Physics 307

Mathematical Physics

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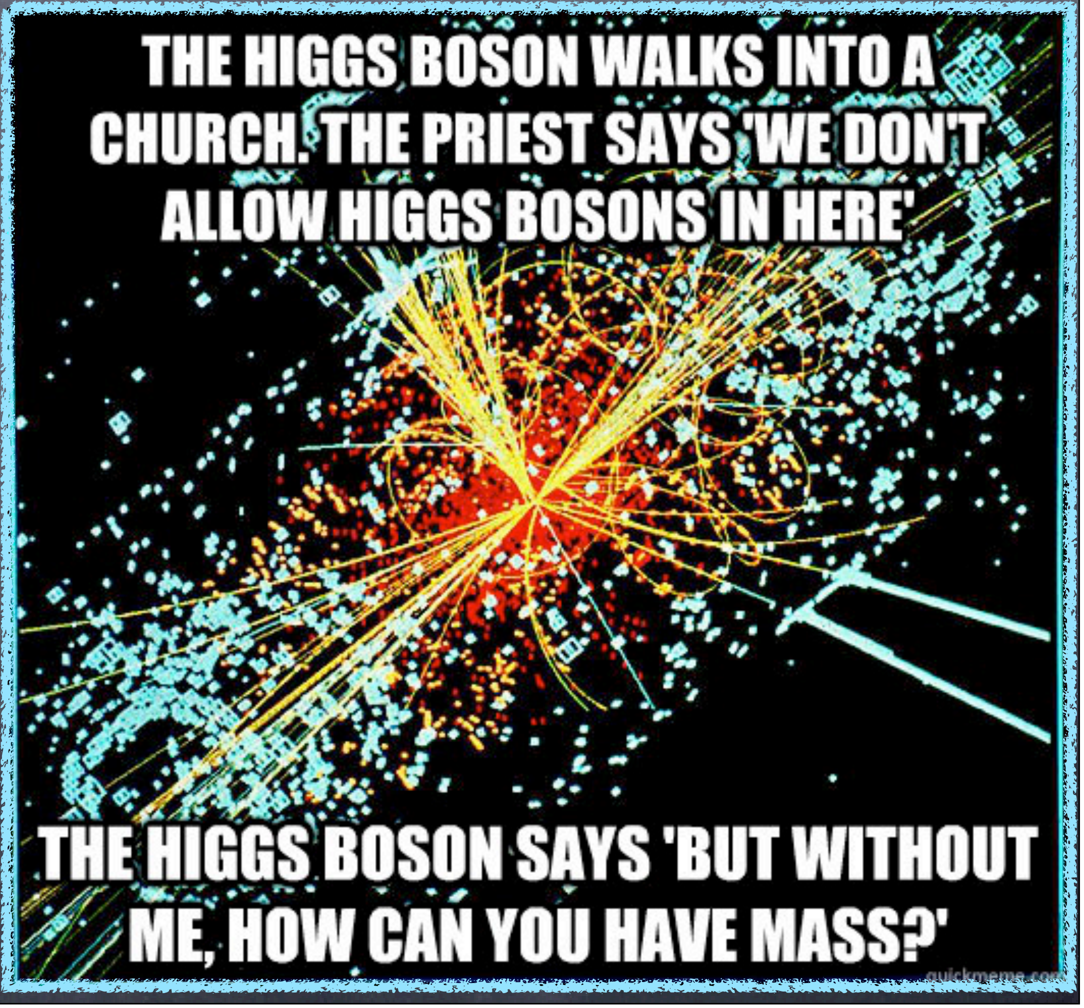
Partial Differential Equations III

4.1 Taxonomy ✓

4.2 Wave Equation ✓

4.3 Diffusion Equation ✓

4.4 Laplace Equation



4.4. Laplace Equation

Today we will discuss canonical form of elliptic equations

Up to lower order terms we found that canonical form is

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

This equation is called Laplace equation

and besides theory of partial differential equations

it is also extremely important in study of complex analysis

More generally ☛ we will consider Laplacian in $\mathbb{R}^n$

$$\nabla^2 = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \qquad (4.4.168.)$$

and Laplace equation

$$\nabla^2 u = 0 \qquad (4.4.169.)$$

Functions satisfying this condition are called harmonic functions

4.4. 1. Harmonics functions

In $\mathbb{R}^2$ Laplacian in polar coordinates is given by

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \qquad (4.4.170.)$$

and for $n > 2$

$$\nabla^2 = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial}{\partial r}\right) + \frac{\nabla_\Omega^2}{r^2} \qquad (4.4.171.)$$

where ∇_Ω^2 is Laplace operator on unit sphere S^{n-1}

For $n = 3$ we have

$$\nabla_\Omega^2 = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \qquad (4.4.172.)$$

Let us seek a harmonic function

with property that it depends only on radial variable r

i.e. ☛ $f(\mathbf{x}) = \phi(r)$ where $r = |\mathbf{x}| = \sqrt{\sum_{j=1}^{n} x_j^2}$

Lemma 4.4.1.

If $f(\mathbf{x}) = \phi(r)$ where $r = |\mathbf{x}|, \mathbf{x} \in \mathbb{R}^n$ then

$$\nabla^2 f(\mathbf{x}) = \phi''(r) + \frac{(n-1)}{r}\,\phi'(r) \qquad (4.4.173.)$$

Proof.

Since $\partial r/\partial x_j = x_j/r$ we have

$$\begin{aligned}
\nabla^2 f(\mathbf{x}) &= \sum_{j=1}^{n} \partial_{x_j}\left[\frac{x_j}{r}\phi'(r)\right] \\
&= \sum_{j=1}^{n}\left[\frac{x_j^2}{r^2}\phi''(r) + \frac{1}{r}\phi'(r) - \frac{x_j^2}{r^3}\phi'(r)\right] \\
&= \phi''(r) + \frac{n}{r}\phi'(r) - \frac{1}{r}\phi'(r) \\
&= \phi''(r) + \frac{(n-1)}{r}\phi'(r) \qquad (4.4.174.)
\end{aligned}$$

Corollary 4.4.1.

If $f(\mathbf{x}) = \phi(r)$ is a radial function on $\mathbb{R}^n$

then f satisfies $\nabla^2 f = 0$ on $\mathbb{R}^n_0$ if and only if:

(i) $\phi(r) = a + b\, r^{2-n}$ for $n > 2$

(ii) $\phi(r) = a + b \ln r$ for $n = 2$

where a, b are constants

Proof.

From (4.4.174.) we have

$$\frac{\phi''(r)}{\phi'(r)} = \frac{1-n}{r} \qquad (4.4.175.)$$

Integrating once we get

$$\ln\left[(\phi'(r)\right] = (1-n) \ln r + \ln c \qquad (4.4.176.)$$

or

$$\phi'(r) = c\, r^{1-n} \qquad (4.4.177.)$$

where c is a constant

One more integration gives desired answer

Next we seek harmonic functions that are products

of radial functions $R(r)$ and angular functions $\Theta(\theta)$

Then ☛ from (4.4.170.) in case $n = 2$ we have

$$r^2 R''(r)\Theta(\theta) + rR'(r)\Theta(\theta) + R(r)\Theta''(\theta) = 0 \quad (4.4.178.)$$

or separating variables

$$\frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{\ddot{\Theta}(\theta)}{\Theta(\theta)} = k^2 \quad (4.4.179.)$$

or

$$r^2 R''(r) + rR'(r) - k^2 R(r) = 0 \quad \text{and} \quad \Theta''(\theta) = -k^2\Theta(\theta) \quad (4.4.180.)$$

We recognize first equation in (4.4.180.) as an Euler equation

Solution is of form r^λ with λ given by

$$\lambda(\lambda - 1) + \lambda - k^2 = 0 \quad (4.4.181.)$$

that is $\lambda = \pm k$

Recall that if $k = 0$

two l.i. solutions are $r^\lambda = r^0 = 1$ and $r^\lambda \ln r = \ln r$

We obtain

$$R_k(r) = \begin{cases} c_1 + c_2 \ \ln \ r & k = 0 \\ c_1 \ r^k + c_2 \ r^{-k} & k \neq 0 \end{cases} \qquad (4.4.182.)$$

angular dependence is given by

$$\Theta_k(\theta) = \begin{cases} c_1 + c_2 \, \theta & k = 0 \\ c_1 \cos(k\,\theta) + c_2 \sin(k\,\theta) & k \neq 0 \end{cases} \qquad (4.4.183.)$$

where c_1 and c_2 are constants

Note that k can be real, imaginary, or complex

If $k = k_r + ik_i$ then

$$r^k = e^{k \ \ln \ r} = e^{k_r \ \ln \ r} \left[\cos(k_i \ln r) + i \sin(k_i \ \ln r)\right] \qquad (4.4.184.)$$

As for examples in rectangular coordinates

we recall some facts from elementary complex analysis

Theorem 4.1.1.

Real and imaginary parts of a complex analytic function are harmonic functions

Proof.

Let $f(z) = f(x,y) = u(x,y) + iv(x,y)$ be analytic on $D \subset \mathbb{C}$

Then since f is analytic on D ☛ it is infinitely differentiable on D and thus u & v have (continuous) partial derivatives of all orders

Furthermore ☛ u and v satisfy Cauchy-Riemann conditions

Therefore ➡

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial v}{\partial y}\right] \\
&= \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \\
&= \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x}\right] = \frac{\partial}{\partial y}\left[-\frac{\partial u}{\partial y}\right] = -\frac{\partial^2 u}{\partial y^2}
\end{aligned}$$

Consequently ☛ $\nabla^2 u = 0$ and u is a harmonic function

We can prove that v is harmonic in much same way

4.4.2 Spherical harmonics

Consider equation $\nabla^2 u = 0$ in a spherically symmetric region

$r_1 \leq r \leq r_2, 0 \leq \theta \leq \pi, 0 \leq \phi \leq 2\pi$

We will use notation $\Omega = (\theta, \phi)$ with $d\Omega = \sin\theta \, d\theta \, d\phi$

In these coordinates ☛ Laplacian is given by (4.4.171) & (4.4.172)

Assuming a solution of the form $u(r, \Omega) = R(r)Y(\Omega)$ we obtain

$$R'' + \frac{2}{r}R' - \frac{k^2}{r^2}R = 0 \quad \text{and} \quad \nabla^2 Y = -k^2 Y \qquad (4.4.187.)$$

$k \equiv$ constant

It is easily seen that the solution of the angular part

is bounded and single-valued only if $k^2 = l(l+1)$ with $l \in \mathbb{N}$

Here ☛ $Y(\Omega) = Y_{lm}(\Omega)$ is spherical harmonic of order l

$$-\nabla^2_\Omega Y_{lm}(\Omega) = l(l+1)Y_{lm}(\Omega), \quad -l \leq m \leq l, \quad l = 0, 1, \ldots \qquad (4.4.188.)$$

with

$$Y_{lm}(\Omega) = (-1)^{(m+|m|)/2} \sqrt{\frac{(2l+1)(l-|m|)!}{4\pi(l+|m|)!}} \, P_l^{|m|}(\cos\theta) \, e^{im\phi} \qquad (4.4.189.)$$

$Y_{lm}(\Omega)$ are normalized eigenfunctions of ∇^2_Ω

$$\int_{S^2} Y_{lm}(\Omega)\, Y^*_{l'm'}(\Omega) d\Omega = \int_0^\pi \int_0^{2\pi} Y_{lm}\, Y^*_{l'm'}\, \sin\theta\, d\theta\, d\phi = \delta_{ll'}\delta_{mm'} \quad (4.4.190.)$$

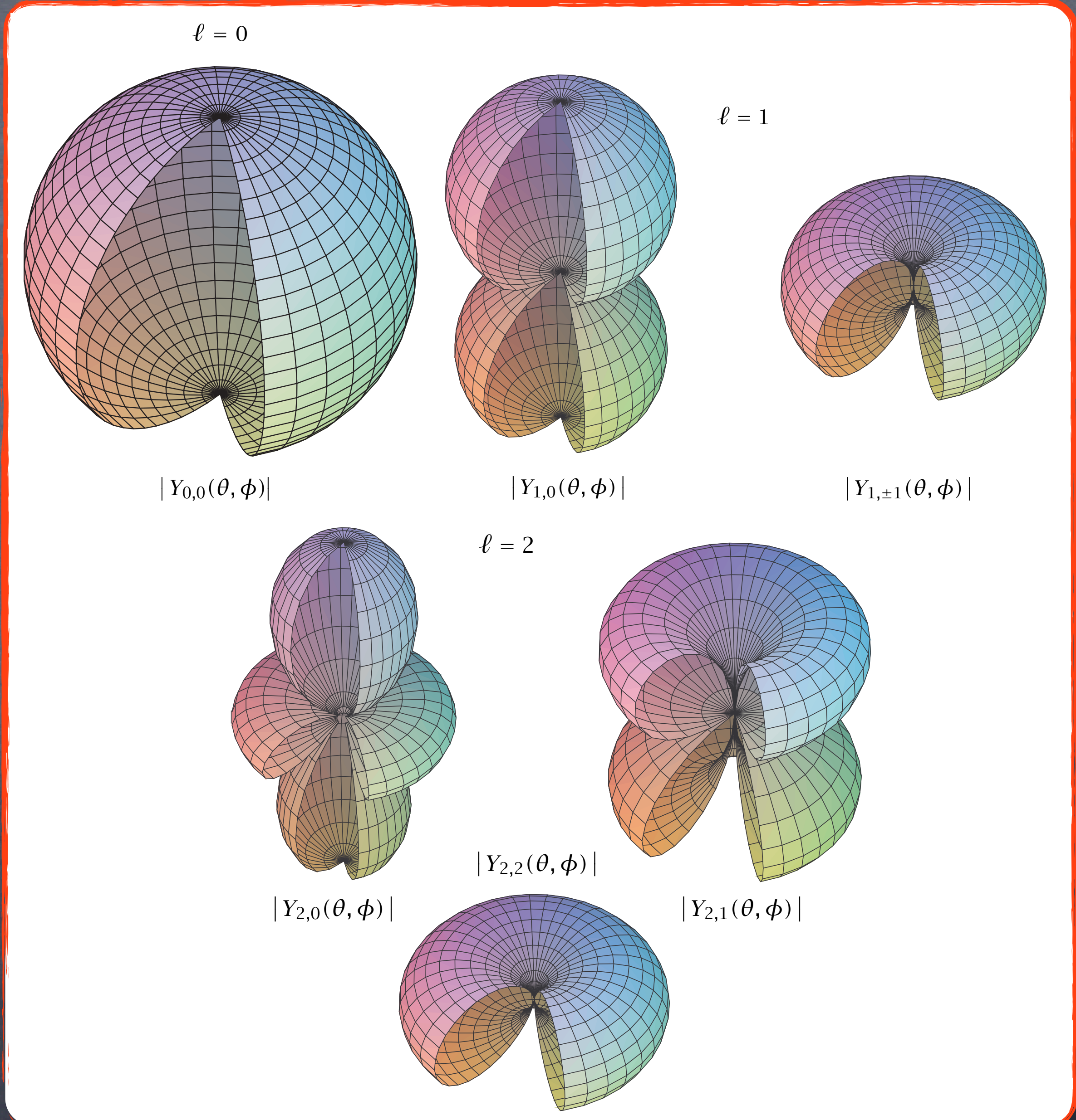
For this phase convention

$$Y^*_{lm}(\Omega) = (-1)^m\, Y_{l-m}(\Omega)$$

For $l = 0, 1, 2$

surfaces $r = |Y_{lm}(\theta,\phi)|$

look like this ☛



Equation for radial part is (as we have seen) of Euler type

solution r^λ and λ determined by $\lambda(\lambda-1)+2\lambda-l(l+1)=0$

$$\lambda = l \text{ and } \lambda = -l-1$$

Product solution is therefore of form

$$R(r)\,Y(\Omega) = \left(a\, r^l + b\, r^{-l-1}\right)\, Y_{lm}(\Omega) \qquad (4.4.191.)$$

Solutions which do not depend on ϕ

(i.e. invariants under rotations about z-axis)

correspond to $m=0$ with arbitrary l

while solutions independent of θ and ϕ

(i.e. invariants under rotations)

are obtained only for $l=0$ and are of form $u(r) = a + b/r$

The general solution takes form

$$u(r,\Omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}\, r^l + \frac{b_{lm}}{r^{l+1}}\right] Y_{lm}(\Omega) \qquad (4.4.192.)$$

For bounded solutions

if $r_1 = 0 \Rightarrow b_{lm} = 0$ while if $r_2 = \infty \Rightarrow a_{lm} = 0$

Example 4.4.1.

Consider problem of determining harmonic function $u(r,\Omega)$

in the interior of a sphere of radius $r_2 = R$

knowing their values on the surface $u(R,\Omega) = f(\Omega)$

Function must be of the form

$$u(r,\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\ r^l\ Y_{lm}(\Omega) \quad (4.4.193.)$$

Boundary condition leads to

$$u(R,\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} R^l Y_{lm}(\Omega) \quad (4.4.194.)$$

which is series expansion of spherical harmonics of $f(\Omega)$

Taking into account (4.4.190.) ☛ coefficients are given by

$$a_{lm} = \frac{1}{R^l} \int_{S^2} Y^*_{lm}(\Omega)\ f(\Omega)\ d\Omega \qquad (4.4.195.)$$

If $f(\Omega) = f(\theta) \Rightarrow a_{lm} = 0$ for $m \neq 0$

and thus $u(r,\Omega) = c_l\ r^l\ P_l(\cos\theta_0)$ with $c_l = a_{l0}\sqrt{(2l+1)/(4\pi)}$

If $f(\Omega) = c \Rightarrow c_l = 0$ for $l \neq 0$

(because of orthogonality of P_l for $l \neq 0$ with $P_0 = 1$)

and therefore $u(r,\Omega) = c$

In general ☛ using (4.4.16.) we obtain

$$u(r,\Omega) = \int_{S^2} \left[\sum_{l=0}^{\infty} \left(\frac{r}{R}\right)^l \sum_{m=-l}^{l} Y_{lm}(\Omega)\, Y^*_{lm}(\Omega') \right] f(\Omega')\ d\Omega' \qquad (4.4.196.)$$

To evaluate this series

let us first introduce theorem of addition of spherical harmonics

$$\sum_{m=-l}^{l} Y_{lm}(\Omega)\ Y^*_{lm}(\Omega') = \frac{2l+1}{4\pi}\ P_l(\cos\theta_0) \qquad (4.4.197.)$$

where θ_0 is angle between directions determined by Ω and Ω'

$$\cos(\theta_0) = \hat{\mathbf{n}}(\Omega)\ \hat{\mathbf{n}}(\Omega') = \cos\theta\ \cos\theta' + \sin\theta\ \sin\theta'\ \cos(\phi - \phi') \quad (4.4.198.)$$

with $\hat{\mathbf{n}}(\Omega) = (\sin\theta\ \cos\phi,\ \sin\theta\ \sin\phi,\ \cos\theta)$

Equation (4.4.197.) reflects fact that first term is a scalar

that depends only on angle θ_0 between Ω and Ω'

In this case ☛ by choosing $\Omega = (\theta, \phi) = (0, 0)$

and given that $P_l^m(1) = \delta_{m0}$

it follows that $Y_{lm}(0,0) = \delta_{m0} Y_{l0}(0,0) = \sqrt{(2l+1)/(4\pi)}$ so we obtain

$$\sum_{m=-l}^{l} Y_{lm}(0,0)\; Y_{lm}(\Omega') = Y_{l0}(0,0)\; Y_{l0}^*(\Omega') = \frac{2l+1}{4\pi} P_l(\cos\theta') \quad (4.4.199.)$$

which leads to (4.4.197) as $\theta_0 = \theta'$ if $\theta = 0$

In addition ☛ (4.4.197.) reflects fact that

as a function of $\Omega, P_l(\cos\theta_0)$ is also eigenfunction of ∇^2_Ω

with eigenvalue $-l(l+1)$

and therefore must be a linear combination of $Y_{lm}(\Omega)$ with same l

$$P_l(\cos\theta_0) = \sum_{m=-l}^{l} c_m\; Y_{lm}(\Omega) \quad (4.4.200.)$$

with $$c_m = \int_{S^2} Y_{lm}^*(\Omega)\, P_l(\cos\theta_0) d\Omega = \frac{4\pi}{2l+1}\; Y_{lm}^*(\Omega') \quad (4.4.201.)$$

We must now evaluate series $\sum_{l=0}^{\infty}(2l+1)\ (r/R)^l\ P_l(\cos\theta_0)$ with $r < R$

To this end we first introduce expansion

$$\frac{1}{d(r,R,\theta_0)} = \sum_{l=0}^{\infty} \frac{r^l}{R^{l+1}}\ P_l(\cos\theta_0), \quad \text{for} \quad r < R \qquad (4.4.202.)$$

with $d(r,R,\cos\theta_0) = \sqrt{R^2 + r^2 - 2Rr\cos\theta_0}$

Relation (4.4.202.) can be derived by noting that

first term of series is 3-dimensional harmonic function of (r, θ_0)

and must therefore be of form $\sum_{l=0}^{\infty} c_l r^l P_l(\cos\theta_0)$

For $\theta_0 = 0, d^{-1}(r,R,0) = (R-r)^{-1} = R^{-1}\sum_{l=0}^{\infty}(r/R)^l$ & so $c_l = 1/R^{l+1}$

If we take derivative of (4.4.202) with respect to r we can write

$$\sum_{l=0}^{\infty} l \frac{r^l}{R^{l+1}} P_l(\cos\theta_0) = r \frac{\partial}{\partial r} \sum_{l=0}^{\infty} \frac{r^l}{R^{l+1}} \ P_l(\cos\theta_0) = -\frac{r(r - R\cos\theta_0)}{(R^2 + r^2 - 2Rr\cos\theta_0)^{3/2}}$$

combining this relation with (4.4.203.) we obtain

$$\sum_{l=0}^{\infty} (2l+1) \ (r/R)^l \ P_l(\cos\theta_0) = \frac{R^2 - r^2}{(R^2 + r^2 - 2Rr\cos\theta_0)^{3/2}}$$

(4.4.203.)

Substituting(4.4.198) and (4.4.204.) into (4.4.197.)

we arrive at the solution for interior of the sphere

$$u(r,\Omega) = \frac{R(R^2 - r^2)}{4\pi} \int_{S^2} \frac{f(\Omega')}{d^3(r, R, \theta_0)} \, d\Omega' \quad (4.4.204.)$$

Example 4.4.2.

We consider now problem of determining the harmonic function on the outside of the sphere ($r > R$) knowing their values on surface $u(R,\Omega)$

From (4.4.193.) we see that if u is harmonic function then

$$v(r,\Omega) = \frac{R}{r}\, u(R^2/r,\Omega) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left[a_{lm}\frac{R^{2l+1}}{r^{l+1}} + b_{lm}\frac{r^l}{R^{2l+1}}\right]Y_{lm}(\Omega)$$

is also harmonic as it is of form (4.4.192.) and satisfies boundary condition $v(R,\Omega) = u(R,\Omega)$

if u is defined for $r < R$ then v is defined for $r > R$

Therefore ☛ the solution for outside of the sphere is

$$v(r,\Omega) = \frac{R(r^2 - R^2)}{4\pi}\int_{S^2}\frac{f(\Omega')}{d^3(r,R,\theta_0)}\, d\Omega' \qquad (4.4.206.)$$

4.4.3 Green function for Laplace operator

Consider boundary value problem

$$\begin{cases} \nabla^2 u(\mathbf{x}) = h(\mathbf{x}) & \text{in } \Omega \\ u(\mathbf{x}) = f(\mathbf{x}) & \text{on } \partial\Omega \end{cases} \quad (4.4.207.)$$

where $\Omega \subset \mathbb{R}^n$ is a normal domain

that is a bounded domain such that:

(i) boundary $\partial\Omega$ consists of a finite number of smooth surfaces

(ii) any straight line parallel to a coordinate axis

either intersects $\partial\Omega$ at a finite number of points

or has a whole interval in common with $\partial\Omega$

Let $\mathbf{x} = \vec{x}$ be a fixed point in $D \subset \mathbb{R}^2$ and let $\vec{\xi}$ be a variable point

Let r be distance from $\mathbf{x}$ to $\vec{\xi}$ ☛ $r = \sqrt{\sum_{j=1}^{n} (x_j - \xi_j)^2}$

Solution of (4.4.207) can be written in terms of Green function

satisfying ☛ $\begin{cases} \nabla^2 G = \delta(r) & \text{in } \Omega \\ G = 0 & \text{on } \partial\Omega \end{cases} \quad (4.4.208.)$

To obtain explicit form of $u(\mathbf{x})$ we make use of Gauss theorem

$$\int_\Omega \vec{\nabla} \, . \, \mathbf{F} \, dV = \int_{\partial\Omega} \mathbf{F} \, . \, \hat{\mathbf{n}} \, dA$$

and write Green's first identity

$$\int_\Omega (G\nabla^2 u + \vec{\nabla} u \, . \vec{\nabla} G) \, dV = \int_{\partial\Omega} G(\vec{\nabla} u \, . \, \hat{\mathbf{n}}) \, dA \quad (4.4.209.)$$

with vector field $\mathbf{F} = G\vec{\nabla} u$

Left side is a volume integral over (n-dimensional) volume Ω

right side is surface integral over boundary of volume Ω

Closed manifold $\partial\Omega$ is quite generally boundary of Ω

oriented by outward-pointing normals

and $\hat{\mathbf{n}}$ is outward pointing unit normal field of boundary $\partial\Omega$

Interchanging G with u and subtracting gives

Green's second identity

$$\int_\Omega (u\nabla^2 G - G\nabla^2 u) \, dV = \int_{\partial\Omega} (u\vec{\nabla} G - G\vec{\nabla} u) \, . \, \hat{\mathbf{n}} \, dA \quad (4.4.210.)$$

Substituting (4.4.207.) and (4.4.208.) into Green's second identity

leads to ☛ $$u(\mathbf{x}) - \int_\Omega G\, h\, dV = \int_{\partial\Omega} f\, \vec{\nabla} G\,.\,\hat{\mathbf{n}}\, dA \qquad (4.4.210.)$$

rearranging we obtain

$$u(\mathbf{x}) = \int_\Omega G\, h\, dV + \int_{\partial\Omega} f\, \vec{\nabla} G\,.\,\hat{\mathbf{n}}\, dA$$
$$= \int_\Omega G\, h\, dV - \int_{\partial\Omega} f\, \frac{\partial G}{\partial \hat{n}}\, dA \qquad (4.4.211.)$$

If we can find G that satisfies (4.4.208) ☛ we can use (4.4.211)

to find the solution $u(\mathbf{x})$ of boundary value problem (4.4.207.)

To find Green's function for a domain $D \subset \mathbb{R}^n$

we first find fundamental function that satisfies $\nabla^2 K = \delta(r)$

We have already seen that $\ln(r = |\mathbf{x}|)$ is harmonic in $\mathbb{R}_0^2$

and r^{2-n} is harmonic in $\mathbb{R}_0^n$ for $n \geq 3$

In terms of these solutions we define fundamental solutions

for Laplace equation with pole at $\mathbf{x} = \vec{\xi}$ by

$$K(\mathbf{x}, \vec{\xi}) = \begin{cases} -\frac{1}{2\pi} \ln |\mathbf{x} - \vec{\xi}| & n = 2 \\ \frac{1}{(n-2)} \, \omega_n \, \left|\mathbf{x} - \vec{\xi}\right|^{2-n} & n \geq 3 \end{cases} \tag{4.4.212.}$$

where ω_n denotes surface area of unit sphere in $\mathbb{R}^n$

that is ☛ $$\omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{4.4.213.}$$

In general ☛ Green's function for a region Ω can be obtained

by adding a harmonic function $v(\mathbf{x}, \vec{\xi})$ i.e. $\nabla^2 v = 0$ in Ω

to fundamental Green's function for complete space ☛ $K(\mathbf{x}, \vec{\xi})$

such that sum satisfies boundary condition $G(\mathbf{x}, \vec{\xi}) = 0$ if $\mathbf{x} \in \partial\Omega$

Of course ☛ v does not need be harmonic outside Ω

We illustrate this idea with some specific examples

Example 4.4.3.

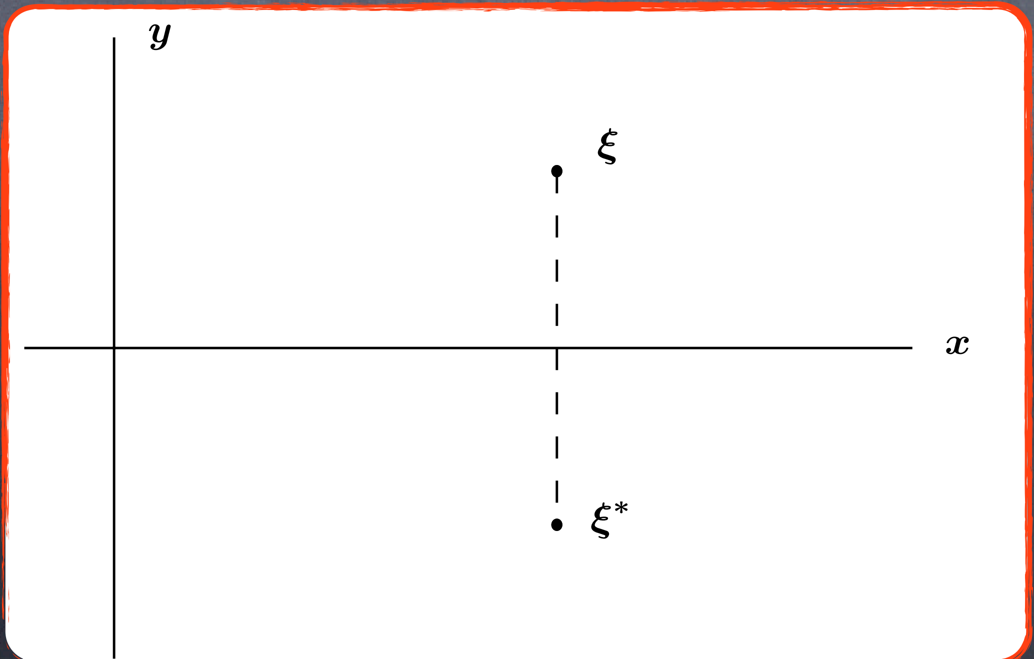
Consider Dirichlet problem for upper half-plane in $\mathbb{R}^2$

$$\begin{cases} \nabla^2 u(x,y) = 0 & \mathbb{R}^2_+ = \{(x,y) : x \in \mathbb{R}, y > 0\} \\ u(x,0) = f(x) & x \in \mathbb{R} \end{cases} \quad (4.4.214.)$$

Green function $G(\mathbf{x}, \vec{\xi})$ must cancel on x-axis $(y = 0)$

$$\mathbf{x} = (x,y), \vec{\xi} = (x', y')$$

This can be achieved using method of images placing in addition to point source at $\vec{\xi} = (x', y')$ with charge 1 another (virtual) at $\vec{\xi}^* = (x', -y')$ with charge -1



Green function is found to be

$$G(\mathbf{x},\vec{\xi}) = \frac{1}{2\pi}\left\{\ln\left[\sqrt{(x-x')^2+(y+y')^2}\right]-\ln\left[\sqrt{(x-x')^2+(y-y')^2}\right]\right\}$$
$$= \frac{1}{2\pi}\ln\left[\frac{\sqrt{(x-x')^2+(y+y')^2}}{\sqrt{(x-x')^2+(y-y')^2}}\right] \qquad (4.4.215.)$$

Clearly ☛ G is harmonic for $(x,y)\neq(x',y')$

and satisfies $G\left((x,0),(x',y')\right)=0$

Normal derivative at $y'=0$ is

$$\left.\frac{\partial G}{\partial n}\right|_{y'=0} = -\left.\frac{\partial G}{\partial y'}\right|_{y'=0} = -\frac{1}{\pi}\frac{y}{(x-x')^2+y^2} \qquad (4.4.216.)$$

Solution for Dirichlet problem in upper half-plane is then given by

$$u(x,y) = \int_{-\infty}^{\infty}\left.\frac{\partial G(\mathbf{x},\vec{\xi})}{\partial y'}\right|_{y'=0} f(x')dx'$$
$$= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(x')}{(x-x')^2+y^2}\,dx' \qquad (4.4.217.)$$

If more generally

$$\mathbf{x} = (x_1, \ldots, x_n) \qquad \text{and} \qquad \vec{\xi} = (\xi_1, \ldots, \xi_n)$$

$$\begin{aligned}
u(x_1, \ldots, x_n) &= \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_{n-1} \left.\frac{\partial G(\mathbf{x}, \vec{\xi})}{\partial \xi_n}\right|_{\xi_n = 0} f(\xi_1, \ldots, \xi_{n-1}) \\
&= \frac{2x_n}{\omega_n} \int_{-\infty}^{\infty} d\xi_1 \ldots \int_{-\infty}^{\infty} d\xi_{n-1} \, f(\xi_1, \ldots, \xi_{n-1}) \\
&\times \frac{1}{(x_1 - \xi_1)^2 + \cdots + (x_{n-1} - \xi_{n-1})^2 + x_n^2} \qquad (4.4.218.)
\end{aligned}$$

Example 4.4.4.

Consider Dirichlet problem

$$\begin{cases} \nabla^2 u(\mathbf{x}) = 0 & B^3(0,R) \\ u(\mathbf{x}) = f(\mathbf{x}) & S^2(0,R) \end{cases} \qquad (4.4.219.)$$

where $B^3(0,R)$ is the ball of radius R centered at the origin

and $S^2(0,R)$ is its 2-dimensional spherical boundary

By placing a $+1$ charge at $\vec{\xi}$ with $|\vec{\xi}| = \xi < R$

and a virtual charge $-R/\xi$ at $\vec{\xi}^* = \vec{\xi} R^2/\xi^2$

with $|\vec{\xi}^*| = \xi^* = R^2/\xi > R$

we obtain $G(\mathbf{x}, \vec{\xi}) = \dfrac{1}{4\pi}\left[\dfrac{1}{d} - \dfrac{R}{\xi}\dfrac{1}{d'}\right]$ (4.4.220.)

where d, d' are distances from $\mathbf{x}$ to $\vec{\xi}$ and $\vec{\xi}^*$:

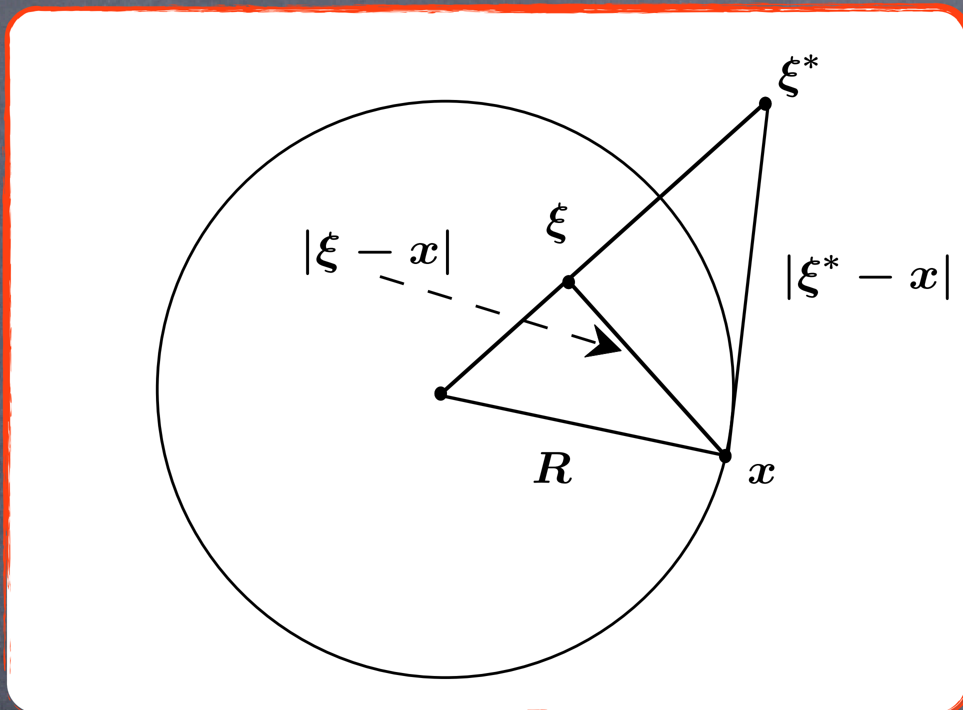
$$d^2 = r^2 + \xi^2 - 2r\xi\cos\theta_0 \quad \text{and} \quad d'^2 = r^2 + \xi^{*2} - 2r\xi^*\cos\theta_0 \qquad (4.4.221.)$$

with $r = |\mathbf{x}|$ and θ_0 angle between $\mathbf{x}$ and $\vec{\xi}$

In this way ☛ if x is at border of sphere $r = R$

the triangle $\triangle(\mathbf{0}, \vec{\xi}, \mathrm{x})$ is similar to the triangle $\triangle(\mathbf{0}, \mathrm{x}, \vec{\xi}^{\,*})$

where $\triangle(a, b, c)$ denotes triangle with vertices a, b, c



Therefore ☛ $d/d' = \xi/R$ and $G(\mathrm{x}, \vec{\xi}) = 0$

If $\xi \to 0$ then $d \to r$ and $d' \to \infty$ with $\xi d' \to R^2$

yielding $$G(\mathrm{x}, \mathbf{0}) = \frac{1}{4\pi}\left(\frac{1}{r} - \frac{1}{R}\right) \quad (4.4.222.)$$

Similarly ☛ in case of a circle $\subset \mathbb{R}^2$

$$G(\mathbf{x}, \vec{\xi}) = -\frac{1}{2\pi}\left[\ln(d) - \ln\left(\frac{d'\xi}{R}\right)\right]$$

$$= -\frac{1}{2\pi}\ln\left[\frac{dR}{d'\xi}\right] \qquad (4.4.223.)$$

if $\xi \to 0$

$$G(\mathbf{x}, \mathbf{0}) = -\frac{1}{2\pi}\ln\left(\frac{r}{R}\right) \qquad (4.4.224.)$$

In both cases ☛ $G(\mathbf{x}, \vec{\xi})$ is of the form $g(d) - g(d'\xi/R)$

We compute normal derivative at $\xi = R$ $\quad (d = d'$ & $\xi = \xi^* = R$)

$$\left.\frac{\partial G(\mathbf{x}, \vec{\xi})}{\partial \xi}\right|_{\xi=R} = g'(d)\left[\frac{\partial d}{\partial \xi} - \frac{\xi}{R}\frac{\partial d'}{\partial \xi} - \frac{d}{R}\right]$$

$$= g'(d)\,\frac{2R^2 - d^2 - 2Rr\cos\theta_0}{dR}$$

$$= g'(d)\frac{R^2 - r^2}{dR} \qquad (4.4.225.)$$

where $$\left.\frac{\partial d}{\partial \xi}\right|_{\xi=R} = -\left.\frac{\partial d'}{\partial \xi}\right|_{\xi=R} = \frac{R - r\cos\theta_0}{d} \qquad (4.4.226.)$$

In case of sphere ☛ $g'(d) = -1/(4\pi d^2)$

and so solution of (4.4.219.) with $u(R,\Omega) = f(\Omega)$ becomes

$$
\begin{aligned}
u(r,\Omega) &= -\int_{S^2} \frac{\partial G}{\partial \xi} f(\Omega')\, dA \\
&= \frac{R(R^2-r^2)}{4\pi} \int_{S^2} \frac{f(\Omega')}{d^3(R,r,\theta_0)}\, d\Omega' \quad (4.4.227.)
\end{aligned}
$$

where we have taken $dA = R^2 d\Omega$ and θ_0 is given by (4.4.198.)

For two dimensional case ☛ $g'(d) = -1/(2\pi d)$

Solution to (4.4.219.) with $u(R,\theta) = f(\theta)$ becomes

$$
\begin{aligned}
u(r,\theta) &= -\int_{S^1} \frac{\partial G}{\partial \xi} f(\theta')\, dA \\
&= \frac{R^2-r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')}{d^2(R,r,\theta_0)} d\theta' \quad (4.4.228.)
\end{aligned}
$$

where we have taken $dA = R d\theta'$ and $\theta_0 = \theta - \theta'$

n-dimensional problem is solved in a similar fashion

Example 4.4.5.

Gravity fields of Earth, Moon, and Mars have been described by Laplace series with real eigenfunctions

$$U(r,\theta,\phi) = \frac{GM}{R}\left\{\frac{R}{r} - \sum_{n=2}^{\infty}\sum_{m=0}^{\infty}\left(\frac{R}{r}\right)^{n+1}\left[C_{nm}Y^{l}_{mn}(\theta,\phi) + S_{nm}Y^{0}_{mn}(\theta,\phi)\right]\right\}$$

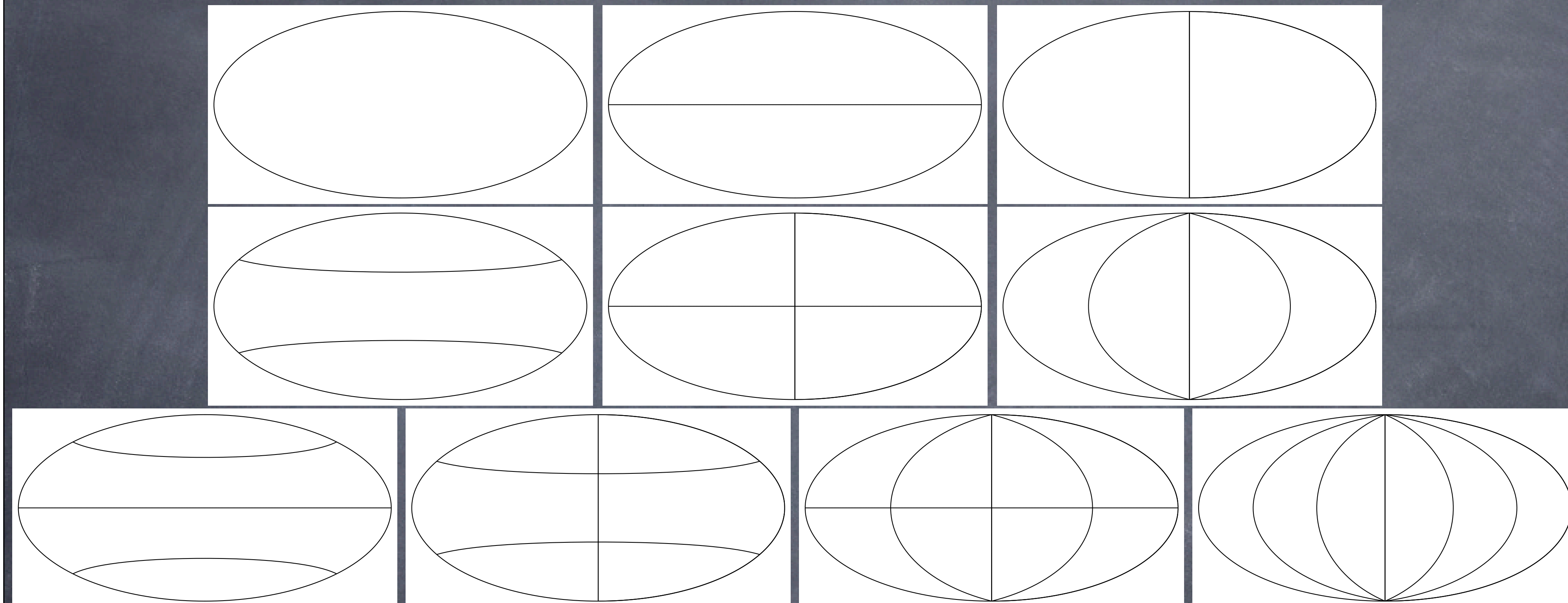
$$Y^{l}_{mn}(\theta,\phi) = P^{m}_{n}(\cos\theta)\,\cos(m\phi)$$

$$Y^{0}_{mn}(\theta,\phi) = P^{m}_{n}(\cos\theta)\,\sin(m\phi)$$

Satellites measurements lead to

Coefficient	Earth	Moon	Mars
C_{20}	1.083×10^{-3}	$(0.200 \pm 0.002) \times 10^{-3}$	$(1.96 \pm 0.01) \times 10^{-3}$
C_{22}	0.16×10^{-5}	$(2.4 \pm 0.5) \times 10^{-5}$	$(-5 \pm 1) \times 10^{-5}$
S_{22}	-0.09×10^{-5}	$(0.5 \pm 0.6) \times 10^{-5}$	$(3 \pm 1) \times 10^{-5}$

NODAL LINES SEPARATING REGIONS OF SPHERE FOR VARIOUS (L, M) PAIRS



Top row shows (0, 0) monopole

and partition of sphere into two dipoles (1, 0) and (1, 1)

Middle row shows quadrupoles (2, 0), (2, 1), and (2, 2)

Bottom row shows the l = 3 partitions, (3, 0), (3, 1), (3, 2), and (3, 3)


THE
END