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Problems set # 2 **Physics 400** February 14, 2019

1. The peak of the radiation curve for a certain blackbody occurs at a wavelength of $\lambda_a = 1~\mu$m. If the temperature is raised so that the total radiated power is increased 16-fold, at what wavelength λ_b will the new intensity maximum be found?

2. *(i)* Show that

$$\langle E \rangle = \frac{\int_0^\infty P(E)\, E\, dE}{\int_0^\infty P(E)\, dE} = \cdots = kT\,,$$

where

$$P(E)\, dE = \frac{e^{-E/kT}}{kT}\, dE\,,$$

$k = 1.38 \times 10^{-23}~\mathrm{J\,K^{-1}}$ is the Boltzmann's constant, and T the absolute temperature. *(ii)* Show that the average energy of an oscillator is given by the discrete sum

$$\langle E \rangle = \frac{\sum_{n=0}^\infty E_n\, P(E_n)}{\sum_{n=0}^\infty P(E_n)} = \cdots = \frac{hc/\lambda}{e^{hc/(\lambda kT)} - 1}\,,$$

where $\Delta E = h\nu$, with $h = 6.626 \times 10^{-34}~\mathrm{J}\,s$ the Planck's constant.

3. Verify that if you integrate

$$B_\lambda(\lambda, T) = \frac{2c}{\lambda^4}\, \langle E \rangle = \frac{2c}{\lambda^4} \frac{hc/\lambda}{e^{hc/(\lambda kT)} - 1}$$

over all wavelengths and solid angles you can reproduce Stefan-Boltzmann law $F(T) = \sigma T^4$.

4. Show that Wien's displacement law can be derived by determining the maximum of

$$B_\lambda(\lambda, T) = \frac{2c}{\lambda^4}\, \langle E \rangle = \frac{2c}{\lambda^4} \frac{hc/\lambda}{e^{hc/(\lambda kT)} - 1}\,.$$

5. A compilation of experimental measurements of the CMB reveals an accurate blackbody spectrum. Actually, according to the FIRAS (Far InfraRed Absolute Spectrometer) instrument aboard the COBE (*Cosmic Background Explorer*) satellite, which measured a temperature of $T = 2.726 \pm 0.010$ K, the CMB is the most perfect blackbody ever seen. *(i)* Write down an integral which determines how many photons per cubic centimeter are contained in the CMB. Estimate the result within an order of magnitude. *(ii)* Convince yourself that the average energy of a CMB photon is $\langle E_\gamma^{\rm CMB} \rangle \approx 6 \times 10^{-4}$ eV *(iii)* Show that a freely expanding blackbody radiation remains described by the Planck formula, but with a temperature that drops in proportion to the scale expansion.

SOLUTIONS

1. The wavelength of the intensity maximum is inversely related to the temperature by Wein's law $\lambda T =$ constant, while the total radiated power is related to the temperature to the fourth power by Stefans law $L = \sigma A T^4$. Thus if the power goes up by a factor of 16, the temperature must have increased by a factor of 2, since $2^4 = 16$. If the temperature doubled, then the wavelength of the peak of the radiation curve must have halved, so $\lambda_b = \lambda_a/2 = 0.5\ \mu$m.

2. $\int_0^\infty P(E)\,dE = \int_0^\infty e^{-E/(kT)}/(kT)\,dE = -(kT)e^{-E/(kT)}/(kT)\big|_0^\infty = 1$. To calculate $\int_0^\infty E\,P(E)\,dE$ we adopt the following change of variables, $u = E$ and $dv = e^{-E/(kT)}dE$, yielding $du = dE$ and $v = -kTe^{-E/(kT)}$. Then, $\int_0^\infty Ee^{-E/(kT)}dE = uv - \int vdu = E(-kT)e^{-E/(kT)}\big|_0^\infty - \int_0^\infty(-kT)e^{-E/(kT)}dE = kT\int_0^\infty e^{-E/(kT)}dE = (kT)^2$. Thus $\langle E\rangle = \int_0^\infty E\,P(E)\,dE = (kT)^2/(kT) = kT$. We now take $E_n = nh\nu = nhc/\lambda$,

$$\langle E\rangle = \frac{\sum_{n=0}^\infty \frac{nhc}{\lambda kT}e^{-nhc/(\lambda kT)}}{\frac{1}{kT}\sum_{n=0}^\infty e^{-nhc/(\lambda kT)}} = kT\frac{\sum_{n=0}^\infty n\alpha e^{-n\alpha}}{\sum_{n=0}^\infty e^{-n\alpha}}\,, \tag{1}$$

where $\alpha = hc/(\lambda kT)$. First we note that

$$-\alpha\frac{d}{d\alpha}\ln\left[\sum_{n=0}^\infty e^{-n\alpha}\right] = \frac{\sum_{n=0}^\infty n\alpha e^{-n\alpha}}{\sum_{n=0}^\infty e^{-n\alpha}}\,, \tag{2}$$

and so substituting (2) into (1) we have

$$\langle E\rangle = kT\left[-\alpha\frac{d}{d\alpha}\ln\left(\sum_{n=0}^\infty e^{-n\alpha}\right)\right] = -\frac{hc}{\lambda}\left[\frac{d}{d\alpha}\ln\left(\sum_{n=0}^\infty e^{-n\alpha}\right)\right] = -\frac{hc}{\lambda}\left\{\frac{d}{d\alpha}\ln\left[(1-e^{-\alpha})^{-1}\right]\right\}\,, \tag{3}$$

where in the last equality we have used the sum of a geometric series $\sum_{n=0}^\infty e^{-n\alpha} = (1-e^{-\alpha})^{-1}$. Now, we calculate

$$\frac{d}{d\alpha}\ln(1-e^{-\alpha})^{-1} = (-1)\frac{(1-e^{-\alpha})^{-2}e^{-\alpha}}{(1-e^{-\alpha})^{-1}} = \frac{-e^{-\alpha}}{1-e^{-\alpha}}\left(\frac{e^\alpha}{e^\alpha}\right) = -\frac{1}{(e^\alpha - 1)}\,. \tag{4}$$

Substituting (4) into (3) we obtain the desired result.

3. In the frequency domain

$$F(T) = \int_0^\infty\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} B_\lambda(\lambda,T)\frac{dA\cos\theta}{dA}\;d\Omega d\lambda$$

can be rewritten as

$$F(T) = \pi\int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)}-1}d\nu\,. \tag{5}$$

This is an integral over frequency alone. Change the variable to $x = h\nu/(kT)$ to obtain an integral of the form $\int[x^3(e^x-1)^{-1}]dx$. Solutions of this type of integral include the Riemann zeta function. It is straightforward to show that the result of the integration can be written as $F(T) = \sigma T^4$, with $\sigma = 2\pi^5k^4/(15c^2h^3) = 5.67\times10^{-8}\ \mathrm{W\,m^{-2}\,K^{-4}}$.

4. The first derivative of

$$B_\lambda(\lambda, T) = \frac{2c}{\lambda^4}\,\langle E\rangle = \frac{2c}{\lambda^4}\frac{hc/\lambda}{e^{hc/(\lambda kT)}-1}$$

in the frequency domain reads

$$\frac{\partial}{\partial\nu}B_\nu(\nu,T) = \frac{2h\nu^2}{c^2\left[e^{h\nu/(kT)}-1\right]^2}\left\{3\left[e^{h\nu/(kT)}-1\right] - \frac{h\nu}{kT}e^{h\nu/(kT)}\right\}\,. \tag{6}$$

This derivative is only equal to zero when the numerator is equal to zero. The corresponding denominator is larger than zero for $0 < \nu < \infty$. Demanding the numerator to vanish we obtain $3(e^{x_\nu} - 1) = x_\nu e^{x_\nu}$, with $x_\nu = h\nu/(kT)$. This transcendental function can only be solved numerically. We find $x_\nu \simeq 2.82$, and in a further step $\nu_{\rm ext}/T = x_\nu k/h$, where $\nu_{\rm ext}$ is the frequency at which the extreme (either a minimum or a maximum) of the Planck function in the frequency domain occurs. It can simply be proved that for this extreme the second derivative fulfills the condition $\partial^2 B_\nu/\partial\nu^2 < 0$ so that the extreme corresponds to a maximum, $\nu_{\rm max}$. The form of Wien's displacement law in terms of maximum spectral emittance per unit wavelength is derived using similar methods, but starting with the form of Planck's law expressed in the wavelength domain. The effective result is to substitute 3 for 5 in the equation for the peak frequency, i.e. $5(e^{x_\lambda} - 1) = x_\lambda e^{x_\lambda}$, where $x_\lambda = hc/(\lambda kT)$. This solves with $x_\lambda = 4.96$, yielding $\lambda_{\rm max} T = ch/(x_\nu k) \simeq 2.90\times 10^{-3}$ m K.

5. The total energy density in the blackbody radiation is

$$u = \int_0^\infty \frac{8\pi hc}{\lambda^5}\,d\lambda\,\frac{1}{e^{hc/\lambda kT}-1}\,. \tag{7}$$

Change the variable $x = hc/\lambda kT$ to obtain an integral of the form $\int[x^3(e^x-1)^{-1}]dx$, which can be solved using the Riemann zeta function; we obatain

$$u = \frac{8\pi^5(kT)^4}{15(hc)^3} = 7.56464\times 10^{-15}\,(T/{\rm K})^4 \text{ erg/cm}^3\,. \tag{8}$$

(Recall that 1 J $\equiv 10^7$ erg $= 6.24\times 10^{18}$ eV.) We can easily interpret the Planck distribution in terms of quanta of light or photons. Each photon has an energy $E_\gamma = hc/\lambda$. Hence the number dn_γ of photons per unit volume in blackbody radiation in a narrow range of wavelengths from λ to $\lambda + d\lambda$ is

$$dn_\gamma = \frac{du}{hc/\lambda} = \frac{8\pi}{\lambda^4}\,d\lambda\,\frac{1}{e^{hc/\lambda kT}-1}\,. \tag{9}$$

Then the total number of photons per unit volume is

$$n_\gamma = \int_0^\infty dn_\gamma = 8\pi\left(\frac{kT}{hc}\right)^3\int_0^\infty \frac{x^2\,dx}{e^x-1}, \tag{10}$$

where $x = hc/(\lambda kT)$. The integral cannot be expressed in terms of elementary functions, but $\int[x^2(e^x-1)^{-1}]dx = \Gamma(3)\zeta(3) \approx 2.4$. Therefore, the number photon density is

$$n_\gamma = 60.42198\left(\frac{kT}{hc}\right)^3 = 20.28\left(\frac{T}{\rm K}\right)^3 \text{ photons cm}^{-3} \approx 400 \text{ photons cm}^{-3}, \tag{11}$$

and the average photon energy is

$$\langle E_\gamma \rangle = u/n_\gamma = 3.73 \times 10^{-16}\,(T/\mathrm{K})\ \mathrm{erg}\ . \tag{12}$$

For a temperature of 2.726 K, the number density of CMB photons is ≈ 410 cm^{-3} and the typical photon energy is $\approx 6 \times 10^{-4}$ eV in agreement with the values adopted in exercise 8.9. *(iii)* Now, let's consider what happens to blackbody radiation in an expanding universe. Suppose the size of the universe changes by a factor f, for example, if it doubles in size, then $f = 2$. As predicted by the Doppler effect, the wavelengths will change in proportion to the size of the universe to a new value $\lambda' = f\lambda$. After the expansion, the energy density du' in the new wavelength range λ' to $\lambda' + d\lambda'$ is less than the original energy density du in the old wavelength range $\lambda + d\lambda$, for two different reasons: (1) since the volume of the universe has increased by a factor of f^3, as long as no photons have been created or destroyed, the number of photons per unit volume has decreased by a factor of $1/f^3$; (2) the energy of each photon is inversely proportional to its wavelength, and therefore is decreased by a factor of $1/f$. It follows that the energy density is decreased by an overall factor $1/f^3 \times 1/f = 1/f^4$:

$$du' = \frac{1}{f^4}\ du = \frac{8\pi hc}{\lambda^5 f^4}\, d\lambda\, \frac{1}{e^{hc/\lambda kT} - 1}. \tag{13}$$

If we rewrite the previous equation in terms of the new wavelengths λ', it becomes

$$du' = \frac{8\pi hc}{\lambda'^5}\, d\lambda'\, \frac{1}{e^{hcf/\lambda' kT} - 1}, \tag{14}$$

which is exactly the same as the old formula for du in terms of λ and $d\lambda$, except that T has been replaced by a new temperature $T' = T/f$. Therefore, we conclude that freely expanding blackbody radiation remains described by the Planck formula, but with a temperature that drops in inverse proportion to the scale of expansion.