitary operators from a given Hilbert space H to itself is sometimes referred to as the Hilbert group of H denoted Hilb (H)

The weaker condition $U^*U = \mathbb{I}$ defines an isometry

Another condition $UU^* = \mathbb{I}$ defines a coisometry

Under a unitarity transformation ☛ linearity requires that any operator T of H satisfies $T(c\varphi) = cT(\varphi)$

An operator $\tilde{T}$ such that $\tilde{T}(c\varphi) = c^*(\tilde{T}\varphi)$ is said to be anti-linear and if it conserves magnitude of scalar product $|\langle \tilde{T}\varphi, \tilde{T}\varphi \rangle| = |\langle \varphi, \varphi \rangle|$ then is called anti-unitary

Proper Lorentz Group

If $\Lambda^{00} > 0$ ☛ transformation is called orthochronous

because it conserves sense of timelike vectors

Additionally ☛ if $\det(\Lambda^{\mu}{}_{\nu}) = 1$ transformation

also conserves sense of Cartesian systems in ordinary space

The ensemble of these transformations forms a group

dubbed proper Lorentz group ☛ It is a Lie group

The crucial property here is that all transformations can be

expressed as a succession of infinitesimal transformations

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu} = (\delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}) x^{\nu}$$

arbitrarily close to identity where quantities $\omega^{\mu}{}_{\nu}$ are infinitesimals

and thus we only keep terms linear in $\omega^{\mu}{}_{\nu}$

Infinitesimal Lorentz transformation

For infinitesimal transformation ☛ condition $g_{\mu\nu}\, {\Lambda^{\mu}}_{\alpha}\, {\Lambda^{\nu}}_{\beta} = g_{\alpha\beta}$ implies

$$g_{\mu\beta}{\omega^{\mu}}_{\sigma} + g_{\sigma\nu}{\omega^{\nu}}_{\beta} = 0$$

i.e. infinitesimals are real antisymmetric tensors $\omega_{\mu\nu} + \omega_{\nu\mu} = 0$

If we now write $U(\Lambda) = e^{i\eta} \longrightarrow$ is hermitian and reduces to zero for identity transformation

for an infinitesimal transformation $\phi(x') = U(\Lambda)\phi(x)U^{\dagger}(\Lambda)$ becomes

$$\phi(x) + i[\eta, \phi(x)] + \cdots = \phi(x^{\mu} + {\omega^{\mu}}_{\nu}x^{\nu})\ldots$$

Expanding the right hand side in terms of ω we obtain

$$\begin{aligned} i[\eta, \phi(x)] &\simeq \phi(x) + {\omega^{\mu}}_{\nu}x^{\nu}\partial_{\mu}\phi - \phi(x) \\ &\simeq {\omega^{\mu}}_{\nu}x^{\nu}\partial_{\mu}\phi \\ &\simeq \frac{1}{2}\omega^{\mu\nu}(x_{\nu}\partial_{\mu} - x_{\mu}\partial_{\nu})\,\phi(x) \end{aligned}$$

where in the last line we have used the antisymmetry of $\omega^{\mu\nu}$

Hints for the calculation

$$\phi(x') = U(\Lambda)\ \phi(x)U^\dagger(\Lambda)$$

taking $U(\Lambda) = e^{i\eta}$ and expanding to first order in η

$$\phi(x') = (1 + i\eta + \dots)\phi(x)(1 - i\eta + \dots)$$

$$\phi(x') = \phi(x) + i[\eta, \phi] + \dots$$

$$\omega^\mu_\nu x^\nu \partial_\mu = \omega^{\mu\nu} x_\nu \partial_\mu$$

the indeces $\mu\nu$ are dummy so we can interchange them to obtain

$$\omega^{\nu\mu} x_\mu \partial_\nu = -\omega^{\mu\nu} x_\mu \partial_\nu$$

$$\omega^{\mu\nu} x_\nu \partial_\mu = \frac{1}{2}(\omega^{\mu\nu} x_\nu \partial_\mu + \omega^{\nu\mu} x_\mu \partial_\nu) = \frac{1}{2}\omega^{\mu\nu}(x_\nu \partial_\mu - x_\mu \partial_\nu)$$

Generators of Lorentz Algebra

identifying $\eta = \frac{1}{2}\omega^{\mu\nu} J_{\mu\nu}$ we obtain

$$[J_{\mu\nu}, \phi(x)] = i(x_\mu \partial_\nu - x_\nu \partial_\mu)\, \phi(x) \equiv L_{\mu\nu}\, \phi(x)$$

Note that for $\mu, \nu = 1,\ 2,\ 3$ quantities $L_1 = L_{23},\ L_2 = L_{31}$ & $L_3 = L_{12}$

are differential operators representing orbital angular momentum

For any continuous group -- transformations that lie infinitesimally

close to identity define a vector space-- called lie algebra of group

basis vectors for this vector space are called generators of lie algebra

e.g. each rotation can be labeled by a set of continuously varying

parameters $(\theta_1, \theta_2, \theta_3)$ that can be regarded as component of a vector

directed along axis of rotation with magnitude given by angle of rotation

Generators of Lorentz Algebra -cont'd-

generators of Lie algebra are angular momentum J^k which satisfy

the commutation relations $[J_i, J_j] = i\epsilon_{ijk} J_k$

$\epsilon_{ijk} = +1(-1)$ if ijk are a cyclic (anticyclic) permutation of 1 2 3

$\epsilon_{ijk} = 0$ otherwise

In lowest-dimension non-trivial representation of rotation group

generators may be written $J_i = \frac{1}{2}\sigma_i$ ☛ where σ_i are Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

basis for this representation is conventionally chosen to be

eigenvectors of σ_3 that is column vectors $\binom{1}{0}$ and $\binom{0}{1}$

describing a spin-$\frac{1}{2}$ particle of spin projection up $(m = \frac{1}{2} \text{ or } \uparrow)$

and spin projection down $(m = -\frac{1}{2} \text{ or } \downarrow)$ along 3-axis, respectively

Generators of Lorentz Algebra -cont'd-

- We will soon see that six $J^{\mu\nu}$ operators generate three boosts and three rotations of Lorentz group
- To determine commutation rules of Lorentz algebra we can now simply compute commutators of differential operators $J^{\mu\nu}$
- The result is

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(g^{\nu\rho}\, J^{\mu\sigma} - g^{\mu\rho}\, J^{\nu\sigma} - g^{\nu\sigma}\, J^{\mu\rho} + g^{\mu\sigma}\, J^{\nu\rho})$$

Any matrices that are to represent this algebra must obey these same commutation rules

Nonrelativistic Quantum Mechanics

We begin by recalling prescription for obtaining Schrodinger equation for free particle of mass m

substitute into classical energy momentum relation

$$\boxed{E = \frac{p^2}{2m}}$$

the differential operators

$$\boxed{E \to i\hbar\frac{\partial}{\partial t} \qquad \mathbf{p} \to -i\hbar\nabla}$$

resulting operator equation acts on complex $\psi(\mathbf{x}, t)$ wavefunction

--with $h \equiv 1$ --

$$\boxed{i\frac{\partial\psi}{\partial t} + \frac{1}{2m}\nabla^2\psi = 0}$$

where we interpret ☛ $\boxed{\rho = |\psi|^2}$ as probability density

$|\psi|^2 d^3x$ gives probability of finding particle in a volume element d^3x

Nonrelativistic Quantum Mechanics (cont'd)

- We are often concerned with moving particles
  for example ☛ the collision of one particle with another
- We therefore need to be able to calculate $\mathbf{j}$
  density flux of a beam of particles
- From conservation of probability ☛
  rate of decrease of number of particles in a given volume
  equals total flux of particles out of that volume

$$-\frac{\partial}{\partial t}\int_V \rho\, dV = \int_S \mathbf{j}\cdot\hat{\mathbf{n}}\, dS = \int_V \nabla\cdot\mathbf{j}\, dV$$

$\hat{n}$ is unit vector along outward normal to element dS of surface S enclosing volume V and last equality is Gauss's theorem

- probability and flux densities are then related by continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0$$

Nonrelativistic Quantum Mechanics (cont'd)

To determine flux we first form $\partial\rho/\partial t$
by subtracting wave equation multiplied by $-i\psi^*$
from complex conjugate equation multiplied by $-i\psi$

We then obtain ☛

$$\boxed{\frac{\partial\rho}{\partial t} - \frac{i}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) = 0}$$

Comparing this with we identify probability flux density as

$$\boxed{\mathbf{j} = -\frac{i}{2m}(\psi^*\nabla\psi - \psi\nabla\psi^*)}$$

For example ☛ a solution of

$$\boxed{\psi = Ne^{i\mathbf{p}\cdot\mathbf{x} - iEt}}$$

which describes a free particle of energy E and momentum $\mathbf{p}$ has

$$\boxed{\rho = |N|^2} \qquad \boxed{\mathbf{j} = \frac{\mathbf{p}}{m}|N|^2}$$

Klein-Gordon Equation

- ✓ Wave equation 🕷 violates Lorentz covariance and is not suitable for particle moving relativistically
- ✓ Relativistic energy-momentum relation is $E^2 = \mathbf{p}^2 + m^2$
- ✓ Recall formulae $\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$ and $\partial_\mu = \left(\frac{\partial}{\partial t}, \nabla\right)$
- ✓ Making the operator substitutions $\underbrace{E \to i\hbar\frac{\partial}{\partial t} \text{ and } \mathbf{p} \to -i\hbar\nabla}_{(p^\mu \to i\partial^\mu)}$

we obtain Klein-Gordon equation ☛ $\boxed{-\frac{\partial^2\phi}{\partial t^2} + \nabla^2\phi = m^2\phi}$

- ✓ by subtracting Klein-Gordon equation multiplied by $-i\phi^*$ from complex conjugate equation multiplied by $-i\phi$ we then obtain

$$\boxed{\partial_t \underbrace{\left[i(\phi^*\,\partial_t\phi - \phi\,\partial_t\phi^*)\right]}_{\rho} + \vec{\nabla}.\underbrace{\left[-i(\phi^*\,\vec{\nabla}\phi - \phi\,\vec{\nabla}\phi^*)\right]}_{\vec{j}} = 0}$$ ⁂

Hints for the calculation

$$\partial_\mu(\phi^*\partial^\mu\phi) = \partial_\mu\phi^*\partial^\mu\phi + \phi^*\partial_\mu\partial^\mu\phi$$

$$\begin{aligned} -i\phi^*\partial_\mu\partial^\mu\phi - i\phi^* m^2\phi + i\phi\partial_\mu\partial^\mu\phi^* + i\phi m^2\phi^* &= -i\phi^*\partial_\mu\partial^\mu\phi + i\phi\partial_\mu\partial^\mu\phi^* \\ &= 0 \end{aligned}$$

Klein-Gordon Equation (cont'd)

By comparison with ✳ we identify probability and flux densities with the terms in square brackets

Example ☛ for a free particle of energy E and momentum $\mathbf{p}$ described by the Klein-Gordon solution

$$\boxed{\phi = N e^{i\mathbf{p}\cdot\mathbf{x} - iEt}}$$

we find from ⁂ that

$$\rho = i(-2iE)|N|^2 = 2E|N|^2$$

$$\mathbf{j} = -i(2i\mathbf{p})|N|2 = 2\mathbf{p}|N|^2$$

We see that probability density is proportional to E

⬇

relativistic energy of particle

Klein-Gordon Equation (cont'd)

- It is advantageous to express these results in 4-vector notation
- Not only are they more concise but also covariance becomes explicit
- Using D'Alembertian operator Klein-Gordon equation becomes

$$\boxed{(\Box^2 + m^2)\phi = 0}$$

- Probability and flux densities form a 4-vector

$$\boxed{j^\mu = (\rho, \mathbf{j}) = i(\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^*)}$$

which satisfies the (covariant) continuity relation

$$\boxed{\partial_\mu j^\mu = 0}$$

- Taking the free particle solution

$$\boxed{\phi = N e^{-ip\cdot x}}$$

we have ☛ $$\boxed{j^\mu = 2p^\mu |N|^2}$$

Klein-Gordon Equation (cont'd)

★ We noted that the probability density ρ

is the time-like component of a 4-vector ☛ $\rho \propto E$

★ Result may be anticipated since under Lorentz boost of velocity v

volume element suffers Lorentz contraction $d^3x \to d^3x\sqrt{1-v^2}$

★ To keep ρd^3x invariant we require

ρ to transform as time-like component of 4-vector $\rho \to \rho/\sqrt{1-v^2}$

So far, so good

★ What are energy eigenvalues of Klein-Gordon equation?

★ Substitution ✪ into ❃ gives ☛ $\boxed{E = \pm(\mathbf{p}^2 + m^2)^{\frac{1}{2}}}$

★ In addition to acceptable $E > 0$ solutions

we have negative energy solutions!!!

Historical Interlude

In 1927 Dirac devised relativistic wave equation linear in $\partial/\partial t$ and ∇

More on this next class...

Prescription for handling negative energy states was proposed by

$\text{Stückelberg}(1941)$ and $\text{Feynman}(1948)$

negative energy solution describes particle propagating backward in time

Expressed most simply ☛ idea is that

positive energy antiparticle propagating forward in time

Consider an electron of energy E, 3-momentum $\mathbf{p}$, and charge $-e$

Electromagnetic 4-vector current is ☛ $j^\mu(e^-) = -2e|N|^2(E,\mathbf{p})$

Now take antiparticle (a positron) with same E and $\mathbf{p}$

Since its charge is $+e$ ☛ $j^\mu(e^+) = +2e|N|^2(E,\mathbf{p})$

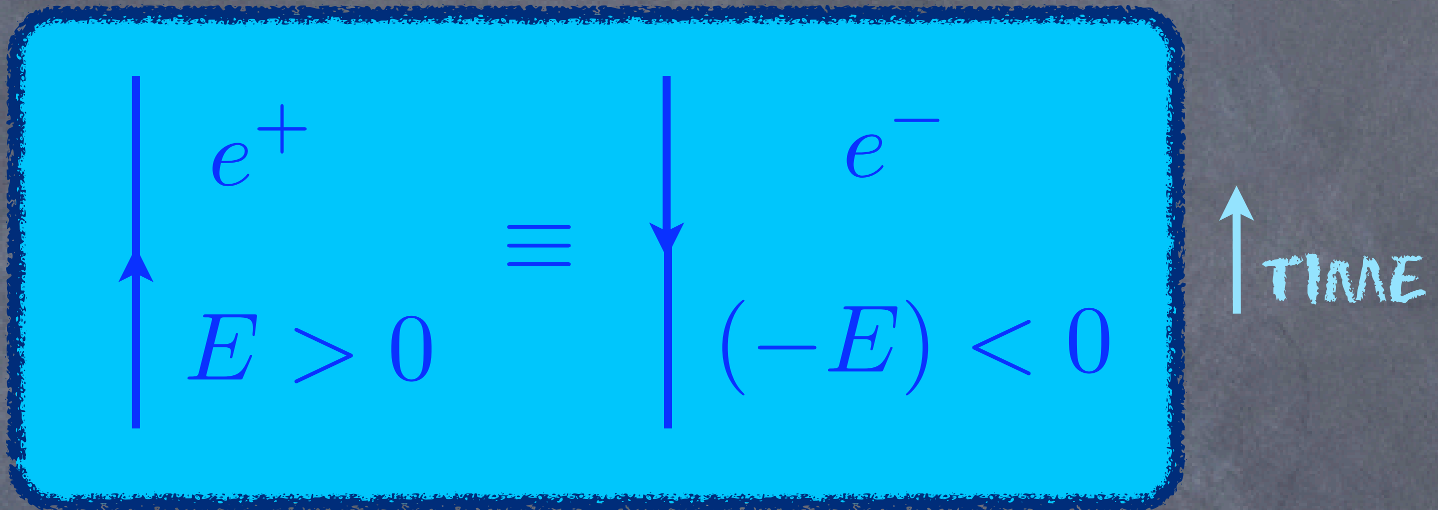
which is exactly same as current j^μ for electron with $-E$ and $-\mathbf{p}$

Historical Interlude (cont'd)

as far as a system is concerned ☛ emission of positron with energy E

is same as the absorption of an electron of energy $-E$

Pictorially we have ☛



In other words ☛

negative-energy particle solutions going backward in time

describe positive-energy antiparticle solutions going forward in time

Of course ☛ reason why this identification can be made is simply because

$$\boxed{e^{-i(-E)(-t)} = e^{-iEt}}$$

TO BE CONTINUED...