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Problems set # 12 **Physics 400** May 9, 2019

1. Consider a 2-dimensional Hilbert space spanned by an orthonormal basis $\{|\uparrow\rangle, |\downarrow\rangle\}$. This corresponds to spin up/down for spin = 1/2. Let us define the operators

$$\hat{S}_x = \frac{\hbar}{2}(\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|), \quad \hat{S}_y = \frac{\hbar}{2i}(\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|), \quad \hat{S}_z = \frac{\hbar}{2}(\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|).$$

(i) Find the matrix representations of these operators in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis. *(ii)* Show that $[\hat{S}_x, \hat{S}_y] = i\hbar\hat{S}_z$, and cyclic permutations. Do this two ways: Using the Dirac notation definitions above and the matrix representations found in *(i)*. *(iii)* Now let $|\pm\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle \pm |\downarrow\rangle$. Show that these vectors form a new orthonormal basis. *(iv)* Find the matrix representations of these operators in the $\{|+\rangle, |-\rangle\}$ basis.

2. A beam of spin 1/2 particles is sent through series of three Stern-Gerlach measuring devices. The first SGz device transmits particles with $\hat{S}_z = \hbar/2$ and filters out particles with $\hat{S}_z = -\hbar/2$. The second device, an SGn device transmits particles with $\hat{S}_n = \hbar/2$ and filters out particles with $\hat{S}_n = -\hbar/2$, where the axis $\hat{n}$ makes an angle θ in the $x - z$ plane with respect to the z-axis. Thus the particles passing through this SGn device are in the state

$$|+\hat{n}\rangle = \cos(\theta/2)|+\hat{z}\rangle + e^{i\varphi}\sin(\theta/2)|-\hat{z}\rangle\,,$$

with the angle $\varphi = 0$. A last SGz device transmits particles with $\hat{S}_z = -\hbar/2$ and filters out particles with $\hat{S}_z = \hbar/2$. *(i)* What fraction of the particles transmitted through the first SGz device will survive the third measurement? *(ii)* How must the angle θ of the SGn device be oriented so as to maximize the number of particles that are transmitted by the final SGz device? What fraction of the particles survive the third measurement for this value of θ? *(iii)* What fraction of the particles survive the last measurement if the SGz device is simply removed from the experiment?

3. *(i)* Consider a composite object such as the hydrogen atom. Will it behave as a boson or fermion? *(ii)* Argue in general that objects containing an even/odd number of fermions will behave as bosons/fermions.

4. Consider two spinless particles of the same mass m, and Hamiltonian given by

$$H = \frac{p_1^2}{2m} + \frac{p_2^2}{2m} + V(r_1, r_2)\,.$$

(i) Define the exchange operator P, such that $P\psi(r_1, r_2) = \psi(r_2, r_1)$, for an arbitrary function ψ. Show that $P^2 = 1$. What are the eigenvalues of P ? *(ii)* Under what circumstances does $[P, H] = 0$? [*Hint*: It may help to multiply by a test function to evaluate this commutator! Be careful with the momentum terms in H!] *(iii)* Suppose that $[P, H] = 0$, and that at time $t = 0$

$$\Psi(r_1, r_2, 0) = \sigma\Psi(r_2, r_1, 0),$$

where $\sigma \pm 1$. Show that

$$\Psi(r_1, r_2, t) = \sigma\Psi(r_2, r_1, t).$$

(iv) Does it make sense to talk about a pair of bosons or fermions if $[P, H] \neq 0$?

SOLUTIONS

1.(*i*)

$$\left[\hat{S}_x\right] = \begin{pmatrix} \langle\uparrow|\hat{S}_x|\uparrow\rangle & \langle\uparrow|\hat{S}_x|\downarrow\rangle \\ \langle\downarrow|\hat{S}_x|\uparrow\rangle & \langle\downarrow|\hat{S}_x|\downarrow\rangle \end{pmatrix}$$
$$= \tfrac{\hbar}{2}\begin{pmatrix} \langle\uparrow|\left(|\uparrow\rangle\langle\downarrow|+|\downarrow\rangle\langle\uparrow|\right)|\uparrow\rangle & \langle\uparrow|\left(|\uparrow\rangle\langle\downarrow|+|\downarrow\rangle\langle\uparrow|\right)|\downarrow\rangle \\ \langle\downarrow|\left(|\uparrow\rangle\langle\downarrow|+|\downarrow\rangle\langle\uparrow|\right)|\uparrow\rangle & \langle\downarrow|\left(|\uparrow\rangle\langle\downarrow|+|\downarrow\rangle\langle\uparrow|\right)|\downarrow\rangle \end{pmatrix} = \tfrac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and similarly,

$$\left[\hat{S}_y\right] = \frac{\hbar}{2i}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \left[\hat{S}_z\right] = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(*ii*)

$$\left[\hat{S}_x, \hat{S}_y\right] = \tfrac{\hbar^2}{4i}\left[\begin{array}{l} \left(|\uparrow\rangle\langle\downarrow|+|\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle\langle\downarrow|-|\downarrow\rangle\langle\uparrow|\right) \\ \qquad -\left(|\uparrow\rangle\langle\downarrow|-|\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle\langle\downarrow|+|\downarrow\rangle\langle\uparrow|\right) \end{array}\right]$$
$$= \tfrac{\hbar^2}{4i}\left[-|\uparrow\rangle\langle\uparrow|+|\downarrow\rangle\langle\downarrow|-|\uparrow\rangle\langle\uparrow|+|\downarrow\rangle\langle\downarrow|\right] = i\hbar\hat{S}_z$$

$$\left[\hat{S}_x, \hat{S}_y\right] = \tfrac{\hbar^2}{4i}\left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$
$$= \tfrac{\hbar^2}{4i}\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\right] = i\hbar\tfrac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\hbar\hat{S}_z$$

(*iii*) Note that these vectors are eigenvectors of $\hat{S}_x$. Clearly, we have

$$\langle + \mid + \rangle = \tfrac{1}{2}\left(\langle\uparrow|+\langle\downarrow|\right)\left(|\uparrow\rangle+|\downarrow\rangle\right) = \tfrac{1}{2}(1+1) = 1 = \langle - \mid - \rangle$$
$$\langle + \mid - \rangle = \tfrac{1}{2}\left(\langle\uparrow|+\langle\downarrow|\right)\left(|\uparrow\rangle-|\downarrow\rangle\right) = \tfrac{1}{2}(1-1) = 0 = \langle - \mid + \rangle$$

so they form an orthonormal basis.

(iv)

$$\left[\hat{S}_x\right]^{\pm} = \frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} \left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle + |\downarrow\rangle\right) & \left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right) \\ \left(\langle\uparrow| - \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle + |\downarrow\rangle\right) & \left(\langle\uparrow| - \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{pmatrix}$$
$$= \frac{\hbar}{4}\begin{pmatrix} 1+1 & -1+1 \\ 1-1 & -1-1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \hat{S}_z$$

Clearly, $\hat{S}_x$ is diagonal in its own basis.

$$\left[\hat{S}_y\right]^{\pm} = \frac{\hbar}{2}\frac{1}{2i}\begin{pmatrix} \left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle + |\downarrow\rangle\right) & \left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right) \\ \left(\langle\uparrow| - \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle + |\downarrow\rangle\right) & \left(\langle\uparrow| - \langle\downarrow|\right)\left(|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{pmatrix}$$
$$= \frac{\hbar}{4i}\begin{pmatrix} 1-1 & -1-1 \\ 1+1 & -1+1 \end{pmatrix} = \frac{\hbar}{2i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\hat{S}_y$$

$$\left[\hat{S}_z\right]^{\pm} = \frac{\hbar}{2}\frac{1}{2}\begin{pmatrix} \left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\left(|\uparrow\rangle + |\downarrow\rangle\right) & \left(\langle\uparrow| + \langle\downarrow|\right)\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right) \\ \left(\langle\uparrow| - \langle\downarrow|\right)\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\left(|\uparrow\rangle + |\downarrow\rangle\right) & \left(\langle\uparrow| - \langle\downarrow|\right)\left(|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|\right)\left(|\uparrow\rangle - |\downarrow\rangle\right) \end{pmatrix}$$
$$= \frac{\hbar}{4}\begin{pmatrix} 1-1 & 1+1 \\ 1+1 & 1-1 \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hat{S}_x$$

2.(*i*) What fraction of the particles transmitted through the first SGz device will survive the third measurement?

We use the $\hat{S}_z$ diagonal basis $|\pm\hat{z}\rangle$. The first measurement corresponds to the projection operator

$$\hat{M}(+\hat{z}) = |+\hat{z}\rangle\langle+\hat{z}|$$

The second measurement is given by the projection operator

$$\hat{M}(+\hat{n}) = |+\hat{n}\rangle\langle+\hat{n}|$$

where

$$|+\hat{n}\rangle = \cos\frac{\theta}{2}|+\hat{z}\rangle + \sin\frac{\theta}{2}|-\hat{z}\rangle$$

so that

$$\hat{M}(+\hat{n}) = \cos^2\frac{\theta}{2}|+\hat{z}\rangle\langle+\hat{z}| + \cos\frac{\theta}{2}\sin\frac{\theta}{2}(|+\hat{z}\rangle\langle-\hat{z}| + |-\hat{z}\rangle\langle+\hat{z}|) + \sin^2\frac{\theta}{2}|-\hat{z}\rangle\langle-\hat{z}|$$

The last measurement corresponds to the projection operator

$$\hat{M}(-\hat{z}) = |-\hat{z}\rangle\langle-\hat{z}|$$

The total or combined measurement is given by the product in the appropriate order

$$\hat{M}_T = \hat{M}(-\hat{z})\hat{M}(+\hat{n})\hat{M}(+\hat{z})$$

The fraction of the particles transmitted through the first SGz device that will survive the third measurement is given by

$$f = \left|\langle-\hat{z}|\hat{M}(-\hat{z})\hat{M}(+\hat{n})|+\hat{z}\rangle\right|^2 = \cos^2\frac{\theta}{2}\sin^2\frac{\theta}{2} = \frac{1}{4}\sin^2\theta$$

(*ii*)

This is maximized by choosing $\theta = \pi/2$ so that $f_{max} = 1/4$.

(iii) If there is no third device, then the fraction surviving is

$$\bar{f} = \left|\langle +\hat{n}|\,\hat{M}(+\hat{n})\,|+\hat{z}\rangle\right|^2 = \left|\langle +\hat{n}|\,(\cos^2\frac{\theta}{2} + \cos\frac{\theta}{2}\sin\frac{\theta}{2})\,|+\hat{z}\rangle\right|^2$$

$$= \left|\cos^3\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin\frac{\theta}{2}\right|^2 = \cos^4\frac{\theta}{2}\,(1+\sin 2\theta)$$

3.(i) To determine whether hydrogen atoms are bosons or fermions it suffices to take two hydrogen atoms and swap them. If the wavefunction changes sign under this exchange operation, the atoms are fermions; if not, they are bosons. Now, since each Hydrogen atom is a bound state of an electron and a proton, the wavefunction for two hydrogen atoms takes the form,

$$\psi(H_1, H_2) \equiv \psi(\underbrace{e_1, p_1}_{\text{1st atom}}\,;\,\underbrace{e_2, p_2}_{\text{2nd atom}}), \tag{4}$$

where e_1 represents the coordinates and the spin of the electron belonging to the first atom, p_2 those of the second proton, and so forth. We want to know what happens to the wavefunction as we exchange H_1 and H_2,

$$\psi(H_1, H_2) = \pm_?\,\psi(H_2, H_1) \tag{5}$$

To exchange the two atoms, we can simply exchange the constituent electrons and protons. But the electrons are ferions, and the protons are fermions, so exchanging them in pairs we find,

$$\psi(e_1, p_1; e_2, p_2) \overset{e_1 \rightleftarrows e_2}{=} (-1)\,\psi(e_2, p_1; e_1, p_2) \overset{p_1 \rightleftarrows p_2}{=} (-1)^2\psi(e_2, p_2; e_1, p_1) \tag{6}$$

so

$$\psi(H_1, H_2) = +\,\psi(H_2, H_1) \tag{7}$$

i.e. Hydrogen atoms are bosons.

(ii) The same logic applies to bound states of N fermions: for each pair of constituent fermions exchanged, the wavefunctions acquires a minus sign; when we are done with exchanging all N constituent fermions between the two bound states, the wavefunction will have acquired a factor of $(-1)^N$. Thus bound states of N fermions are bosons if N is even (the wavefunction is even under the exchange of two such bound states) and fermions if N is odd (the wavefunction changes sign under exchange)

Note: In quantum field theory there is a theorem (the spin-statistics theorem) which states that all half-integer spin particles are fermions and all integer spin particles are bosons. This theorem can be used to construct an alternate proof. The total spin of the composite system of a proton and an electron is the vectorial sum of the individual spins:

$$\vec{S}_H = \vec{S}_p + \vec{S}_e, \tag{8}$$

from which follows that a measurement of the total spin along an arbitrary direction $\hat{n}$:

$$\hat{n} \cdot \vec{S}_H = \hat{n} \cdot \vec{S}_p + \hat{n} \cdot \vec{S}_e \tag{9}$$

can have only the results:

$$\hat{n} \cdot \vec{S}_H : \left\{ \frac{1}{2} + \frac{1}{2}, \frac{1}{2} - \frac{1}{2}, -\frac{1}{2} + \frac{1}{2}, -\frac{1}{2} - \frac{1}{2} \right\} = \{1, 0, 0, -1\}. \tag{10}$$

We see that all possible eigenvalues are integers, so the hydrogen atom is a boson.

Generally for an arbitrary number N of fermions the possible eigenvalues for the total spin have the form:

$$\sum_{i=1}^{N} \frac{2m_i + 1}{2} = \underbrace{\sum_{i=1}^{N} m_i}_{integer} + \frac{1}{2} N = \begin{cases} half-integer & N \ odd \\ integer & N \ even \end{cases}, \tag{11}$$

where m_i's are integers.

4.(*i*)

$$P^2\Psi(\mathbf{r}_1,\mathbf{r}_2) = PP\Psi(\mathbf{r}_1,\mathbf{r}_2) = P\Psi(\mathbf{r}_2,\mathbf{r}_1) = \Psi(\mathbf{r}_1,\mathbf{r}_2),$$

thus $P^2 = 1$. The eigenvalues σ must obey $P^2\Psi = \sigma^2\Psi = \Psi$, so we get $\sigma^2 = 1$ or $\sigma = \pm 1$. The eigenvalue $\sigma = 1$ corresponds to a bosonic wave function, and $\sigma = -1$ to a fermionic wave function.

(*ii*) One thing we need to be clear about is that $\nabla_1^2\Psi(\circ,\circ)$ *always* acts on the first slot of the wave function, and ∇_2^2 always acts on the second slot of the wave function. Let's evaluate

$$\begin{aligned}
PH\Psi(\mathbf{r}_1,\mathbf{r}_2) &= P\left[-\frac{\hbar^2}{2m}\left(\nabla_1^2\Psi(\mathbf{r}_1,\mathbf{r}_2) + \nabla_2^2\Psi(\mathbf{r}_1,\mathbf{r}_2)\right) + V(\mathbf{r}_1,\mathbf{r}_2)\Psi(\mathbf{r}_1,\mathbf{r}_2)\right] \\
&= -\frac{\hbar^2}{2m}\left(\nabla_2^2\Psi(\mathbf{r}_2,\mathbf{r}_1) + \nabla_1^2\Psi(\mathbf{r}_2,\mathbf{r}_1)\right) + V(\mathbf{r}_2,\mathbf{r}_1)\Psi(\mathbf{r}_2,\mathbf{r}_1) \\
HP\Psi(\mathbf{r}_1,\mathbf{r}_2) = H\Psi(\mathbf{r}_2,\mathbf{r}_1) &= \left[-\frac{\hbar^2}{2m}\left(\nabla_1^2\Psi(\mathbf{r}_2,\mathbf{r}_1) + \nabla_2^2\Psi(\mathbf{r}_2,\mathbf{r}_1)\right) + V(\mathbf{r}_1,\mathbf{r}_2)\Psi(\mathbf{r}_2,\mathbf{r}_1)\right].
\end{aligned}$$

These equations are equivalent when $V(\mathbf{r}_1,\mathbf{r}_2) = V(\mathbf{r}_2,\mathbf{r}_1)$.

(*iii*) Here is a simple way to do this:

$$\begin{aligned}
P\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathrm{d}t) &= P\left(1+\frac{H}{\mathrm{i}\hbar}\mathrm{d}t\right)\Psi(\mathbf{r}_1,\mathbf{r}_2,0) = \left(1+\frac{H}{\mathrm{i}\hbar}\mathrm{d}t\right)P\Psi(\mathbf{r}_1,\mathbf{r}_2,0) \\
&= \sigma\left(1+\frac{H}{\mathrm{i}\hbar}\mathrm{d}t\right)\Psi(\mathbf{r}_1,\mathbf{r}_2,0) = \sigma\Psi(\mathbf{r}_1,\mathbf{r}_2,\mathrm{d}t)
\end{aligned}$$

In an infinitesimal time step, thus the wave function stays symmetric/antisymmetric. Thus add up many of these small time steps, and we conclude that $P\Psi(\mathbf{r}_1,\mathbf{r}_2,t) = \sigma\Psi(\mathbf{r}_1,\mathbf{r}_2,t)$.

A bosonic/fermionic wave function will stay bosonic/fermionic for all times, and thus it makes sense to demand symmetry/antisymmetry of the wave function.

(iv) No, because from part (c) if $[P, H] \neq 0$, then the wave function will not stay symmetric or antisymmetric. Alternatively, we can distinguish the particles by seeing "which potential" they feel, roughly speaking.