Quantum Mechanics

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Lesson II
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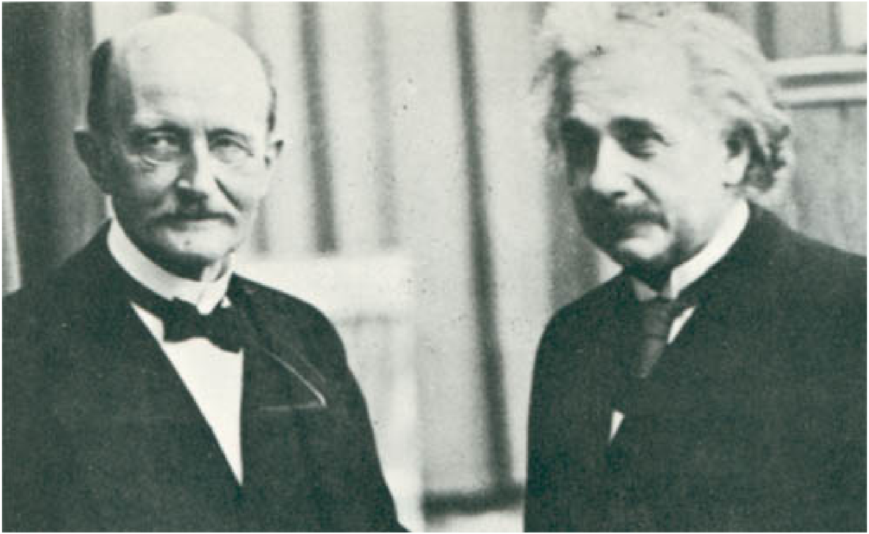
Table of Contents



- Quantum mechanics was born in early 20th century
  due to collapse of deterministic classical mechanics
  driven by Euler-Lagrange equations

- Collapse resulted from the discovery of various phenomena
  which are inexplicable with classical physics

- Pathway to quantum mechanics invariably begins with Planck
  and his analysis of blackbody spectral data

Stefan-Boltzmann law

- Rate at which objects radiate energy ☞ $L \propto AT^4$
- At normal temperatures ☞ ≈ 300 K
  not aware of this radiation because of its low intensity
- At higher temperatures ☞ sufficient IR radiation to feel the heat
- At still higher temperatures ☞ $\mathcal{O}(1000~\mathrm{K})$
  objects actually glow such as a red-hot electric stove burner
- At temperatures above 2000 K
  objects glow with a yellow or whitish color ☞ filament of lightbulb
- Blackbody ☞ idealized object that absorbs all incident radiation regardless of frequency or angle of incidence
- bolometric luminosity: $L = \sigma\, A\, T^4$ ☞ $\sigma = 5.67 \times 10^{-8}~\mathrm{W\,m^{-2}\,K^{-4}}$
- Radiant flux ☞ total power leaving $1~\mathrm{m}^2$ of blackbody surface @ T

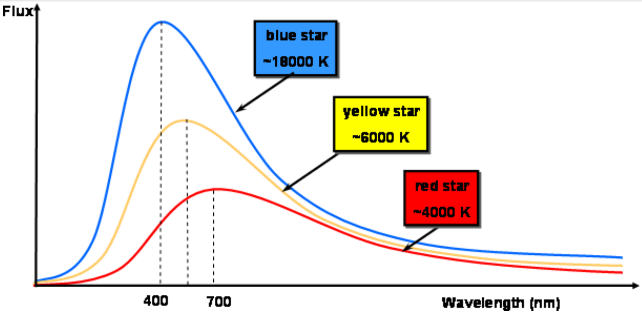
$$F(T) = L/A = \sigma T^4 \tag{1}$$

Wien's displacement law

- Wavelength $\lambda_{\max}$ at which spectral emittance reaches maximum decreases as T is increased in inverse proportion to T

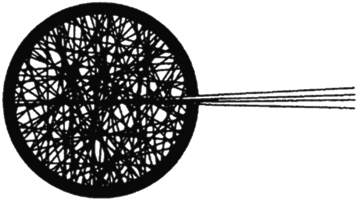
$$\lambda_{\max} T = 2.90 \times 10^{-3}\,\mathrm{m\,K} \tag{2}$$

- Qualitatively consistent with observation that
  heated objects first begin to glow with red color
  and at higher temperatures color becomes more yellow



Approximate realization of blackbody surface

- Consider hollow metal box with walls in thermal equilibrium @ T
- Cavity is filled with radiation forming standing waves
- Suppose there is small hole in one wall of box
  which allows some radiation to escape
- It is the hole and not the box itself that is the blackbody
- Radiation from outside that is incident on hole gets lost inside box and has a negligible chance of reemerging from the hole
  no reflections occur from blackbody (the hole)
- Radiation emerging from hole ☞ sample of radiation inside box



Radiation inside box

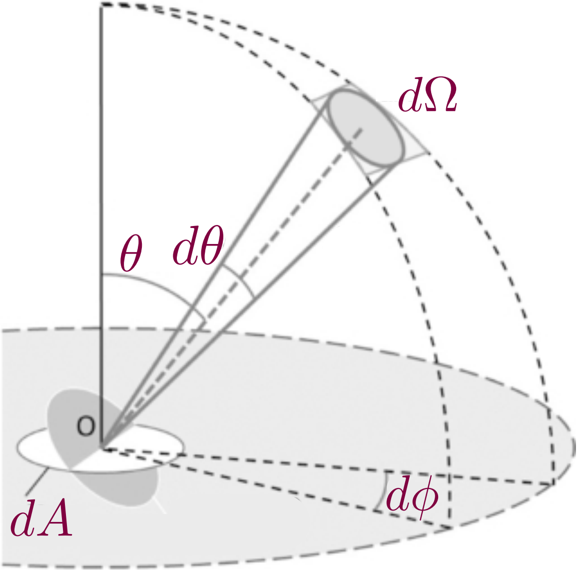
- $u(T)$ ☞ energy density (energy per unit volume) $[\mathrm{J\,m^{-3}}]$
- $\frac{du}{d\lambda} \equiv u_\lambda(\lambda, T)$ ☞ spectral energy density $[\mathrm{J\,m^{-3}\,nm^{-1}}]$
- If we look into interior of box and measure spectral energy density with wavelengths between λ and $\lambda + d\lambda$ in small volume element result would be ☞ $u_\lambda(\lambda, T)\, d\lambda$

Surface brightness (or spectral emittance)

- $B_\lambda(\lambda, T)$ ☞ spectral radiant flux per sterradian
  emitted from unit surface that lies normal to view direction
- Because photons of all wavelengths travel at c
  wavelength dependence of u_λ equals that of B_λ
- It does not matter whether radiation sampled is:
  that in $1\ \mathrm{m}^3$ @ fixed time or that impinging on $1\ \mathrm{m}^2$ in 1 s

$$B_\lambda(\lambda, T) = \frac{c}{4\pi} u_\lambda(\lambda, T) \tag{3}$$

Surface element ☞ different for each view direction



Radiant flux through unit area of fixed surface immersed in blackbody

$$dF_\lambda(\lambda, T) = B_\lambda(\lambda, T) \cos\theta d\Omega \tag{4}$$

$$F(T) = \int_0^\infty \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} B_\lambda(\lambda, T) \frac{dA \cos\theta}{dA} \, d\Omega d\lambda \tag{5}$$

$$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \cos\theta \sin\theta d\theta d\phi = \pi \tag{6}$$

$$F(T) = \int_0^\infty B_\lambda(\lambda, T) \, d\lambda = \sigma T^4 \tag{7}$$

$$u(T) = \int_0^\infty u_\lambda(\lambda, T) d\lambda = a T^4 \tag{8}$$

$$a = 4\sigma/c = 7.566 \times 10^{-16} \, \mathrm{J\, m^{-3}\, K^{-4}}$$

- Wave can be characterized by:
  wavelength λ, speed c, period $T = \lambda/c$, frequency $\nu = 1/T = c/\lambda$

$$\omega = 2\pi\nu = 2\pi c/\lambda \tag{9}$$

- Spectral energy density within $(\lambda, \lambda + \Delta\lambda)$

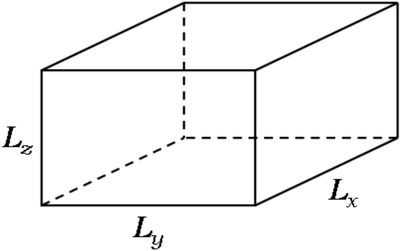
$$u_\lambda(\lambda, T)\, d\lambda = u_\omega(\omega, T)\, d\omega \tag{10}$$

- (9) and (10) yield

$$u_\lambda(\lambda, T) = u_\omega(\omega, T) \left|\frac{d\omega}{d\lambda}\right| = u_\omega(\omega, T)\,\frac{2\pi c}{\lambda^2} \tag{11}$$

- Consider box in thermal equlibrium @ T
- Spectral energy density of radiation with frequencies between ω and $\omega + d\omega$ in small volume element

$$\underbrace{u_\omega(\omega,T)}_{\substack{\text{average energy}\\ \text{per } \omega \text{ interval}\\ \text{per volume}}} \; d\omega = \left(\underbrace{\frac{\text{number of states inside box within interval } (\omega, \omega + d\omega)}{\text{volume of the box}}}_{N(\omega,T)} \right) \cdot \left(\underbrace{\text{average energy of one radiation mode of frequency } \omega}_{\langle E \rangle} \right)$$



☞ $V = L_x L_y L_z$

Estimate of $N(\omega, T)$

- Take $Z(\omega)$ ☞ # of standing waves up to ω in box

$$\left(\begin{array}{c} \text{number of states inside the box} \\ \text{within the interval } (\omega, \omega + d\omega) \end{array} \right) = \frac{dZ}{d\omega} d\omega \tag{12}$$

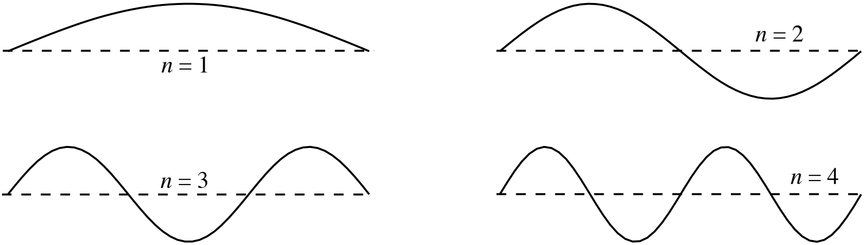
- Assume allowed frequencies of the radiation are spaced evenly

$$Z(\omega) = \varkappa \underbrace{\left(\frac{\omega}{\omega_{\min,x}}\right)}_{\substack{\text{number of waves} \\ \text{in } x \text{direction}}} \underbrace{\left(\frac{\omega}{\omega_{\min,y}}\right)}_{\substack{\text{number of waves} \\ \text{in } y \text{direction}}} \underbrace{\left(\frac{\omega}{\omega_{\min,z}}\right)}_{\substack{\text{number of waves} \\ \text{in } z \text{direction}}}$$

- Minimum frequency exists because there is maximum wavelength

$$\lambda_{\max,x} = 2L_x \tag{13}$$

that can exist between walls located L_x units apart



$$\omega_{\min,j} = \frac{2\pi c}{2L_j} = \frac{\pi c}{L_j} \qquad j = \{x, y, z\} \tag{14}$$

- Next 3 wavelengths ☞ $2L_x/2$, $2L_x/3$, $2L_x/4$
- Corresponding frequencies ☞ $2\omega_{\min,x}$ $3\omega_{\min,x}$, $4\omega_{\min,x}$
- This justifies our assumption
  that frequencies of radiation in box are spaced evenly

- Using (14)

$$Z(\omega) = \varkappa \frac{\omega^3}{(\pi c)^3/(L_x L_y l_z)} = \varkappa \frac{\omega^3}{(\pi c)^3} L_x L_y L_z \tag{15}$$

- Photons have two independent polarizations ☞ $\varkappa = \pi/3$
(2 states per wave vector $\vec{k} = (\omega/c)\,\hat{n}$ propagating in direction $\hat{n}$)
- From (15)

$$\begin{pmatrix} \text{number of states inside the box} \\ \text{within the interval } (\omega, \omega + d\omega) \end{pmatrix} = \frac{\omega^2}{\pi^2 c^3} L_x L_y L_z$$

and finally

$$\frac{\begin{pmatrix} \text{number of states inside the box} \\ \text{within the interval } (\omega, \omega + d\omega) \end{pmatrix}}{\text{volume of the box}} = \frac{\omega^2 d\omega}{\pi^2 c^3} \tag{16}$$

- Energy of each standing wave (normal mode) distributed according to Maxwell-Boltzmann distribution

$$P(E)\,dE = \frac{e^{-E/kT}}{kT}\,dE \tag{17}$$

$k = 1.38 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ ☞ Boltzmann's constant

- Classical Rayleigh-Jeans prediction

$$\langle E \rangle = \frac{\int_0^\infty P(E)\,E\,dE}{\int_0^\infty P(E)\,dE} = \cdots = kT \tag{18}$$

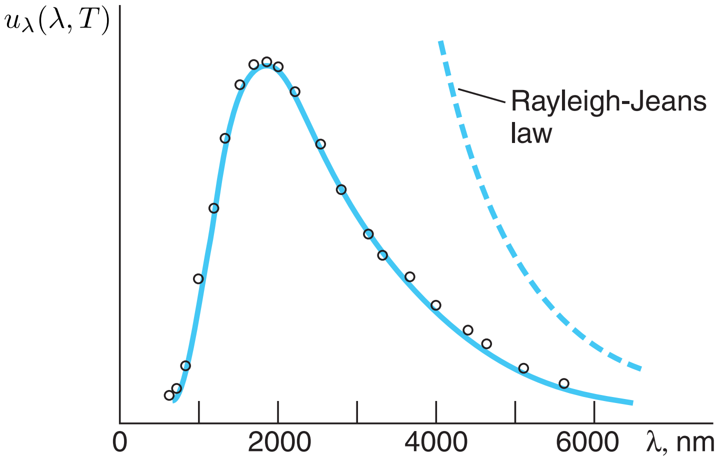
- Energy density and surface brightness become

$$u_\lambda(\lambda, T)\,d\lambda = \frac{N(\lambda)\,d\lambda}{V}\,kT = \frac{8\pi}{\lambda^4}\,kT\,d\lambda \tag{19}$$

and

$$B_\lambda(\lambda, T) = \frac{c}{4\pi}\,u_\lambda(\lambda, T) = \frac{2c}{\lambda^4}\,kT \tag{20}$$

UV catastrophe



$$\int_0^\infty u_\lambda(\lambda T)\, d\lambda \to \infty \tag{21}$$

At short wavelengths classical theory is absolutely not physical

Planck proposed a solution to this problem...

- Energy is *not* a continuous variable
- Each oscillator emits or absorb only in integer multiples $\Delta E = h\nu$

$$E_n = n\,\Delta E, \quad \text{with} \quad n = 0, 1, 2, 3, \cdots . \tag{22}$$

$h = 6.626 \times 10^{-34}\,\mathrm{J\,s} = 4.136 \times 10^{-15}\,\mathrm{eV\,s}$ Planck's constant

- Average energy of an oscillator is then given by the discrete sum

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n\, P(E_n)}{\sum_{n=0}^{\infty} P(E_n)} = \cdots = \frac{hc/\lambda}{e^{hc/(\lambda kT)} - 1} . \tag{23}$$

- Multiplying this result by number of oscillators per unit volume
☞ spectral emittance distribution function of radiation inside cavity

$$B_\lambda(\lambda, T) = \frac{2c}{\lambda^4}\,\langle E \rangle = \frac{2c}{\lambda^4} \frac{hc/\lambda}{e^{hc/(\lambda kT)} - 1} \tag{24}$$

Relevant limits

- Classical limit $h \to 0$ and $\Delta E \to 0$ ☞ For $x = hc/(\lambda kT) \ll 1$ exponential in (24) can be expanded using $e^x \approx 1 + x + \cdots$

$$e^{hc/(\lambda kT)} - 1 \approx \frac{hc}{\lambda kT} \quad \text{and so } \langle E \rangle = \frac{hc/\lambda}{e^{hc/(\lambda kT)} - 1} = kT \tag{25}$$

- For long wavelength ☞ Rayleigh-Jeans formula

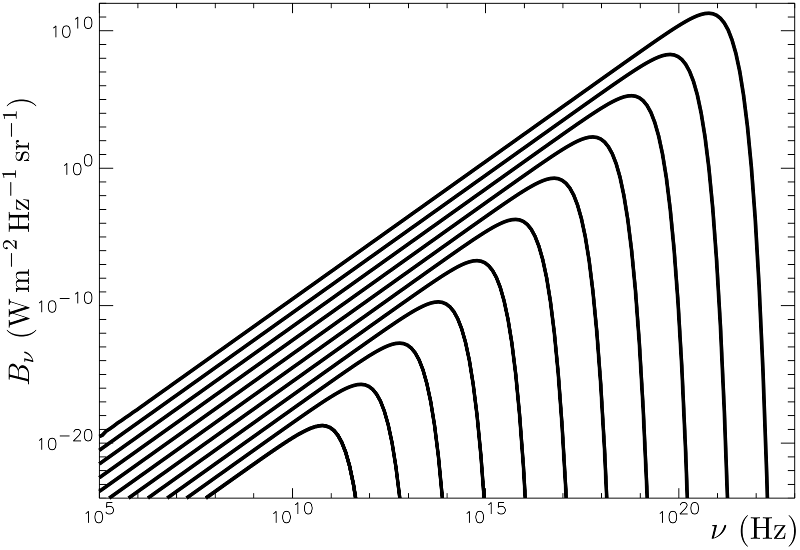
$$\lim_{\lambda \to \infty} u_\lambda \to \frac{8\pi}{\lambda^4} kT \tag{26}$$

- Quantum regime $\lambda \to 0$ (i.e. high photon energy) $e^{hc/(\lambda kT)} \to \infty$ exponentially faster than $\lambda^5 \to 0$ so

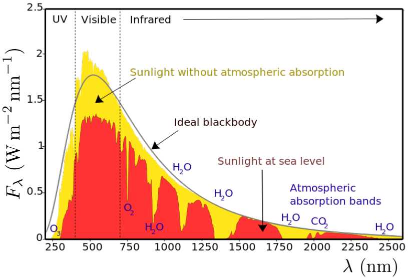
$$\lim_{\lambda \to 0} \frac{1}{\lambda^5 (e^{hc/(\lambda kT)} - 1)} \to 0 \tag{27}$$

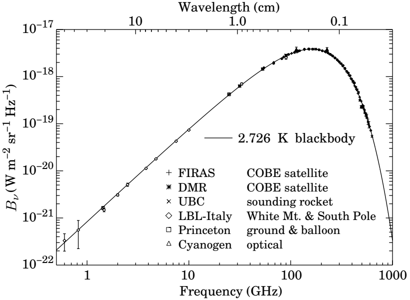
There is no ultraviolet catastrophe in the quantum limit !!!

Planck spectrum of blackbody radiation for 10^0 K, 10^1 K, $\cdots$, 10^{10} K



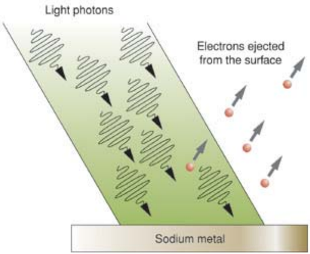
Specific examples: the Sun and the CMB

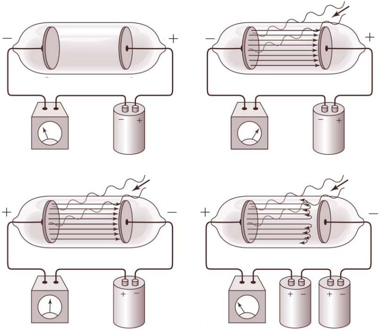




- Success of Planck's idea immediately raises question:
  why is it that oscillators in walls
  can only emit and absorb energies in multiples of $h\nu$?
- Explanation supplied by Einstein:
  light is composed of particles called photons
  and each photon has an energy $E_\gamma = h\nu$
- Photoelectric effect is the observation that a beam of light
  can knock electrons out of metal surface
- Electrons emitted from surface are called photoelectrons
- What is surprising about photoelectric effect?
  Energy of photoelectrons independent of intensity of incident light
- If frequency of light is swept
  find minimum frequency ν_0 below which no electrons are emitted
- Energy $\varphi = h\nu_0$ corresponding to this frequency
  is called work function of surface

- Beam of light can knock electrons out of metal surface
- Photoelectrons collected on detector which forms part of electrical circuit
- Current measured in circuit $\propto$ # electrons striking detector plate
- To measure kinetic energy of e^-'s apply static retarding potential V
- Only electrons with $K > eV$ will reach the plate
- Any electrons with $K < eV$ will be repelled and won't be detected





Why classical electromagnetism fails to explain photoelectric effect

In classical electromagnetism...

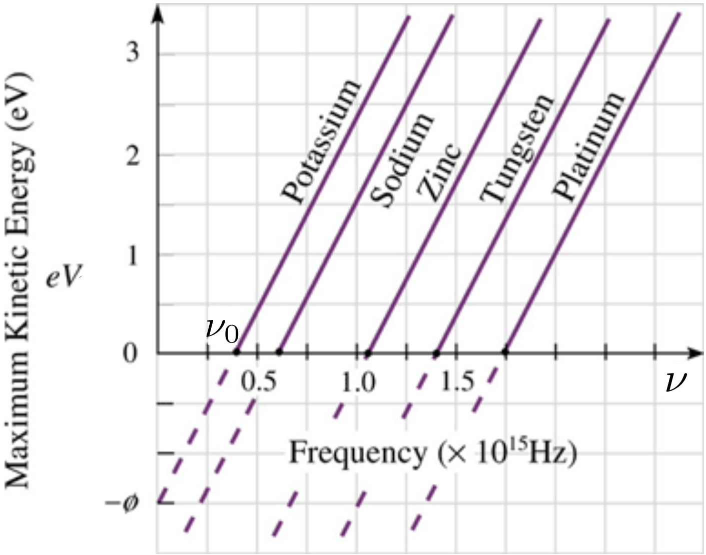
1. Increasing intensity I of beam
   increases amplitude of oscillating electric field $\vec{E}$
   Since force incident beam exerts on electron is $\vec{F} = e\vec{E}$
   theory predicts photoelectron energy increases with increasing I
   However ☞ V_0 is independent of light intensity
2. As long as intensity of light is large enough
   photoelectric effect should occur at any frequency
   in direct contradiction with experiment
   showing clear cutoff ν_0 below which no electrons are ejected
3. Energy imparted to e^- must be "soaked up" from incident wave
   if very weak light is used ☞ expected measurable time delay
   between light striking surface and e^- emission
   This has never been observed

Why all problems are solved by quantum mechanics

In quantum mechanics...

1. Doubling intensity doubles number of photons
   but doesn't change their energy
   total number of photons striking surface
   is immaterial in determining energy of ejected electron
2. Frequency of light determines photon energy
   photons with $h\nu < \varphi$ don't have enough energy to leave surface
3. Photoelectric effect is viewed as single collisional event
   and no time delay is predicted
4. When $K_{\rm max}$ is plotted as function of frequency $\nu > \nu_0$
   experimental data fit straight line whose slope equals h

$$\boxed{K_{\max} = h\nu - \varphi}$$



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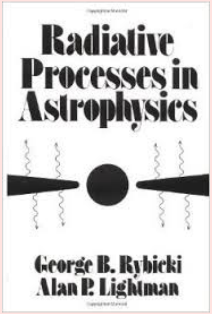

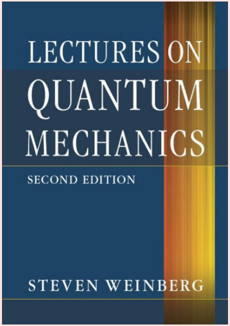

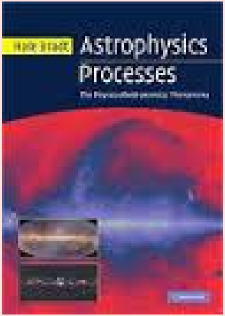

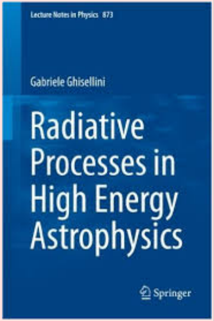

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