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Problems set # 1 **Physics 400** January 31 and February 7, 2019

1. Let V be a vector space over F and let T be a linear transformation of the vector space V to itself. A nonzero element $\boldsymbol{x} \in V$ satisfying $T(\boldsymbol{x}) = \lambda \boldsymbol{x}$ for some $\lambda \in F$ is called an eigenvector of T, with eigenvalue λ. Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of T with eigenvalue λ together with $\mathbf{0}$ forms a subspace of V, that is, a subset of the vector space V that is closed under addition and scalar multiplication.

2. *(i)* Show that $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions.
(ii) Find all unit vectors lying in $\text{span}\{(3,4)\}$.

3. Consider the following matrices:

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}$$

$$\mathbb{B} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\mathbb{C} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix}$$

find the following:
(i) $\det(\mathbb{AB}) = |\mathbb{AB}|$, *(ii)* $\mathbb{AC}$, *(iii)* $\mathbb{ABC}$, and *(iv)* $\mathbb{AB} - \mathbb{B}^T\mathbb{A}^T$.

4. A matrix is orthogonal if its transpose is equal to its inverse: $\mathbb{A}^T = \mathbb{A}^{-1}$. Show that the product of 2 orthogonal matrices is also an orthogonal matrix.

5. A matrix $\mathbb{A} \in \mathbb{C}^{n\times n}$ is nilpotent if $\mathbb{A}^k = 0$ for some integer $k > 0$. Prove that the only eigenvalue of a nilpotent matrix is zero.

6. *(i)* Determine whether the function $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that $T(x, y) = (x^2, y)$ is linear?
(ii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(1,0,0) = (2,4,-1), \quad T(0,1,0) = (1,3,-2), \quad T(0,0,1) = (0,-2,2);$$

compute $T(-2,4,-1)$.
(iii) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T(x_1,x_2,x_3) = (2x_1 + x_2, 2x_2 - 3x_1, x_1 - x_3), \quad \boldsymbol{x} = (x_1,x_2,x_3) \in \mathbb{R}^3;$$

compute $T(-4,-5,1)$.
(iv) Let $T:\mathbb{R}^5\to\mathbb{R}^2$ be a linear transformation $T(\boldsymbol{x})=\mathbb{A}\boldsymbol{x}$, with

$$\mathbb{A}=\begin{pmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{pmatrix};$$

compute $T(1,0,-1,3,0)$.
(v) Let $T(x,y,z)=(3x-2y+z,2x-3y,y-4z)$. Write down the matrix representation of T in the standard basis and use it to find $T(2,-1,-1)$.
(vi) Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be given by $T(a_1,a_2,a_3)=(3a_1-2a_3,a_2,3a_1+4a_2)$. Prove that T is an isomorphism and find T^{-1}.

7. *(i)* Show that if $T:\mathbb{R}^2\to\mathbb{R}^2$ is the counterclockwise rotation by a fixed angle θ, then $T(x,y)=\mathbb{A}\boldsymbol{x}=\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$.
(ii) Let T be the counterclockwise rotation in $\mathbb{R}^2$ by an angle $120°$, write down the matrix of T and compute $T(2,2)$.
(iii) Prove that if θ is not an integer multiple of π there does not exist a real valued matrix $\mathbb{B}$ such that $\mathbb{B}^{-1}\mathbb{A}\mathbb{B}$ is a diagonal matrix.

8. Let $\boldsymbol{x}\in\mathbb{R}^n$ be a vector. Then, for $\boldsymbol{y}\in\mathbb{R}^n$, define $\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})=\frac{\boldsymbol{x}\cdot\boldsymbol{y}}{\|\boldsymbol{x}\|^2}\,\boldsymbol{x}$. The point of such projections is that any vector $\boldsymbol{y}\in\mathbb{R}^n$ can be written uniquely as a sum of a vector along $\boldsymbol{x}$ and another one perpendicular to $\boldsymbol{x}$: $\boldsymbol{y}=\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})+[y-\mathrm{proj}_{\boldsymbol{x}}[\boldsymbol{y})]$. It is easy to check that $[y-\mathrm{proj}_{\boldsymbol{x}}[\boldsymbol{y})]\perp\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})$.
(i) Show that $\mathrm{proj}_{\boldsymbol{x}}:\mathbb{R}^n\to\mathbb{R}^n$ is a linear transformation.
(ii) Let T be the projection on to the vector $\boldsymbol{x}=(1,-5)\in\mathbb{R}^2:T(\boldsymbol{y})=\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})$; find the matrix representation in the standard basis and compute $T(2,3)$.

9. *(i)* Show that the eigenvalues of a symmetric linear operator A are real.
(ii) Prove that the eigenvectors of a symmetric linear operator A associated to different eigenvalues are mutually orthogonal.

10. *(i)* Show that Hermitian matrices satisfy the following properties $(\mathbb{A}\mathbb{B})^\dagger=\mathbb{B}^\dagger\mathbb{A}^\dagger$.
(ii) Prove that the inverse of a Hermitian matrix is again a Hermitian matrix.

11. Find the eigenvalues and normalized eigenvectors of the Pauli matrices:

$$\sigma_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\qquad \sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

12. Show that the Pauli matrices obey the following commutation and anticommutation relations: $[\sigma_i,\sigma_j]=2i\sum_{k=1}^3\epsilon_{ijk}\sigma_k$ and $\{\sigma_i,\sigma_j\}=2\delta_{ij}\mathbb{1}_2$.

13. Show that $\{\mathbb{1},\sigma_1,\sigma_2,\sigma_3\}$ is an appropriate basis to describe the space of operators on a two-Hilbert space. *(i)* Show that $\{\mathbb{1},\sigma_1,\sigma_2,\sigma_3\}$ are linearly independent. *(ii)* Prove that $\{\mathbb{1},\sigma_1,\sigma_2,\sigma_3\}$

form a basis in 2×2 matrix space, by showing that any arbitrary matrix

$$\mathbb{M} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

can be written on the form $\mathbb{M} = a_0 \mathbb{1} + \vec{a} \,.\, \vec{\sigma}$, where $a_0 = \frac{1}{2}\mathrm{Tr}\,(M)$, $\vec{a} = \frac{1}{2}\mathrm{Tr}\,(\mathbb{M}\vec{\sigma})$, and $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is the Pauli vector.

14. Evaluate *(i)* $\int_{-\infty}^{\infty} [f(x)\delta(x-1) + f(x)\delta(x+2)]\, dx$; *(ii)* $\int_{-\infty}^{\infty} f(x)\, \delta'(x) dx$ (to do this integral use integration by parts); *(iii)* $\int_{-\infty}^{\infty} [f(x)\delta(x-a) - f(x)\delta''(x)] dx$; *(iv)* $\int_{-\infty}^{\infty} \Theta(x)\, \Theta(1-x)\, f(x)\, dx$; *(v)* $\int_{-\infty}^{\infty} \Theta(x)\, \Theta(b-x)\, x\, f(x)\, dx$; *(vi)* $\int_{-\infty}^{\infty} [f(x)\, \delta(x-\pi) - f(x)\delta'(x-2\pi) + f(x)\delta''(x-b)] dx$.

15. Show that: *(i)* $\frac{d}{dx}|x| = \mathrm{sgn}\ x = \Theta(x) - \Theta(-x)$, where $|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$; *(ii)* $\frac{d^2}{dx^2}|x| = \frac{d}{dx}\mathrm{sgn}\ x = 2\delta(x)$.

SOLUTIONS

1. Let $\lambda \in F$, and let V_λ denote the set of eigenvectors for λ, together with $\mathbf{0}$. You have to show that V_λ is a subspace of V. By construction, $\mathbf{0} \in V_\lambda$. Suppose $\boldsymbol{x}, \boldsymbol{y} \in V_\lambda$, then $T(\boldsymbol{x}) = \lambda \boldsymbol{x}$ and $T(\boldsymbol{y}) = \lambda \boldsymbol{y}$. Hence $T(\boldsymbol{x}+\boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}) = \lambda\boldsymbol{x} + \lambda\boldsymbol{y} = \lambda(\boldsymbol{x}+\boldsymbol{y})$, so $\boldsymbol{x}+\boldsymbol{y} \in V_\lambda$. Similarly, if $c \in F$,then $T(c\boldsymbol{x}) = cT(\boldsymbol{x}) = c\lambda\boldsymbol{x} = \lambda(c\boldsymbol{x})$, so $c\boldsymbol{x} \in V_\lambda$. Therefore V_λ is a subspace.

2 *(i)* In order to prove this collection is linearly independent, you need to show that if $c_1 t + c_2 \sin t + c_3 \cos 2t + c_4 \sin t \cos t = 0$ for all t, then $c1 = c2 = c3 = c4 = 0$. Firstly, plug in $t = 0$, and find that $0 + 0 + c_3 + 0 = 0$, so that $c_3 = 0$. Plugging this into the original equation, you now have $c_1 t + c_2 \sin t + c_4 \sin t \cos t = 0$, for all t. Secondly, plug in $t = \pi$, and find that $c_1\pi + 0 + 0 = 0$, so that $c_1 = 0$. Plugging this into the original equation, you now have $c_2 \sin t + c_4 \sin t \cos t = 0$ for all t. Thirdly, plug in $t = \pi/2$, and find that $c_2 + 0 = 0$, so that $c_2 = 0$. Plugging this into our original equation, you now have: $c_4 \sin t \cos t = 0$ for all t. Finally, plug in $t = \pi/4$, to find $c_4(\sqrt{2}/2)(\sqrt{2}/2) = 0$, or $c_4 = 0$. So, overall, you have proven that $c_1 = c_2 = c_3 = c_4 = 0$, and thus that the given collection of functions is linearly independent. *(ii)* Every element of span $\{(3,4)\}$ has the form $(3t, 4t)$, where $t \in \mathbb{R}$. An element of span $\{(3,4)\}$ would then be a unit vector if and only if $\|(3t, 4t)\| = 1$; in other words, if $(3t)^2 + (4t)^2 = 1$. This equation has two solutions, $t = \pm 1/5$. Therefore, span $\{(3,4)\}$ has two unit vectors, $(3/5, 4/5)$ and $(-3/5, -4/5)$.

3. *(i)*

$$\mathbb{AB} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \\ 1 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 1 \\ 1 & -2 & 9 \\ 5 & 3 & 3 \end{pmatrix}$$

Expand by the first row

$$|\mathbb{AB}| = 1 \begin{vmatrix} -2 & 9 \\ 3 & 3 \end{vmatrix} + 2 \begin{vmatrix} -1 & 9 \\ 5 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -2 \\ 5 & 3 \end{vmatrix} = -104$$

(ii)

$$\mathbb{AC} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 4 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 13 & 9 \\ 5 & 2 \end{pmatrix}$$

(iii)

$$\mathbb{ABC} = \mathbb{A}(\mathbb{BC}) = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 8 & 5 \\ -2 & -3 \\ 9 & 4 \end{pmatrix} = \begin{pmatrix} -5 & -5 \\ 3 & -5 \\ 25 & 14 \end{pmatrix}$$

(iv)

$$\mathbb{B}^T\mathbb{A}^T = \begin{pmatrix} 2 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 2 & 3 & 0 \\ -1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 5 \\ -2 & -2 & 3 \\ 1 & 9 & 3 \end{pmatrix}$$

$$\mathbb{AB} - \mathbb{B}^T\mathbb{A}^T = \begin{pmatrix} 0 & -3 & -4 \\ 3 & 0 & 6 \\ 4 & -6 & 0 \end{pmatrix}$$

4. The product of two orthogonal matrices $\mathbb{A}\mathbb{B}$ will be an orthogonal matrix $\mathbb{C} \Leftrightarrow \mathbb{C}^T = \mathbb{C}^{-1}$. Now, $C_{ij} = \sum_k A_{ik}B_{kj}$ and $(C^T)_{ij} = C_{ji} = \sum_k A_{jk}B_{ki} = \sum_k B_{ki}A_{jk}$. Identifying $B_{ki} = (B^T)_{ik}$ and $A_{jk} = (A^T)_{kj}$ we have $(C^T)_{ij} = \sum_k (B^T)_{ik}(A^T)_{kj}$, or equivalently $\mathbb{C}^T = (\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T$. Then $(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T$, but because $\mathbb{A}$ and $\mathbb{B}$ are orthogonals $\mathbb{A}^T = \mathbb{A}^{-1}$ and $\mathbb{B}^T = \mathbb{B}^{-1}$. Multiplying the expression $(\mathbb{A}\mathbb{B})^T = \mathbb{B}^T\mathbb{A}^T$ by $\mathbb{A}\mathbb{B}$ from the right we have $(\mathbb{A}\mathbb{B})^T\mathbb{A}\mathbb{B} = \mathbb{B}^T\mathbb{A}^T\mathbb{A}\mathbb{B} = \mathbb{B}^T\mathbb{B} = \mathbb{1}$. Then $\mathbb{1} = (\mathbb{A}\mathbb{B})^{-1}\mathbb{A}\mathbb{B}$ therefore $(\mathbb{A}\mathbb{B})^T = (\mathbb{A}\mathbb{B})^{-1}$ and the matrix $\mathbb{C}$ is orthogonal.

5. Note that a matrix $\mathbb{A} \in \mathbb{C}^{n\times n}$ is nilpotent of degree k, if k is a positive integer such that $\mathbb{A}^p = 0_{n\times n}$ for $p \geq k$, and $\mathbb{A}^p \neq 0_{n\times n}$ for $0 < p < k$. Suppose $\lambda \neq 0$ is an eigenvalue corresponding to the eigenvector $\boldsymbol{x} \neq \mathbf{0}$. It follows that $\mathbb{A}\boldsymbol{x} = \lambda\boldsymbol{x}$ and $\mathbb{A}^k\boldsymbol{x} = \lambda^k\boldsymbol{x}$. However, by the nilpotent assumption $\mathbb{A}^k = 0_{n\times n}$ and therefore $\mathbb{A}^k\boldsymbol{x} = 0_{n\times n}\boldsymbol{x} = \mathbf{0} = \lambda^k\boldsymbol{x}$. Since $\boldsymbol{x} \neq \mathbf{0}$, it follows that $\lambda = 0$, which is a contradiction. Therefore all λ must be zero.

6. *(i)* Note that $T((x,y)+(z,w)) = T(x+z, y+w) = ((x+z)^2, y+w) \neq (x^2,y)+(z^2,w) = T(x,y)+T(z,w)$. So, T does not preserve additivity. So, T is not linear. *(ii)* Note that $(-2,4,-1) = -2(1,0,0)+4(0,1,0)-(0,0,1)$, so $T(-2,4,-1) = -2T(1,0,0)+4T(0,1,0)-T(0,0,1) = (-4,-8,2)+(4,12,-8)+(0,-2,2) = (0,6,-8)$. *(iii)* $T(-4,5,1) = (2\times(-4)-5, 2\times(-5)-3\times(-4), -4-1) = (-13,2,-5)$. *(iv)* $T(1,0,-1,3,0) = \begin{pmatrix} -1 & 2 & 1 & 3 & 4 \\ 0 & 0 & 2 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \end{pmatrix}$. *(v)* With $\boldsymbol{e_1} = (1,0,0)^T$, $\boldsymbol{e_2} = (0,1,0)^T$, $\boldsymbol{e_3} = (0,0,1)^T$ it follows that $T(\boldsymbol{e_1}) = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$, $T(\boldsymbol{e_2}) = \begin{pmatrix} -2 \\ -3 \\ 1 \end{pmatrix}$, $T(\boldsymbol{e_3}) = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$, so the matrix representation in the standard basis is $\begin{pmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{pmatrix}$, and $T(2,-1,-1) = \begin{pmatrix} 3 & -2 & 1 \\ 2 & -3 & 0 \\ 0 & 1 & -4 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ 3 \end{pmatrix}$. *(vi)* Relative to the standard basis, the matrix of T is $\begin{pmatrix} 3 & 0 & -2 \\ 0 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$. It is sufficient to prove that this matrix is invertible. Its determinant is, using the column expansion for the last column, $-2\times(0\times 4 - 1\times 3) = 6 \neq 0$. Therefore, the matrix is invertible because its column vectors are linearly independent. The inverse matrix is $\begin{pmatrix} 0 & -\frac{4}{3} & \frac{1}{3} \\ 0 & 1 & 0 \\ -\frac{1}{2} & -2 & \frac{1}{2} \end{pmatrix}$, so that T^{-1} is given by: $T^{-1}(a_1,a_2,a_3) = (-\frac{4}{3}a_2 - \frac{1}{3}a_3, a_2, -\frac{1}{2}a_1 - 2a_2 + \frac{1}{2}a_3)$.

7. *(i)* Write $x = r\cos\phi$ and $y = r\sin\phi$, where $r = \sqrt{x^2+y^2}$ and $\tan\phi = y/x$. By definition $T(x,y) = (r\cos(\phi+\theta),\ r\sin(\phi+\theta))$. Using trigonometric formulas $r\cos(\phi+\theta) = r\cos\phi\cos\theta -$

$r\sin\phi\,\sin\theta = x\cos\theta - y\sin\theta$ and $r\sin(\phi+\theta) = r\sin\phi\,\cos\theta + r\cos\phi\,\sin\theta = y\cos\theta + x\sin\theta$. Thus, $T(x,y) = \mathbb{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$. *(ii)* The matrix representation in the standard basis $\mathbb{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$, and therefore $T(2,2) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -1-\sqrt{3} \\ -1+\sqrt{3} \end{pmatrix}$. *(iii)* $\det(\mathbb{A} - \lambda\mathbb{1}) = \begin{pmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{pmatrix} = \lambda^2 - 2\lambda\cos\theta + 1$, then $\lambda_{1,2} = \cos\theta \pm \sqrt{\cos^2\theta - 1} \Rightarrow \lambda_{1,2} = a \pm ib,\ b \neq 0$, *i.e.* $\lambda_{1,2} \notin \mathbb{R}$. Hence, the eigenvectors are not real and $\nexists\mathbb{B} \in M(\mathbb{R})$, such that $\mathbb{B}^{-1}A\mathbb{B}$ is diagonal.

8. *(i)* Let $\boldsymbol{w} \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$, then you can easily check by direct substitution that $\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y} + \boldsymbol{w}) = \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y}) + \mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{w})$ and $\mathrm{proj}_{\boldsymbol{x}}(\mu\,\boldsymbol{y}) = \mu\,[\mathrm{proj}_{\boldsymbol{x}}(\boldsymbol{y})]$. *(ii)* $T(y_1,y_2) = \mathrm{proj}_{\boldsymbol{x}}(y_1,y_2) = \frac{\boldsymbol{x}\,.\,(y_1,y_2)}{\|\boldsymbol{x}\|^2}\,\boldsymbol{x} = \frac{(1,-5)\,.\,(y_1,y_2)}{\|(1,-5)\|^2}\,(1,-5) = \frac{y_1-5y_2}{26}\,(1,-5) = \left(\frac{y_1-5y_2}{26}, \frac{-5y_1+25y_2}{26}\right)$. Thus, with $\boldsymbol{e_1} = (1,0)^T$, $\boldsymbol{e_2} = (0,1)^T$ you obtain $T(\boldsymbol{e_1}) = \begin{pmatrix} \frac{1}{26} \\ -\frac{5}{26} \end{pmatrix}$, $T(\boldsymbol{e_2}) = \begin{pmatrix} -\frac{5}{26} \\ \frac{25}{26} \end{pmatrix}$, so the "standard matrix" is $\mathbb{A} = \begin{pmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{pmatrix}$, and therefore $T(2,3) = \begin{pmatrix} \frac{1}{26} & -\frac{5}{26} \\ -\frac{5}{26} & \frac{25}{26} \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{65}{26} \end{pmatrix}$.

9. *(i)* Assume $A\boldsymbol{x} = \lambda\boldsymbol{x}$; then it follows that

$$\lambda\|\boldsymbol{x}\|^2 = \langle x, Ax\rangle = \langle Ax, x\rangle = \lambda^*\|\boldsymbol{x}\|^2 \Rightarrow \lambda^* = \lambda\,.$$

(ii) Assume $A\boldsymbol{x} = \lambda\boldsymbol{x}$ and $A\boldsymbol{y} = \mu\boldsymbol{y}$, with $\lambda \neq \mu$. It follows that,

$$(\lambda - \mu)\langle \boldsymbol{y}, \boldsymbol{x}\rangle = \langle \boldsymbol{y}, A\boldsymbol{x}\rangle - \langle A\boldsymbol{y}, \boldsymbol{x}\rangle = 0 \Rightarrow \langle \boldsymbol{y}, \boldsymbol{x}\rangle = 0\,.$$

Therefore, $\boldsymbol{x} \perp \boldsymbol{y}$ if $\lambda \neq \mu$.

10. *(i)* Derive this using matrix multiplication $(AB)_{ij} = \sum_{k=1}^n A_{ik}B_{kj}$, where $(AB)_{ij}$ denotes the $(i,j)^{\rm th}$ entry of $(\mathbb{AB})$, and likewise for $\mathbb{A}$ and $\mathbb{B}$. Then $(AB)^\dagger_{ji} = (A^*B^*)_{ji}{}^T = (A^*B^*)_{ij} = \sum_{k=1}^n A^*_{ik}B^*_{kj} = \sum_{k=1}^n (A^*_{ki})^T(B^*_{jk})^T = \sum_{k=1}^n (B^*_{jk})^T(A^*_{ki})^T = \sum_{k=1}^n B^\dagger_{jk}A^\dagger_{ki}$. The product on the right is the $(j,i)^{\rm th}$ entry of $\mathbb{B}^\dagger\mathbb{A}^\dagger$, while $(AB)^\dagger_{ji}$ is the $(j,i)^{\rm th}$ entry of $(\mathbb{AB})^\dagger$. Therefore, $(\mathbb{AB})^\dagger = \mathbb{B}^\dagger\mathbb{A}^\dagger$. *(ii)* If $\mathbb{A}$ is Hermitian, then $\mathbb{A} = \mathbb{UDU}^\dagger$, where $\mathbb{U}$ is unitary and $\mathbb{D}$ is a real diagonal matrix. Therefore, $\mathbb{A}^{-1} = (\mathbb{UDU}^\dagger)^{-1} = (\mathbb{U}^\dagger)^{-1}\mathbb{D}^{-1}\mathbb{U}^{-1} = \mathbb{UD}^{-1}\mathbb{U}^\dagger$, because $\mathbb{U}^{-1} = \mathbb{U}^\dagger$. Note that $\mathbb{D}^{-1}$ is just the diagonal matrix with entries λ_i^{-1} (where the λ_i are the entries in $\mathbb{D}$). Hence, $(\mathbb{A}^{-1})^\dagger = (\mathbb{UD}^{-1}\mathbb{U}^\dagger)^\dagger = \mathbb{U}(\mathbb{D}^{-1})^\dagger\mathbb{U}^\dagger = \mathbb{UD}^{-1}\mathbb{U}^\dagger = \mathbb{A}^{-1}$, because $\mathbb{D}^{-1}$ is a real matrix, so that $\mathbb{A}^{-1}$ is Hermitian.

11. For σ_1 the eigenvalue relation

$$\det\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0$$

leads to $\lambda^2 = 1$, which implies $\lambda = \pm 1$. For $\lambda = 1$, we have

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -x_1 + x_2 = 0 \\ x_1 - x_2 = 0 \end{cases},$$

yielding $x_1 = x_2 = 1/\sqrt{2}$. For $\lambda = -1$, we have

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + x_2 = 0 \\ x_1 + x_2 = 0 \end{cases},$$

yielding $x_1 = -x_2 = 1/\sqrt{2}$.

For σ_2 the eigenvalue relation

$$\det \begin{pmatrix} -\lambda & -i \\ i & -\lambda \end{pmatrix} = 0$$

leads to $\lambda^2 = 1$, which implies $\lambda = \pm 1$. For $\lambda = 1$, we have

$$\begin{pmatrix} -1 & -i \\ i & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 + ix_2 = 0 \\ ix_1 - x_2 = 0 \end{cases},$$

yielding $x_1 = i/\sqrt{2}$ and $x_2 = -1/\sqrt{2}$. For $\lambda = -1$, we have

$$\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x_1 - ix_2 = 0 \\ ix_1 + x_2 = 0 \end{cases},$$

yielding $x_1 = i/\sqrt{2}$ and $x_2 = 1/\sqrt{2}$.

For σ_3, the eigenvalue relation

$$\det \begin{pmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix} = 0$$

leads to $-(1-\lambda)(1+\lambda) = 0$, which implies $\lambda = \pm 1$. For $\lambda = 1$, we have

$$\begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 0 = 0 \\ -2x_2 = 0 \end{cases},$$

yielding $x_1 = 1$ and $x_2 = 0$. For $\lambda = -1$, we have

$$\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} 2x_1 = 0 \\ 0 = 0 \end{cases},$$

yielding $x_1 = 0$ and $x_2 = 1$.

All in all, each Pauli matrix has eigenvalues 1 and -1. The normalized eigenvectors are

$$\sigma_1 \quad \Rightarrow \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix};$$

$$\sigma_2 \quad \Rightarrow \quad \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix};$$

$$\sigma_3 \quad \Rightarrow \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

12. It is straightforward to check that the multiplication of two different Pauli matrices yields the third one multiplied by the (positive or negative) imaginary unit, i.e., $\sigma_1\sigma_2 = i\sigma_3$, $\sigma_1\sigma_3 = -i\sigma_2$, $\sigma_2\sigma_3 = i\sigma_1$, $\sigma_2\sigma_1 = -i\sigma_3$, $\sigma_3\sigma_1 = i\sigma_2$, $\sigma_3\sigma_2 = -i\sigma_1$. This may be expressed in more compact form for all cyclic permutations of $i, j, k \in \{1, 2, 3\}$ as

$$\sigma_i\sigma_j = \delta_{ij}\mathbb{1}_2 + i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k \,.$$

As a direct consequence of this last relation the commutation and anticommutation relations for Pauli spin matrices are given by

$$[\sigma_i, \sigma_j] = 2i\sum_{k=1}^{3}\epsilon_{ijk}\sigma_k \quad \text{and} \quad \{\sigma_i, \sigma_j\} = 2\delta_{ij}\mathbb{1}_2 \,.$$

13. Suppose

$$\alpha\mathbb{1} + \beta\sigma_1 + \zeta\sigma_2 + \xi\sigma_3 = \begin{pmatrix} \alpha+\xi & \beta - i\zeta \\ \beta + i\zeta & \alpha - \xi \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} . \tag{1}$$

Then $\alpha = -\xi$ and $\alpha = \xi \Leftrightarrow \alpha = \xi = 0$. Similarly, $\beta = -i\zeta$ and $\beta = i\zeta$, which implies $\beta = \zeta = 0$. *(ii)* Now we show that the vectors $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ span the 2×2 matrix space. Let

$$\begin{aligned}
\mathbb{M} &= \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \\
&= \frac{1}{2}(m_{11} + m_{22})\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}(m_{11} - m_{22})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&+ \frac{1}{2}(m_{12} + m_{21})\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{i}{2}(m_{12} - m_{21})\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\
&= \frac{1}{2}(m_{11} + m_{22})\mathbb{1} + \frac{1}{2}(m_{12} + m_{21})\sigma_1 + \frac{i}{2}(m_{12} - m_{21})\sigma_2 \\
&+ \frac{1}{2}(m_{11} - m_{22})\sigma_3 \,.
\end{aligned} \tag{2}$$

Note that

$$\frac{1}{2}\text{Tr }[\mathbb{M}] = \frac{1}{2}(m_{11} + m_{22}) \tag{3}$$

and so the first term in (2) can be written as $\frac{1}{2}\text{Tr }[\mathbb{M}]\,\mathbb{1}$. Now,

$$\begin{aligned}
\frac{1}{2}\text{Tr }[\mathbb{M}\sigma_1] &= \frac{1}{2}\text{Tr}\begin{pmatrix} m_{12} & m_{11} \\ m_{22} & m_{21} \end{pmatrix} = \frac{1}{2}(m_{12} + m_{21}) \\
\frac{1}{2}\text{Tr }[\mathbb{M}\sigma_2] &= \frac{1}{2}\text{Tr}\begin{pmatrix} i\,m_{12} & -i\,m_{11} \\ i\,m_{22} & -i\,m_{21} \end{pmatrix} = \frac{1}{2}(m_{12} - m_{21}) \\
\frac{1}{2}\text{Tr }[\mathbb{M}\sigma_3] &= \frac{1}{2}\text{Tr}\begin{pmatrix} m_{11} & -m_{12} \\ m_{21} & -m_{22} \end{pmatrix} = \frac{1}{2}(m_{11} - m_{22}) \,.
\end{aligned}$$

We define, $\mathbb{M}\boldsymbol{\sigma} = (\mathbb{M}\sigma_1, \mathbb{M}\sigma_2, \mathbb{M}\sigma_3)$ so that the last three terms in (2) can be written as $\frac{1}{2}\mathrm{Tr}\,[\mathbb{M}\boldsymbol{\sigma}]\cdot\boldsymbol{\sigma}$. Therefore, any 2×2 matrix can be written as $\mathbb{M} = a_0\mathbb{1}+\boldsymbol{a}\cdot\boldsymbol{\sigma}$, where $a_0 = \frac{1}{2}\mathrm{Tr}\,[\mathbb{M}]$ and $\boldsymbol{a} = \frac{1}{2}\,\mathrm{Tr}\,[\mathbb{M}\boldsymbol{\sigma}]$.

14. *(i)* $\int_{-\infty}^{\infty}[f(x)\delta(x-1) + f(x)\delta(x+2)]\,dx = f(1) + f(-2)$; *(ii)* $\int_{-\infty}^{\infty} f(x)\,\delta'(x)dx = -f'(0)$; *(iii)* $\int_{-\infty}^{\infty}[f(x)\delta(x-a) - f(x)\delta''(x)]dx = f(a) - f''(0)$; *(iv)* $\int_{-\infty}^{\infty}\Theta(x)\,\Theta(1-x)\,f(x)\,dx = \int_0^1 f(x)\,dx$; *(v)* $\int_{-\infty}^{\infty}\Theta(x)\,\Theta(b-x)\,x\,f(x)\,dx = \int_0^b x\,f(x)\,dx$; *(vi)* $\int_{-\infty}^{\infty}[f(x)\,\delta(x-\pi) - f(x)\delta'(x-2\pi) + f(x)\delta''(x-b)]dx = f(\pi) + f'(2\pi) + f''(b)$.

15. The signum function is the derivative of the absolute value function (up to the indeterminacy at zero), $|x| = \begin{cases} x & \text{if } x>0 \\ -x & \text{if } x<0 \end{cases} \Rightarrow |x|' = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases} = \mathrm{sgn}(x)$; on the other hand, $\Theta(x) = \begin{cases} 0 & \text{if } x<0 \\ 1 & \text{if } x>0 \end{cases}$, hence $\Theta(x) - \Theta(-x) = \begin{cases} 1 & \text{if } x>0 \\ -1 & \text{if } x<0 \end{cases} = \mathrm{sgn}(x) = |x|'$. *(ii)* The signum function is differentiable with derivative zero everywhere except at zero. It is not differentiable at zero in the ordinary sense, but under the generalised notion of differentiation in distribution theory you may write $[\mathrm{sgn(x)}]' = [\Theta(\mathrm{x}) - \Theta(-\mathrm{x})]' = 2\delta(\mathrm{x})$. Then $|x|'' = [\mathrm{sgn}(x)]' = 2\delta(x)$.